IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 4 (Jul. - Aug. 2013), PP 62-74
www.iosrjournals.org

A Stability Theorem for Large Solutions of Three Dimensional Incompressible Magnetohydrodynamic Equations

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Abstract: *In this paper we prove a stability theorem for large solutions of three dimensional incompressible Magnetohydrodynamic equations under suitable initial & boundary conditions and appropriate boundedness assumptions on the initial data. We use Sobolev spaces as function space for various quantities.*

Keywords: *Magnetohydrodynamic equations, stability of solutions, strong solution,Sobolev spaces, generalized Gronwall's inequality.*

I. Introduction

Magnetohydrodynamics (MHD) is the study of flows of fluids which are electrically conducting and move in a magnetic field. The simplest example of an electrically conducting fluid is a liquid metal like mercury or liquid sodium. The major use of MHD is in plasma physics. Plasma is a hot ionized gas containing free electrons and ions. It is not obvious that plasmas are regarded as fluids since the mean free paths of collision between the electrons and the ions are macroscopically long. However collective interactions between large number of plasma particles can isotropize the particles velocity distributions in some local mean reference frame. This makes it sensible to describe the plasma macroscopically by a mean density, velocity and pressure. These main quantities obey the same conservation laws of mass, momentum and energy. As a result, a fluid description of a plasma is often reasonably accurate. MHD has technological applications also, like description of space within the solar system and astrophysical plasmas beyond the solar system.

The equations describing the motion of a viscous incompressible conducting fluid moving in a magnetic field are derived by coupling Navier Stokes equations with Maxwell's equations together with expression for the Lorentz force. The domain Ω in which the fluid is moving is either a bounded subset of R^3 or the whole space R^3. In this paper we restrict our considerations to a domain Ω which is a bounded subset of R^3. The equations of motion for three dimensional MHD flows are given by:

$$u_t + u\cdot\nabla u - b\cdot\nabla b + \text{grad } p = f + \nu \Delta u \quad ..(1.1)$$

$$b_t - \lambda \Delta b + b\cdot\nabla u - u\cdot\nabla b = \lambda \Delta b \quad ..(1.2)$$

$$\nabla \cdot u = 0 \qquad \nabla\cdot b = 0 \; ...(1.3)$$

$$\left.\begin{array}{l} ((u(x,0), b(x,0)) = (u_0(x), b_0(x)) \\ (u, b)|_{\partial\Omega} = (0,0) \end{array}\right\} \quad .... (1.4)$$

Where

u = u(x,t) is the velocity field ,b= b(x,t) is the magnetic field, ν is the kinematic coefficient of viscosity, λ is coefficient of magnetic diffusivity, p = p(t, x) is the pressure, f = f (t, x) is the internal force or volume force applied on the fluid, $(u_o(x), b_o(x)) = (u_o, b_o)(x)$, $x\in R^3$ is the initial condition & $\partial\Omega$ denotes the boundary of Ω .

Here we prove that for any global strong solution (u, b, p) of (1.1) − (1.4) satisfying

$$\int_0^\infty \|(\nabla u(t), \nabla b(t))\|^4 \, dt < \infty \quad ...(1.5),$$

there is a neighborhood of (u_o, b_o, f) such that (1.1) – (1.4) has a global strong solution for any data taken from this neighborhood. Here $\|(\cdot)\|$ denotes the natural product norm in $L^2(\Omega)$ X $L^2(\Omega)$, $L^2(\Omega)$ being the space of square integrable functions on Ω.

The idea to be followed is that if the reference solution decays to zero and another solution is sufficiently close initially, then the perturbed solution should remain close to the reference solution. As stated above, $\Omega \subset R^3$ is a bounded domain with boundary $\partial\Omega$ uniformly of class C^3. Thus for any point $x \in \partial\Omega$ there are positive constants r & M such that $\partial\Omega \cup B(x)$ consists of functions whose derivatives upto order 3 are bounded by M.

The scalar product in $L^2(\Omega)$, $H^m(\Omega)$, $H_0^m(\Omega)$ are denoted by:

$$(u, v) = \int_\Omega u \cdot v \, dx$$

$$(u, v) = \sum_{|\alpha|\le M} (D^\alpha u, D^\alpha v)$$

$$((u, v)) = \sum_{i=1}^n (D_i u, D_i v) \text{ respectively.}$$

& corresponding norms are denoted by

$$|u| = (u, u)^{1/2}$$

$$|u|_m = [(u_m, u_m)]^{1/2}$$

$$\|u\| = ((u, v))^{1/2}$$

Here α is the multiindex notation. Let P denote the Helmholtz projection

$$P : L^2(\Omega) \to H = \overline{\{u \in H_0^1(\Omega) : \nabla \cdot u = 0\}}^{L^2\Omega}$$

We denote by $U = \{u \in D(\Omega) | \text{div } u = 0\}$

V = Closure of U in $H_0^1(\Omega)$

$= \{u \in H_0^1(\Omega) : \nabla \cdot u = 0\}$

H = Closure of U in $L^2(\Omega)$

$= \{u \in L^2(\Omega) | \text{ div } u = 0\}$

By using Helmhotz projection we reformulate the problem (1.1)- (1.4) as:

By using Helmhotz projection we reformulate the problem (1.1) − (1.4) as :

$$\left.\begin{array}{l} u_t + Au + P((u \cdot \nabla)u) - P((b \cdot \nabla)b) = Pf \\ b_t + Ab + P((u \cdot \nabla)b) - P((b \cdot \nabla)u) = 0 \end{array}\right\} \quad \text{….(1.6)}$$

$$(u, b) = P(u, b) \quad \text{…(1.7)}$$

$$(u, b)|_{\partial\Omega} = (0, 0) \quad \text{…(1.8)}$$

$$((u(x,0), b(x,0)) = (u_0(x), b_0(x)) \text{ in } \Omega \quad \text{…(1.9)}$$

where $A = -P\Delta$. For the sake of convenience we take $\lambda = \nu = 1$. Thus A is a Stokes operator with domain $D(A) = H^2(\Omega) \cap V$, $V = H_0^1(\Omega) \cap H$

Where $H^2(\Omega)$, $H_0^1(\Omega)$ denote the Sobolev space as above for those domains Ω which satisfy the Poincare inequality $\|g\| \leq C \|\nabla g\|$ …(1.10)

$\forall g \in H_0^1(\Omega)$.

Over a period of past four decades, there have been existence study of qualitative properties of MHD flows of incompressible fluids and many results have been proved giving existence uniqueness and regularity of solutions of such flows see for example E. Sanchez Palencia [1] & M Sermange & R. Temam [2]. The functions spaces used are either Holder spaces or Sobolev spaces and methods used are from non-linear functional analysis such as Galerkin Approximation or fixed point theorems. The method of monotone operators is also useful in proving such result (see for example Zeidler [3]). The issue of stability of such solution is an important one, since solution of any dynamical system is supposed to be physically reasonable only if it is stable. There are a number of ways in which stability can be examined. For example, one can change the initial data slightly (with respect to norm on a suitable function space in which the solution lies) & study the corresponding change in the solution. If this change is also small, then we say that the solution is stable.

Another way is to study linear stability by methods of numerical analysis as in J.Priede, S.Aleksandrova & S.Molokov [4]. As far as Navier -Stokes equations are concerned there are number of result regarding stability of solutions, for example G.Ponce, R.Racke, T.C.Sideris, E.S.Titi [5], K.J.Ilin & V.A.Vladimorov[6].

In this paper we show that the result on stability proved in [5] are extendible to MHD flows. The function spaces used in this work are standard Sobolev spaces. Thus, in Section 2 we prove our first stability theorem for MHD flows. In Section 3, we state (without proof) some more theorems, which follow as consequences of the theorem proved in Section 2. In Section 4 we compare our results with the work by other authors & give concluding remarks commenting on probable future work.

II Section

In this section we state & prove our first stability theorem ,

We follow the notations from R. Temam [07]. Thus $L^{\infty}_{loc}(([0,\infty),V)$ denotes the space of locally integrable L^{∞} functions on $[0, \infty)$ with values in V where V is as defined above. With this we prove the following theorem.

Theorem 1.1: Let $(u^1,b^1) \in (L^{\infty}_{loc}((0,\infty); V\times V)) \cap (L^2_{loc}((0,\infty); D(A) \times D(A))$, be a strong solution of (1.1) − (1.4) with initial data

$((u^1(x,0), b^1(x,0)) = (u^1_0(x), b^1_0(x)) \in V\times V$, $Pf_1(\cdot,\mathbf{t}) \in H$ satisfying

$\int_0^{\infty}(\|\nabla u^1(t)\|^4_{L^2} + \|\nabla b^1(t)\|^4_{L^2})dt < \infty$ …(1.11)

(i) Let Ω be a domain which satisfies (5.10) & let $Pf_1 \in L^2([0,\infty); H)$ Suppose there exist a $\delta > 0$ such that if $(u_0, b_0) \in V \times V$, $Pf \in L^2([0,\infty); H)$ with

$$\|(\nabla u_0 - \nabla u^1_o,\ \nabla b_0 - \nabla b^1_o)\| + \int_0^{\infty}\|Pf(t) - Pf_1(t)\|^2\, dt < \delta$$

… (1.12)

then there is a unique global strong solution of (1.6) – (1.9) with data $((u_o, b_o, P, f)$ moreover there is an $M = M(\delta)$ with $M(\delta) \to 0$ as $\delta \to 0$ such that

$\sup_{t>0} \|(u(t),b(t)) - (u^1(t),b^1(t))\|_{H^1\times H^1} \le M(\delta)$ and if $\|(\nabla u^1(t), \nabla b^1(t))\|$ & $\|Pf(t) - Pf_1(t)\|$ decay to zero exponentially then $(\nabla u(t), \nabla b(t))$ decay to zero exponentially as $t \to \infty$.

(ii) Let $\Omega \subset R^3$ be a general domain & let $P f_1 \in L^1 \cap L^2([0,\infty); H)$ Suppose there exists a $\delta > 0$ such that if $(u_o, b_o) \in V \times V$ & $P f_1, \in L^1 \cap L^2([0,\infty); H)$ with

$\|(u_0, b_0) - (u^1_o, b^1_o)\|_{H^1\times H^1} + \int_o^{\infty}[\|Pf(t) - Pf(t)\| + \|Pf(t) - Pf_1(t)\|^2]\, dt < \delta$ …(5.13) then there is a unique global strong solution (u, b) of (5.6)−(5.9) with data $((u_0, b_0, P, f)$ & M (δ) → 0 as δ → 0 such that $\sup_{t>0} \|(u(t), b(t)) - (u^1(t), b^1(t))\|_{H^1 \times H^1} \le M(\delta)$.

Proof : Following the method illustrated in R. Temam [07] it can be easily proved that there exists a local strong solution $(u,b) \in (L^\infty((0, T); V \times V) \cap (L^2(0, T); D(A) \times D(A)$ of (1.6)−(1.9) for some

$T = T(\|(\nabla u, \nabla b)(t)\|$ uniformly on the interval of local existence. Towards this we let $w = u - u^1$, $v = b - b^1$ so that we have $(w, v) = (u, b) - (u^1, b^1)$.

Then we see that w & v satisfy

$$w_t + Aw + P((w \cdot \nabla)w + (w \cdot \nabla)u^1 + (u^1 \cdot \nabla)w)$$

$$- P((v \cdot \nabla) v + (v \cdot \nabla)b^1 + (b^1 \cdot \nabla)v) = Pf - Pf_1 \qquad \ldots(1.14A) \quad \text{and}$$

$$v_t + Av + P((w \cdot \nabla)v + (w \cdot \nabla)b^1 + (u^1 \cdot \nabla)v) - P((v \cdot \nabla)w + (v \cdot \nabla)u^1 + (u^1 \cdot \nabla)w) = 0 \qquad \ldots(1.14\text{ B})$$

with initial value $(w(0), v(0)) = (w_o, v_o)$ we observe that (u, b) does not appear explicitly in (1.14A) and (1.14B). Multiplying (1.14A) by Aw(t) & (1.14B) by Av(t) and integrating over Ω we get..

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + \|\nabla Aw\|^2 + \underbrace{\int_\Omega P(w\cdot\nabla)w\cdot Aw}_{I_1} + \underbrace{\int_\Omega P(w\cdot\nabla)u^1\cdot Aw}_{I_2} + \underbrace{\int_\Omega P(u^1\cdot\nabla)w\cdot Aw}_{I_3}$$

$$- \underbrace{\int_\Omega P(v\cdot\nabla)v\cdot Aw}_{I_4} - \underbrace{\int_\Omega P(v\cdot\nabla)b^1\cdot Aw}_{I_5} - \underbrace{\int_\Omega P(b^1\cdot\nabla)v\cdot Aw}_{I_6}$$

$$= \underbrace{\int_\Omega (Pf - Pf_1)\cdot Aw}_{I_7} \qquad \ldots(1.15A)$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \|\nabla Av\|^2 + \underbrace{\int_\Omega P(w\cdot\nabla)v\cdot Av}_{I_8} + \underbrace{\int_\Omega P(w\cdot\nabla)b^1\cdot Av}_{I_9} + \underbrace{\int_\Omega P(u^1\cdot\nabla)v\cdot Av}_{I_{10}}$$

$$-\underbrace{\int_\Omega P(v\cdot\nabla)w\cdot Av}_{I_{11}}-\underbrace{\int_\Omega P(v\cdot\nabla)u^1\cdot Av}_{I_{12}}-\underbrace{\int_\Omega P(b^1\cdot\nabla)w\cdot Av}_{I_{13}}=0 \quad \ldots(1.15B)$$

We now estimate the integrals I_1- I_{13} for this we need interpolation inequalities for functions g $\in$ D (A) Using the results from J.G.Heywood [08] ,under the assumption that $\partial\,\Omega$ is uniformly C^3 we have $\|\partial_{ij}^2 g\| \le C\,(\|Ag\| + \|\nabla g\|)$ using $D(A) \subset PL^2(\Omega)$ & integrating by parts & using Cauchy-Schwartz inequality we get.

$$\|\nabla g\|^2 = -\int_\Omega \nabla g g = \int_\Omega \nabla g P g = -\int_\Omega P\nabla g P g \le \|Ag\|\,\|g\| \le \|Ag\|^2 + \|g\|^2 \quad \ldots(1.16)$$

Therefore, in general we have the following inequality

$$\|\partial_{ij}^2 g\| \le C\,(\|Ag\| + \|g\|) \quad \ldots(1.17)$$

Now in the case where (1.10) holds it is clear that from (5.16) argument that $\|g\| + \|\nabla g\| \le C\|Ag\|$ can be improved when Ω satisfies (1.10) to $\|\partial_{ij}^2 g\| \le C\,\|Ag\|$ $\ldots(1.19)$

this holds in R^3 also. Next we have

$$\|g\|_{L_6} \le C\,\|\nabla g\| \ldots(1.20)$$

which holds for any domain Ω

Now by the Gagliardo - Nirenberg inequality in R^3 (1.19) & (1.20) we have

$$\|g\|_{L_\infty} \le \|g\|_{L_6}^{1/2}\,\|\partial_{ij}^2 g\|^{1/2} \le C\,\|g\|_{L_6}^{1/2}\,\|Ag\|^{1/2}$$
$$\le \|\nabla g\|^{1/2}\,\|Ag\|^{1/2} \ldots(1.21)$$

Now by using the Calderon extension theorem and then (1.17) we obtain

$$\|g\|_{L_\infty} \le C\|\nabla g\|^{\frac{1}{2}}(\|Ag\|^{\frac{1}{2}} + \|g\|^{1/2}) \,..(1.22)$$

In the case when (1.10) and hence also (1.18) are true and we see that (1.14) holds. Then from Gagliardo-Nirenberg and (1.19) we get

$$\|\nabla g\|_{L_3} \le C\|\partial_{ij}^2 g\|^{\frac{1}{2}}\|\nabla g\|^{\frac{1}{2}} \le C\|\nabla g\|^{\frac{1}{2}}\|Ag\|^{\frac{1}{2}} \ldots(1.23)$$

As the general case we can apply the Calderon Extension theorem

$$\|\nabla g\|_{L_3} \le C\|\nabla g\|^{\frac{1}{2}}(\|g\|^{\frac{1}{2}} + \|Ag\|^{\frac{1}{2}}) \ldots(1.24)$$

Which exactly as above improves to (1.23) in the case (1.10) is available. We first consider case (i) where (1.10) is valid .Hence Ω is such that (1.4) and (1.23) hold. Using these inequalities $I_1 - I_{13}$ are estimated as follows:

$$|I_1| = |\int_\Omega P(w\cdot\nabla)w\cdot Aw| = |\int_\Omega (w\cdot\nabla)w\cdot Aw|$$

$\leq \|w\|_{L_6} \quad \|\nabla w\|_{L_3} \ \|Aw\|$

$\leq C \ \|\nabla w\|^{3/2}\|Aw\|^{3/2}$

$\leq C_\epsilon \|\nabla w\|^6 \ + \epsilon \ \|Aw\|^2$ …(1.25)

$|I_2| \quad = |\int_\Omega P(w\cdot\nabla)u^1 \cdot Aw| = |\int_\Omega (w\cdot\nabla)u^1 \cdot Aw|$

$\leq \|w\|_{L_\infty} \quad \|\nabla u^1\| \ \|Aw\|$

$\leq C_\epsilon \|\nabla u^1\|^4 \ \|\nabla w\|^2 + \epsilon \ \|Aw\|^2$ …(1.26)

$|I_3| \quad = |\int_\Omega P(u^1\cdot\nabla)w \cdot Aw| = |\int_\Omega (u^1\cdot\nabla)w \cdot Aw|$

$\leq \|u^1\|_{L_6} \ \|\nabla w\|_{L_3} \ \|Aw\|$

$\leq C \ \|\nabla u^1\| \ \ \|\nabla w\|^{1/2} \ \ \|Aw\|^2$

$|I_3| \quad \leq C_\epsilon \|\nabla u^1\|^4 \ \|\nabla w\|^2 + \epsilon \ \|Aw\|^2$ …(1.27)

$|I_4| \quad = |\int_\Omega P(v\cdot\nabla)v \cdot Aw| = |\int_\Omega (v\cdot\nabla)v \cdot Aw|$

$\leq \|v\|_{L_6} \ \|\nabla v\|_{L_3} \ \|Aw\|$

$\leq C \ \|\nabla v\|^{3/2} \ \ \|Av\|^{1/2} \ \ \|Aw\|$

$\leq C_\epsilon \|\nabla v\|^6 + \epsilon \ \|Av\|^2 + \epsilon \ \|Aw\|^2$ …(1.28)

$|I_5| \quad = |\int_\Omega P(v\cdot\nabla)b^1 \cdot Aw| = |\int_\Omega P(v\cdot\nabla)b^1 \cdot Aw|$

$\leq \|v\|_{L_\infty} \quad \|\nabla b^1\| \ \ \|Aw\|$

$\leq C \ \|\nabla v\|^{1/2} \ \ \|Av\|^{1/2} \ \|\nabla b^1\| \ \ \|Aw\|$

$\leq C_\epsilon \|\nabla v\|^2 \ \|\nabla b^1\|^4 + \epsilon \ \|Av\|^2 + \epsilon \ \|Aw\|^2$ …(1.29)

$|I_6| \quad = |\int_\Omega P(b^1\cdot\nabla)v \cdot Aw|$

$= |\int_\Omega (b^1\cdot\nabla)v \cdot Aw|$

$\leq \|b^1\|_{L_6} \quad \|\nabla v\|_{L_3} \ \ \|Aw\|$

$\leq C \ \|\nabla b^1\| \ \ \|\nabla v\|^{1/2} \ \ \|Av\|^{1/2} \ \ \|Aw\|$

$\leq C_\epsilon \|\nabla b^1\|^4 \ \|\nabla v\|^2 + \epsilon \ \|Av\|^2 + \epsilon \ \|Aw\|^2$ …(1.30)

$|I_7| \quad = |\int_\Omega (Pf - Pf_1) \cdot Aw|$

$\leq C_\epsilon \|Pf - Pf_1\|^2 \ + \epsilon \ \|Aw\|^2$ …(1.31)

$|I_8| \quad = |\int_\Omega P(w\cdot\nabla)v \cdot Av|$

$= |\int_\Omega (w\cdot\nabla)v \cdot Av|$

$\leq \|w\|_{L_6} \ \|\nabla v\|_{L_3} \ \|Av\|$

$\leq C_\in \|w\|^4 \|\nabla v\|^2 + \in \|Av\|^2$

$\leq C_\in \|w\|^6 + C_\in \|\nabla v\|^4 + \in \|Av\|^2$ …(1.32)

$|I_9| = |\int_\Omega P(w\cdot\nabla)b^1 \cdot Av|$

$\leq \|w\|_{L_\infty} \|\nabla b^1\| \|Aw\|$

$\leq C_\in \|\nabla b^1\|^4 \|\nabla w\|^2 + \in \|Aw\|^2 + \in \|Aw\|^2$ …(1.33)

$|I_{10}| = |\int_\Omega P(u^1\cdot\nabla) \cdot Av|$

$\leq \|u^1\|_{L_6} \|\nabla v\|_{L_3} \|Av\|$

$\leq C_\in \|\nabla u^1\|^4 \|\nabla v\|^2 + \in \|Av\|^2$ …(1.34)

$|I_{11}| = |\int_\Omega P(v\cdot\nabla)w \cdot Av|$

$\leq \|v\|_{L_6} \|\nabla w\|_{L_3} \|Av\|$

$\leq C_\in \|v\|^4 \|\nabla w\|^2 + \in \|Av\|^2$

$\leq C_\in \|\nabla v\|^6 + C_\in \|\nabla w\|^4 + \in \|Aw\|^2 + \in \|Av\|^2$ …(1.35)

$|I_{12}| = |\int_\Omega P(v\cdot\nabla)u^1 \cdot Av|$

$\leq \|v\|_{L_\infty} \|\nabla u^1\| \|Av\|$

$\leq C_\in \|\nabla u^1\|^4 \|\nabla v\|^2 + \in \|Av\|^2$ …(1.36)

$|I_{13}| \leq |\int_\Omega P(b^1\cdot\nabla)w \cdot Av|$

$\leq \|b^1\|_{L_6} \|\nabla w\|_{L_3} \|Av\|$

$\leq C_\in \|\nabla b^1\|^4 \|\nabla w\|^2 + \in \|Aw\|^2 + \in \|Av\|^2$ …(1.37)

Choosing $\in$ sufficiently small, using (1.18), (1.15), (5.25)-(5.37) we get

$\frac{d}{dt}\|\nabla w\|^2 + C_o\|\nabla w(t)\|^2$

$\leq C_1[\|\nabla w\|^6 + \|\nabla v\|^6 + \|\nabla u^1\|^4 \|\nabla w\|^2 + \|\nabla b^1\|^4 \|\nabla v\|^2 + \|Pf - Pf_1\|^2]$ … (1.38)

and

$\frac{d}{dt}\|\nabla v\|^2 + C_2\|\nabla v(t)\|^2$

$\leq C_3[\|\nabla w\|^8 + \|\nabla v\|^8 + \|\nabla v\|^4 + \|\nabla w\|^4 + \|\nabla b^1\|^4 \|\nabla w\|^2 + \|\nabla u^1\|^4 \|\nabla v\|^2]$ … (1.39)

Now adding (1.38) & (1.39)

$\frac{d}{dt}\|\nabla w\|^2 + \frac{d}{dt}\|\nabla v\|^2 + C_4[\|\nabla v(t)\|^2 + \|\nabla w(t)\|^2]$

$\leq C_5[\|\nabla v(t)\|^8 + \|\nabla w\|^8 + \|\nabla v(t)\|^6 + \|\nabla w\|^6 + \|\nabla v(t)\|^4 + \|\nabla w\|^4 +$
$(\|\nabla u^1\|^4 + \|\nabla b^1\|^4) \|\nabla w\|^2 +$

$(\|\nabla u^1\|^4 + \|\nabla b^1\|^4)\ \|\nabla v\|^2 + \|Pf - Pf_1\|^2]$

$\leq C_5[\ \|\nabla v(t)\|^8 + \|\nabla w(t)\|^8 + \|\nabla v(t)\|^6 + \|\nabla w\|^6 + \|\nabla v(t)\|^4 +$

$+\|\nabla w\|^4 + (\|\nabla u^1\|^4 + \|\nabla b^1\|^4)\,(\|\nabla w\|^2 + \|\nabla v\|^2) +$

$\|Pf(t) - Pf_1(t)\|^2]$

using product norm we get

$\frac{d}{dt}(\|(\nabla w(t), \nabla v(t))\|^2) + C_4\,[\|(\nabla w(t), \nabla v(t))\|^2]$

$\leq\ C_5\ [(\|(\nabla w(t), \nabla v(t))\|^8 + \|(\nabla w(t), \nabla v(t))\|^6\ + \|(\nabla w(t), \nabla v(t))\|^4 +$
$(\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)\,(\|(\nabla w(t), \nabla v(t))\|^2\ +\ \|Pf(t) - Pf_1(t)\|^2]$

Let $h(t) = \|(\nabla w(t), \nabla v(t))\|^2$ using product norm the above inequality becomes

$\frac{d}{dt}h(t) + C_4\, h(t) \leq C_5[\ h(t)^4 + h(t)^3 + h(t)^2$

$+\ (\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)\, h(t) +\ \|Pf(t) - Pf_1(t)\|^2] +$ ….(5.40)

With $C_4 = \mathrm{Min}\,\{C_o, C_2\}$ & $C_5 = \mathrm{Max}\,\{C_1, C_3\}$ …(1.41)

Let , $\lambda = C_5 \sup_{s>0} e^{-C_4\frac{s}{2}} \int_0^s e^{-C_4\frac{\tau}{2}} + \|Pf(\tau) - Pf_1(\tau)\|^2]\, d\tau$ …(1.42)

Observe that

$\lambda \quad \leq C_5 \int_0^\infty \|Pf(\tau) - Pf_1(\tau)\|^2\ d\tau$ holds

If now :

$$\|(\nabla w_0, \nabla v_0)\|^2 + \frac{1}{2\max\left(1, \exp C_5 \int_0^\infty (\|\nabla u^1(\tau)\|^4 + \|\nabla b^1(\tau)\|^4)\, d\tau\right)}\lambda \qquad \leq \left(\frac{C_4}{2C_5}\right)^{\frac{1}{2}}$$

Is true then we infer that :

$$\|(\nabla w, \nabla v)\ (s)\|^2 \leq \left(\frac{C_4}{2C_5}\right)^{\frac{1}{2}}$$

Holds for $0 \leq s \leq t_1$ & some $t_1 \geq 0$ and by using standard norm we get :

$\frac{d}{dt}h(s) + C_6\, h(s)$

$\leq C_5\ \{[\|\nabla u^1(s)\|^4 + \|\nabla b^1(s)\|^4]\ \ \|h(s)\|^2 + \|Pf(s) - Pf_1(s)\|^2\}$ Consequently by a generalized Gronwall Lemma in S.Dragomir [09]

$h(s) \qquad \leq$

$$e^{-C_6\left(\frac{s}{2}\right)}(h(0) + C_5 \int_0^s e^{C_4\frac{\tau}{2}} \|Pf(\tau) - Pf_1(\tau)\|^2)\ \exp\left\{C_5 \int_0^s (\|\nabla u^1(\tau)\|^4 + \|\nabla b^1(\tau)\|^4)\ d\tau\right\}.$$

for $0 \leq s \leq t_1$

Hence we conclude that

$\|(\nabla w(t), \nabla v(t))\|^2 \leq \left(e^{-C_6\left(\frac{t}{2}\right)}\|(\nabla w(t), \nabla v(t))\|^2 + \lambda\right)$ exp

$$\left\{C_5 \int_0^{\infty} (\|\nabla u^1(\tau)\|^4 + \|\nabla b^1(\tau)\|^4)\ d\tau \right\}$$

Right hand side is uniformly bounded by assumption (1.11) since $\|(\nabla u^1(t), \nabla b^1(t))\|$ is also uniformly bounded so we conclude that $\|(\nabla u(t), \nabla b(t))\|$ is uniformly bounded. Thus (u, b) exists globally. We choose δ in (1.12) according to equation (1.42) & thus the remaining statements in (i) of Theorem (1.1) are immediate consequences of above justification.

We now go to the general case (ii).Multiplying the differential equation (1.15A) for w by w (t), and multiplying the (1.15B) for v by v (t) & integrating over Ω and adding and using bilinear property we get

$$\frac{d}{dt}\ [\|w\|^2+\|v\|^2]+\ C_4\ [\|\nabla w(t)\|^2\ + \|\nabla v(t)\|^2]\ \leq\ \int_\Omega P(w\cdot\nabla)u^1\cdot w\ -\int_\Omega P(v\cdot\nabla)b^1\cdot w$$

$$+\int_\Omega P(w\cdot\nabla)b^1\cdot v - \int_\Omega P(v\cdot\nabla)u^1\cdot v\ + \int_\Omega (Pf - Pf_1)\cdot w \quad \ldots(1.43)$$

Now

$$\begin{aligned}
|I_2| &= \left|\int_\Omega P(w\cdot\nabla)u^1\cdot w\right| \\
&= \left|\int_\Omega (w\cdot\nabla)\, u^1\cdot w\right| \\
&\leq C\ \|w\|_{L_4}^2\ \|\nabla u^1\| \\
&\leq C\|\nabla u^1\|\ \|w\|^{1/2}\ \|w\|^{3/2}{}_{L_6} \\
&\leq C\|\nabla u^1\|\ \|w\|^{1/2}\ \|\nabla w\|^{3/2} \\
&\leq C_\epsilon\|u^1\|^4\ \|w\|^2 + \epsilon\ \|\nabla w\|^2
\end{aligned}$$

$$\begin{aligned}
|I_5| &= \left|\int_\Omega P(v\cdot\nabla)b^1\cdot w\right| = \left|\int_\Omega (v\cdot\nabla)b^1\cdot w\right| \\
&\leq C\ \|v\|_{L_4}\ \|\nabla b^1\|\ \|w\|_{L_4} \\
&\leq C\ \|v\|^{1/4}\ \|w\|^{1/4}\ \|\nabla b^1\|\ \ \|v\|^{3/4}{}_{L_6}\ \ \|w\|^{3/4}{}_{L_6} \\
&\leq C\ \|v\|^{1/4}\ \|w\|^{1/4}\ \|\nabla b^1\|\|\nabla v\|^{3/4}\|\nabla w\|^{3/4} \\
&\leq C_\epsilon\ \|b^1\|^4\ \|w\|\|v\|\ \ + \epsilon\ \|\nabla w\|\|\nabla v\|
\end{aligned}$$

Similarly,

$$\begin{aligned}
|I_9| &= \left|\int_\Omega P(w\cdot\nabla)b^1\cdot v\right| \\
&= \left|\int_\Omega (w\cdot\nabla)b^1\cdot v\right| \\
&\leq C\ \|w\|_{L_4}\ \|\nabla b^1\|\ \ \|v\|_{L_4} \\
&\leq C_\epsilon\ \|b^1\|^4\ \|w\|\|v\|\ \ + \epsilon\ \|\nabla w\|\|\nabla v\|
\end{aligned}$$

And $|I_{12}| = \left|\int_\Omega P(v\cdot\nabla)u^1\cdot v\right|$

$$= |\int_\Omega (v\cdot\nabla)u^1\cdot v|$$
$$\leq C\,\|v\|_{L_4}^2\,\|\nabla u^1\|$$
$$\leq C_\epsilon\|u^1\|^4\,\|v\|^2 + \epsilon\,\|\nabla v\|^2$$

Here we have used Holder inequality,Gagliardo-Nirenberg inequality and Young Inequality. Thus using above estimates our equation (1.43) becomes:

$$\frac{d}{dt}[\|w(t)\|^2 + \|v(t)\|^2] + C_4\,[\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2]$$
$$\leq C_5[(\|\nabla u^1\|^4\|w\|^2 + \|\nabla u^1\|^4\|v\|^2) + \|\nabla b^1\|^4\|w\|\,\|v\| \quad]$$
$$+ \|Pf(t) - Pf_1(t)\|\;\|w(t)\|$$

Now, using product norm above inequality becomes

$$\frac{d}{dt}\|(w(t),v(t))\|^2 + C_4\,\|(\nabla w(t),\nabla v(t))\|^2$$
$$\leq C_5[\|\nabla u^1\|^4 + \|\nabla b^1\|^4](\|(w(t),v(t))\|^2)\;] +$$
$$\|Pf(t) - Pf_1(t)\|\;\|(w(t),v(t))\| \quad \text{….(1.44)}$$

By using Gronwall Lemma we get from (1.44)

$$\|(w(t),v(t))\| \leq \left(\|(w,v)(0)\| + \int_0^\infty \|Pf(\tau) - Pf_1(\tau)\|\,d\tau\right)\exp\{C_5\int_0^\infty\left(\|\nabla u^1\|^4 + \|\nabla b^1(t)\|^4\right) d\tau\}$$

for $t > 0$ …(1.45)

Observing that the estimates I_1 to I_{13} carry over except for the addition (in the case$\Omega\neq R^3$) of a term of the form C $\|w\|^2$, we get:

$$\frac{d}{dt}[\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2] + C_4\,[\|Aw(t)\|^2 + \|Av(t)\|^2]$$
$$\leq C_5\,[(\|\nabla w(t)\|^8 + \|\nabla v(t)\|^8 + \|\nabla w(t)\|^6 + \|\nabla v(t)\|^6\;\|\nabla w(t)\|^4 + \|\nabla v(t)\|^4 + (\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)\;(\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2) + \|Pf(t) - Pf_1(t)\|^2 + \|w(t)\|^2 + \|v(t)\|^2] \quad \text{…(1.46)}$$

Since $\|w(t)\|^2 \leq \text{const}\;(\|Aw(t)\|^2 + \|w(t)\|^2)$

And $\|v(t)\|^2 \leq \text{const}\;(\|Av(t)\|^2 + \|v(t)\|^2$

So we get:

$$\frac{d}{dt}(\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2) + \widetilde{C_4}\,[\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2]$$
$$\leq C_5\,[(\|\nabla w(t)\nabla v(t)\|^8 + \|\nabla v(t)\|^8 + \|\nabla w(t)\|^6 + \|\nabla v(t)\|^6 + \|\nabla w(t)\|^4 + \|\nabla v(t)\|^4 + (\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)\,\|\nabla w(t)\|^2 + \|\nabla v(t)\|^2) + \|Pf(t) - Pf_1(t)\|^2]$$

Using product norm we get

$\frac{d}{dt}(\|(\nabla w(t), \nabla v(t))\|^2) + \widetilde{C_4}[\|(\nabla w(t), \nabla v(t))\|^2]$

$\leq C_5 [(\|(\nabla w(t), \nabla v(t))\|^8 + \|(\nabla w(t), \nabla v(t))\|^6 + \|(\nabla w(t), \nabla v(t))\|^4 +$
$(\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)(\|(\nabla w(t), \nabla v(t))\|^2 + \|Pf(t) - Pf_1(t)\|^2]$

and like before we define h(t) = $\|(\nabla w(t), \nabla v(t))\|^2$ and

M = exp $\{C_5 \int_0^\infty (\|\nabla u^1(\tau)\|^4 + \|\nabla b^1(\tau)\|^4)\, d\tau\}$ we have using (1.13), (1.45) & (1.47),

$$\frac{d}{dt} h(t) + \widetilde{C_4}\, h(t) \leq C_5 [\, h(t)^4 + h(t)^3 + h(t)^2 + (\|\nabla u^1(t)\|^4 + \|\nabla b^1(t)\|^4)\, h(t) + \|Pf(t) - Pf_1(t)\|^2 + (\delta M)^2] \qquad \ldots (1.48)$$

Thus the boundedness of h(t) now follows by Gronwall's Inequality, for δ sufficiently small. This completes the proof of Theorem 1.

III.Section

Other Theorems As Consequences Of Theorem 1

Here we state (without proof) some theorems which are consequences of Theorem 1 proved in previous section.

Theorem 1.2 : Suppose that Ω is R^3 or a domain for which (1.10) holds & that the external force is zero. For strong solution (u^1, b^1) of (1.1) − (1.4), in the space $L^q([0, \infty), L^p(\Omega))$, where

$3/p+2/q = 1$, $3 < p < \infty$ is equivalent to the condition (1.11) described in theorem 1.1

Proof : The proof of this theorem can be given by proceeding analogously as in G.Ponce et.al [05].

Theorem 1.3: - (i) Let Ωbe obtained by rotation about the x_3 – axis of a planar_ domains D lying in the half plane $\{x_2 = 0, x_1 > 0\}$ at a positions distance from the Pf_1 – axis. Let $Pf_1 = 0$ suppose that $(u_o, b_o) \in D(A)$ is rotationally symmetric i.e. expressed in cylindrical co-ordinates (u_o, b_o) is independent of the angle of rotation ϕ & about the x_3 – axis Then there is a $\delta > 0$ such that for any $(u_o, b_o) \in V$ & $Pf \in L_2([0, \infty), H)$ Satisfying

$\|(u_o, b_o) - (u_o^1, b_o^1)\|_{H^1} + \int_0^\infty \|Pf(t)\|^2\, dt < \delta$ then there is a unique global strong solution (u, b) of (1.6) − (1.9).

(ii) let $\Omega = R^3$ Let $Pf_1 = 0$ & suppose that $(u_o, b_o) \in D(A) \cap H^4(R^3)$ is axially symmetric is expressed in cylindrical co-ordinates (u_o^1, b_o^1) is independent of the angle of rotation ϕ about the x_3 – axis and the components of (u_o, b_o) in the ϕ − direction is zero. Then There is a $\delta > 0$

s · t. for any $(u_o, b_o) \in V$ & Pf $\in L^1 \cap L^1$ ((0, ∞), H) satisfying $\|(u_o - u_o^1, b_o - b_o^1)\|_{H^1}$ $+\int_0^\infty [\|Pf(t) + Pf(t)\|^2]\, dt < \delta$ there is a unique, global strong solution (u, b) of (1.6)–(1.9).

Theorem 1.4: Let $(u_o^1, b_o^1) = (u_{o1}^1, b_{o1}^1), (u_{o2}^1, b_{o2}^1) \in L^1(R^2) \cap H^1(R^2)$

with $\nabla \cdot (u_o^1, b_o^1)$ there is a $\delta > 0$ s · t.

if $(w_0, v_0) \in V$ & $\|(w_o, v_o)\|_{H^1(R^3)} < \delta$ then (5.1) – (5.4)

has a unique global strong solution with data

$(u_o, b_o, f) = ((\widetilde{u_o^1, b_o^1}) + (w_o, v_o), 0)$

where $(u_o^1, b_o^1)(x_1, x_2, x_3) = ((u_{o1}^1, b_{o1}^1)(x_1, x_2), (u_{o2}^1, b_{o2}^1)(x_1, x_2), 0)$

IV.Conclusion

There have been more recent results on stability of viscous incompressible fluid flows [10],[11],[12] and with some modifications, it should be possible to extend these results to MHD case, at least in the case of bounded domain in R^3. The regularity results on Navier-Stokes equations have been sharpened with the use of Besov Spaces [13] & there are already some regularity & partial regularity results available in MHD- case. In our future work, we wish to study these results on regularity & stability, and extend them to MHD-case.

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