

## **Local Minimum Characterization of Interacting time Series Data: Application in Stability And stabilization**

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**Abstract:** *The dynamics of the best-fit mathematical model which can be used to describe the interaction between cowpea and groundnut is an important scientific problem. In this study, we propose to use the popular Euclidean norm or 2-norm cost function method to select the best-fit model parameters. Having selected the best-fit mathematical model, we intend to determine the important model parameters of this interacting system. The application of stability and stabilization of this proposed mathematical model will be studied and presented. Our novel results which we have not seen elsewhere will be presented and discussed with the expectation of providing further insights about these crop growth time series data.*

**Key words and phrases:** *Best-fit parameters, agricultural data, 2-norm, stability, stabilization.*

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### **I. Introduction**

This present study is based on the current data which were collected by two experts in microbiology who are working in the Niger Delta Region of Nigeria ([1]). Extensive literatures which relate to plant-plant interactions, modelling biological interacting populations, equations of determinate growth and plant growth analysis are ([3]; [4]; [5]; [6]; [7]; [8]; [9]; [10]; [11]; [12]). Given these data on the growth of legumes ([1]), it is a challenging scientific problem to construct a mathematical model which describes the dynamics of any two interacting legumes within an uncontaminated environmental setting. Attempting to develop these distinct nonlinear model equations will provide crop scientists with useful information such as the growth rates for two types of legumes, the intraspecific coefficients of legumes and their mass law action or interspecific coefficients as is popularly known in mathematical biology and molecular physics. The information about the doubling times of the two interacting legumes is an important insight which will assist further research in the stabilization of the mathematical model of interacting legumes and other land equivalent ratio studies in livelihood analysis and possibly about the numerical simulation of mutualism due to increasing sea level rise from a competition interaction model between two legumes in severely affected communities in the Niger Delta Region of Nigeria. For the purpose of our present study, we propose to utilize the penalty-function selection methods to select the best-fit model parameters from these uncontaminated agricultural data. As far as we know, this novel innovative numerical simulation technique has not been previously implemented computationally to tackle this interesting proposed scientific problem in the biological sciences.

### **II. Methodology**

In this study, we will focus on the application of the three popular mathematical norms ([2]) to select the best-fit parameters which are expected to characterize the dynamics of the interaction between two legumes such as the cowpea and groundnut over the growth season in weeks. The main thrust of this method concerns how well to select the parameters which will provide the best-fit between the provided data and our simulated data. We will deduce this characteristic of these data if we can successfully find the local minimum from a sequence of the 1-norm, 2-norm and infinity-norm monotonic values. Without a detailed familiar simplification about the notions of mathematical norms, we will simply present our calculations subsequently which we have obtained in the context of selecting the best-fit model parameters from some uncontaminated data of cowpea and groundnut. The core part of our method depends on an appropriate set up of a single initial value logistic model with only two parameters namely the intrinsic growth rate for the growing cowpea as well as the two similar parameters for the growing groundnut over a growing period in weeks. The data of [1] were collected every two weeks. Our proposed simulation technique is constructed using the logic that if the beginning of the first week starts on the first day, then the second week will start on day 8. In our context of the length of the growing period, subsequently after every two weeks the growth data will be obtained by simulation on day 22, followed by day 36, then on day 50, on day 64 and on day 78. By this proven procedure, our simulated data and the provided data will have six data points. Therefore, we can now measure the error between the provided data and our simulated data provided that we can successfully find the local minimum from the penalty-functions sequence of monotonic precise values.

In the following sequence of results, our aim is to find the best-fit model parameters using the three popular mathematical norms. The estimated daily growth rate for the cowpea data is 0.0225. This value is calculated by dividing the second data value of 1.81 by the first data value of 1.32 and taking the logarithm to base e. This estimated growth rate of 0.315695108679455 is called the weekly growth rate. Our expected daily growth rate of 0.022549650619961 can now be obtained by dividing this weekly growth rate by 14 since the data provided were collected every two weeks. For the purpose of this study, we prefer to consider only the estimated daily growth rate which is approximated to only 4 decimal places.

From Table 1, we observe that the local minimum for the 2-norm is 0.4691. A similar analysis was conducted for the groundnut data to find the local minimum and hence deduce its corresponding logistic model. We have found a smaller local minimum value of 0.4673 using the 2-norm selection of best-fit parameters which correspond to the two values of the intraspecific coefficients 0.007500 and 0.007475 when the steady-state values are 3.00 and 3.01.

Therefore, the 2-norm penalty function selected interspecific interaction model between cowpea and groundnut under a realistic assumed value of the interspecific coefficient is

**Examples Calculation of the local minimum**

no a b ss 2 – norm

1	0.0225	0.0132	1.71	1.4881
2	0.0225	0.0115	1.96	1.1881
3	0.0225	0.0102	2.21	0.9271
4	0.0225	0.0091	2.46	0.6940
5	0.0225	0.0083	2.71	0.5419
6	0.0225	0.0076	2.96	0.4691
7	0.0225	0.0070	3.21	0.5010
8	0.0225	0.0065	3.46	0.6046
9	0.0225	0.0061	3.71	0.7288
10	0.0225	0.0057	3.96	0.8824

Table 1. Calculation of the local minimum for the Cowpea data:

the notation ss stands for the steady state

$$dG(t)/dt = G(t)(0.0446 - 0.0121G(t) - 0.002C(t)).$$

where the initial biomasses are chosen appropriately.

**Application in Stability**

In this section, we are interested to study the stability behaviour of the steady-state solutions which define the dynamics of the above derived Lotka-Volterra system of continuous nonlinear first order ordinary differential equations.

Without loss of generality, following the mathematical theories of steady-state solutions and stability, our numerical mathematical analysis of the above Lotka-Volterra system is reported as follows:

**Examples Calculation of the stability behaviour**

no steady-state solutions  $\lambda$

1

$\lambda$

**2 Each Type of Stability**

1 (0, 0) 0.0225 0.0446 Unstable

2 (0, 3.686) -0.0446 0.0078 Unstable

3 (3, 0) -0.0225 0.0386 Unstable

4 (1.134, 3.50) -0.0076 -0.043 Stable

Table 2. Numerical calculations of the steady-state solutions and

their stability

It is very clear from this study that our best-fit mathematical model has only four steady-state solutions out of which three are unstable ( and will require further stabilizations) while the only unique positive co-existence steady-state solution is said to be stable. The trivial steady-state solution implies that the two interacting populations of cowpea and groundnut will go into the ecological risk of extinction while the semi-trivial steady-state solutions will imply the theory of competitive exclusion, that is, one population will survive at its carrying capacity while the second population will be driven into extinction.

**4. Application in Stabilization**

In this section, we are interested in studying the extent of stabilizing the unstable

steady-state solutions which we have determined in the last section. Our results will be presented for the unstable trivial steady-state solution and for the unstable semi-trivial steady-state solutions. These results clearly compliment the theory of stabilizing a mathematical model of population system ([13]) but differ in the sense that the stabilization of these unstable steady-state solutions is systematically conducted for a varying final time. The number of loops, that is, the number of repeated simulations which guarantees the stabilization for a given  $T_{final}$  and the step length  $k$  can be calculated using the simulation formula in which the specified value of the  $T_{final}$  is divided by the step length value of 0.01. As such for a given simulation run, the number of loops can be calculated. It is worth mentioning that there is a perfect correlation between the chosen values of the  $T_{final}$  and the number of loops for a fixed step length  $k$  of 0.01. We also observed that for the range of  $T_{final}$  values between 5 days and 100 days, stabilization of the trivial unstable steady-state solution is delayed while near stabilization will occur when the  $T_{final}$  takes values between 200 days and 300 days. Full stabilization will occur when  $T_{final}$  takes values between 400 days and 1000 days.

4.1. Stabilization of the unstable trivial steady-state solution. For a fixed step length value of 0.01 and varying values of the  $T_{final}$  in the unit of days, we are interested to find out if the unstable trivial steady-state solution can exhibit qualitative characteristics of delayed stabilization, near stabilization and full stabilization. Our novel contributions in this scenario are displayed next for the starting biomasses of cowpea and groundnut having the values of 1.5 grams and 1.4 grams.

Examples Calculation of the convergence point  
no  $T_{final}$   $N1e$   $N2e$

1	5	0.24877070752202	0.180209205013358
2	10	0.205525124041950	0.148683265740912
3	15	0.170740304731888	0.123519836168339
4	20	0.142053785867621	0.102767738484468
5	25	0.118331800315851	0.085606790411431
6	30	0.098671214286834	0.071383773075409
7	35	0.082346428010054	0.059573852811441
8	40	0.068770567026368	0.049752522513139
9	45	0.057466274204769	0.041574482640261
10	50	0.048043420448214	0.034757523617704
11	60	0.033618098133540	0.024321464556882
12	70	0.023550803135566	0.017038190666189
13	80	0.016511363870808	0.011945422572436
14	90	0.011582473792370	0.008379545586418
15	100	0.008128092440119	0.005880417712843
16	200	0.000237096554952	0.000171532217878
17	300	0.000006930173296	0.000005013772002
18	400	0.000000202576333	0.000000146557886
19	500	0.000000005921532	0.000000004284050
20	1000	0.000000000000000	0.000000000000000

Table 3. Calculation of the stabilization values for the unstable trivial steady-state solution

The main result which we have obtained from this table is the fact that the unstable trivial steady-state solution can be fully stabilized when the  $T_{final}$  is 1000 days or approximately two and half years. Although this result is mathematically tractable because as the time variable approaches infinity ([13]),  $N1e$  converges to zero and  $N2e$

converges to zero, however this result does not provide meaningful scientific insight. On the other hand, the delayed and near stabilization scenarios can provide useful information for growing two legumes such as cowpea and groundnut and hence enhance their productions. These insights are capable of guiding the legume farmers in terms of the timing of planting and the timing of harvesting.

4.2. Stabilization of the unstable semi trivial steady-state solution (0, 3.686).

For a fixed step length value of 0.01 and varying values of the Tfinal in the unit of days, we are interested to find out if the unstable semi-trivial steady-state solution (0, 3.686) can similarly exhibit qualitative characteristics of delayed stabilization, near stabilization and full stabilization. Our novel contributions in this scenario are displayed next for the starting biomasses of cowpea and groundnut having the values of 1.5 grams and 1.4 grams.

Examples Calculation of the convergence point  
no Tfinal N1e N2e

1	5	0.692001887753313	0.664624424891831
2	10	0.678379256028512	0.721628597491191
3	15	0.663690715434823	0.787055725688576
4	20	0.647123454411545	0.860788751410527
5	25	0.628571449214457	0.943271757752760
6	30	0.607964770749119	1.034786913533901
7	35	0.585280944969797	1.135401407696008
8	40	0.560556480387913	1.244917553499621
9	45	0.533896550332677	1.362831894897302
10	50	0.505481206514983	1.488310482449745
11	60	0.444479720900382	1.75698906488650
12	70	0.380376761316555	2.038304210991296
13	80	0.316625455389549	2.317035597442157
14	90	0.256597835282702	2.578537850872500
15	100	0.202911753909992	2.811634643751604
16	200	0.009715540711291	3.644373053901081
17	300	0.000363693620234	3.684440657470661
18	400	0.000024762974161	3.685892372365511
19	500	0.000012626847138	3.685944353536691
20	1000	0.000012176352934	3.685946283082031

Table 4. Calculation of the stabilization values for the unstable semi-trivial steady-state solution (0, 3.686)

In this scenario, we can consider the unstable semi-trivial steady-state solution (0, 3.686) to be relatively fully stabilized when the final time is 1000 days. The same semi trivial steady-state solution has shown qualitative characteristics of delayed stabilization and near stabilization.

4.3. Stabilization of the unstable semi trivial steady-state solution (3, 0)

For a fixed step length value of 0.01 and varying values of the Tfinal in the unit of days, we are interested to find out if the unstable semi-trivial steady-state solution (3, 0) can similarly exhibit qualitative characteristics of delayed stabilization, near stabilization and full stabilization. Our novel contributions in this scenario are displayed next for the starting biomasses of cowpea and groundnut having the values of 4.8 grams and 1.5 grams. In this scenario, when the final time is 5 days, the steady-state solution (N1e, N2e) converges to the point

(3.222786822990921, 0.042295457256194) while when the final time is 10 days, the steady-state solution (N1e, N2e)

) converges to the point (3.180492351348109, 0.033515607993592).

Both delayed and near stabilizations continue to exist until this unstable semi-trivial steady-state solution becomes fully stabilized when the final time is 1000 days as the steady-state solution (N1e, N2e)

) converges to the point (3.000000000000893, 0.000000000000162).

5. Concluding Remarks and Further Research

In this study, we have used the technique of local minimum for 2-norm penalty function to select the best-fit model parameters which form the best-fit candidate model. The stability of this model was studied. The qualitative behaviour of the steady-state solutions consistently display the ecological theory of extinction,

competitive exclusion and co-existence. For the unstable steady-state solutions, we have developed a controller using the Riccati formulation to stabilize them over a variation of the final time in the unit of days. We will expect these novel contributions to provide insights for the growth and production of legumes which will ultimately enhance food production security in Nigeria.

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