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Common Fixed Point Theorems for Sequence of Mappings in Generalisation of Strict Contractive Conditions

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Abstract: *The main purpose of this paper is to obtain fixed point theorems for sequence of mappings in strict contractive conditions which generalizes Theorem 1 of Aamri [1].*
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I. Introduction

In metric fixed point theory, strict contractive condition do not ensure the existence of common fixed point unless the space is assumed to be compact or the strict condition is replaced by stronger conditions as in [4-6]. In 1986, Jungck [3] proved common fixed point theorem by introducing the notion of compatible mappings. This concept was frequently used to prove the existence theorems in common fixed points of noncompatible mappings and is also very interesting. Work along these lines has recently been initiated by Pant [6, 7].Section 2 is devoted to definitions and known results which make the paper self reliant. In Section 3 we have proved a common fixed point theorem for sequence of mappings that generalizes the Theorem 2.8 of Aamri [1].

II. Preliminaries

Before proving our results, we need the following definitions and known results in this sequel.
Definition 2.1 ([3]). Let T and S be two self mappings of a metric space (X, d). T and S are said to be compatible if $\lim_{n\to\infty} d(ST_{x_n}, TS_{x_n}) = 0$ whenever $\{x_n\}$ is a sequence on X such that $\lim_{n\to\infty} S_{x_n} = \lim_{n\to\infty} T_{x_n}$ = t for some t ∈X.
Remark 2.2. Two weakly commuting maps are compatible, but the converse is not true as in shown in [3].
Definition 2.3 ([3]). Two self mappings T and S of a metric space X are said to be weakly compatible if $T_u = S_u$ for some u∈ X, then $ST_u = TS_u$.
Note 2.4. Two compatible maps are weakly compatible.
M. Aamri [1] introduced the concept property (E.A) in the following way.
Definition 2.5 (Aamri [1]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$=t for some t ∈X.
Definition 2.6 (Aamri [1]). Two self mappings S and T of a metric space (X, d) will be non compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(ST_{x_n}, TS_{x_n})$ is either nonzero or non-existent.
Remark 2.7. Two noncompatible self mappings of a metric space (X, d) satisfy the property (E.A).
Aamri [1] proved the following theorems.
Theorem 2.8. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that (i) T and S satisfy the property (E.A), (ii)$d(T_x,T_y) < \text{Max}\{d(S_x,S_y), [d(T_x,S_x)+ d(T_y,S_y)]/2, [d(T_y,S_x)+ d(T_x,S_y)]/2$ ∀x≠ y ∈ X, (iii) TX ⊂ SX. If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

III. Main Results

In this section we prove common fixed point theorem for sequence of mappings that generalizes Theorem 2.8.
Theorem 3.1 Suppose that $\{A_i\},\{T_i\}$ be two weakly compatible self mappings of a metric space (X, d) such that (1) For every i, $A_iX \subset T_iX$ (2) A_i andT_i satisfies the property (E.A).
(3) $d(A_ix,A_iy) < \text{Max}\{d(T_ix,T_iy),[d(A_ix,T_ix)+[d(A_iy,T_iy)]/2,[d(A_ix,T_iy)+[d(A_iy,T_ix)]/2\}$ for every x≠y ∈ X and for every i.If T_iX or S_iX is a complete subspace of X, then A_i and T_i have a unique common fixed point.
Proof: Suppose that A_i and T_i satisfies the property (E.A) there exists in X a sequence (x_n) satisfying $\lim_{n\to\infty} A_ix_n = \lim_{n\to\infty} T_ix_n$=t for some t ∈X,for every i.
Suppose that T_iX is complete. Then $\lim_{n\to\infty} T_ix_n = T_i$ a for some a ∈X.
Also $\lim_{n\to\infty} A_ix_n = T_i$a.We show that A_i a=T_i a.
Suppose that A_i a ≠ T_i a. Condition (3) implies
$d(A_ix_n,A_ia) < \text{Max}\{d(T_ix_n,T_ia),[d(A_ix_n,T_ix_n)+[d(A_ia,T_ia)]/2,[d(A_ix_n,T_ia)+[d(A_ia,T_ix_n)]/2\}$

Letting $n\to+\infty$ yields.
$d(T_ia,A_ia)<\text{Max}\{d(T_ia,T_ia),[d(T_ia,T_ia)+[d(A_ia,T_ia)]/2,[d(T_ia,T_ia)+[d(A_ia,T_ia)]/2\}= d(A_ia,T_ia)/2$. Which is a contradicition. Hence A_i a=T_i a for every i.
Since A_i and T_i are weakly compatible, $A_iT_ia=T_iA_ia$ and therefore $A_iA_ia=A_iT_ia=T_iA_ia= T_iT_ia$. $\forall i$, i=1,2,......,n.
Finally, we show that A_ia is a Common fixed point of A_i and, $T_i \forall i$, i=1,2,......,n.
Suppose that $A_ia\neq A_iA_ia$. Then we have, $d(A_ia,A_iA_ia) \leq \text{Max}\{ d(T_ia,T_iA_ia), [d(A_ia,T_ia)+d(A_iA_ia,T_iA_ia)]/2, [d(T_iA_ia,A_ia)+d(T_ia,A_iA_ia)]/2\} \forall i$,
$= \text{Max}\{ d(A_ia,A_iA_ia),0, [d(A_ia,A_iA_ia)+ d(A_ia,A_iA_ia)]/2\}$
$= \text{Max}\{ d(A_ia,A_iA_ia),d(A_i,A_iA_ia)\}= d(A_ia,A_iA_ia)$
Therefore $d(A_i,A_iaA_ia) < d(A_i,A_iaA_ia)$ Which is a contradicition. Hence $A_iA_ia=A_ia$ $\forall i$, and $T_iA_ia=A_iA_ia=A_ia$.
The proof is similar when A_iX is assumed to be a complete subspace of X. Since $A_iX \subset T_iX$. $\forall i$.
Uniquenesss: Suppose u,v are two fixed points A_i and T_i $\forall i$.
Then $A_iu=T_iu=u$ $\forall i$ and $A_iv=T_iv=v$ $\forall i$
$d(u,v)=d(A_iu,A_iv)<\text{Max}\{d(T_iu,T_iv),[d(A_iu,T_iu) + d(A_iv,T_iv)]/2,[d(A_iu,T_iv)+d(A_iv,T_iu)]/2\}$ $\forall i$.
$=\text{Max}\{d(u,v), [d(u,u)+ d(v,v)]/2, [d(v,u)+d(u,v)]/2\}$
$=\text{Max}\{d(u,v),0,d(u,v)\}=d(u,v)$
Therefore, $d(u,v)<d(u,v) \Rightarrow$ <=when $u\neq v$. Hence u=v.
Therefore A_i and T_i have unique common fixed point for all i.

Now we give an example to support our result.

Example for theorem 3.1: Let $X = [1, \infty)$ with the usual metric $d(x, y)=|x-y|$.
Define $A_i,T_i : X\to X$ $\forall i$ by $A_ix=3x-1$ and $T_ix=x^2+1$ $\forall x \in X$ and $\forall i$
Then (1) Ai and T_i satisfy the property (E.A) for the sequence $x_n=2+2/n$, n=1, 2,....., n
For, to prove if $\lim_{n\to\infty}A_ix_n = \lim_{n\to\infty}T_ix_n$ =t for some $t \in X$.
$\lim_{n\to\infty}A_ix_n = \lim_{n\to\infty}3x_n-1= \lim_{n\to\infty}3(2 + 2/n)-1=6-1=5\forall i$--------(i)
and $\lim_{n\to\infty}T_ix_n = \lim_{n\to\infty}x_n^2+1= \lim_{n\to\infty}(2+2/n)^2+1=4+1=5\forall i$--------(ii)
From (i) and (ii) we get A_i and T_i $\forall i$ satisfy the property (E.A).
(2) T_i and A_i $\forall i$ are weakly compatible. (3) A_i and T_i satisfy for all $x\neq y$, $\forall i$
$d(A_ix,A_iy) <\text{Max}\{d(T_ix,T_iy),[d(A_ix,T_ix)+d(A_iy,T_iy)]/2,[d(A_ix,T_iy)+d(A_iy,T_ix)]/2\}$
for all $x\neq y$, $\forall i$
(4) $T_i1=A_i1=2$. For, $T_i1=1^2+1=2$ and $A_i1=3(1)-1=2$.

Theorem 3.2. Let A_i,B_i,T_i and S_i be self maps of a metric space (X,d) such that
(1) $A_iX \subset T_iX$ and $B_iX \subset S_iX$ for every i. (2) (A_i,S_i),$\forall$ i is weakly compatible.
(3) (A_i,S_i) or (B_i,T_i), $\forall i$ satisfies the property (E.A).
(4) $d(A_ix,B_iy) <\text{Max}\{d(S_ix,T_iy),d(A_ix,S_ix),d(B_iy,T_iy),[d(S_ix,B_iy)+d(A_ix,T_iy)]/2\}\forall i$
If the range of the one of the mappings A_i,B_i,S_i or T_i $\forall i$ is a complete subspace of X,
Then (I) A_i and S_i $\forall i$ have a common fixed point. (II) B_i, T_i $\forall i$ have a common fixed point provided that (B_k, T_i) $\forall i$ for every k is weakly compatible.
(III) Ai, B_i, S_i and T_i $\forall i$ have a unique common fixed point provided that (I) and (II) are true.
Proof: Suppose that (B_i,T_i) $\forall i$ satisfies the property (E.A) => There exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} B_ix_n = \lim_{n\to\infty} T_ix_n$=t for some $t \in X$.for every i.
Since, $B_iX \subset S_iX$ $\forall i$, there exists a sequence $\{y_n\}$ in X such that $B_ix_n=S_iy_n$.
Therefore $\lim_{n\to\infty} B_ix_n = \lim_{n\to\infty} S_iy_n$=t (Since, $\lim_{n\to\infty} B_ix_n$ =t)
Let us prove that $\lim_{n\to\infty} A_iy_n = t$.
It is enough to prove that $A_iy_n=B_ix_n$ $\forall i$, and for sufficiently large n.
Suppose not, then using (4), $d(A_iy_n,B_ix_n) < \text{Max}\{d(S_iy_n,T_ix_n),d(A_iy_n,S_iy_n),d(B_ix_n,T_ix_n), [d(S_iy_n,B_ix_n) +d(A_iy_n,T_ix_n)]/2\}\forall i = \text{Max}\{ d(B_ix_n,T_ix_n),d(A_iy_n,B_ix_n),d(B_ix_n,T_ix_n), [d(B_ix_n, B_ix_n) + d(A_iy_n,B_ix_n)]/2\}$ $\forall i$
For sufficiently large n, $d(A_iy_n,B_ix_n) < \text{Max}\{d(A_iy_n,B_ix_n),1/2d(A_iy_n,B_ix_n)\}$,
$< d(A_iy_n,B_ix_n)$
=><= when $A_iy_n\neq B_ix_n\forall i$
Therefore $A_iy_n=B_ix_n$ as $n\to\infty$ $\forall i$
Therefore $\lim_{n\to\infty} A_iy_n = t$. $\forall i$ (Since, $\lim_{n\to\infty} B_ix_n = t$. $\forall i$)
Suppose, S_iX $\forall i$ is a complete subspace of X.
=>t= S_iu $\forall i$ for some $u\in X$.
Therefore $\lim_{n\to\infty} A_iy_n = \lim_{n\to\infty} B_ix_n = \lim_{n\to\infty} T_ix_n = \lim_{n\to\infty} S_iy_n$=$S_iu\forall i$

$d(A_iu,B_ix_n) < Max\{d(S_iu,T_ix_n),d(A_iu,S_iu),d(B_ix_n,T_ix_n), [d(S_iu,B_ix_n) + d(A_iu,T_ix_n)]/2\} \forall i$.
For sufficiently large n, $d(A_iu,B_ix_n) < Max\{d(A_iu,S_iu),1/2d(A_iu,S_iu)]/2\} \forall i$
$d(A_iu,S_iu)<d(A_iu,S_iu) \forall i$
$=><=$ when $A_iu \neq S_iu \forall i$.
Therefore $A_iu=S_iu \forall i$. This means that A_i and S_i $\forall i$ have coincidence point.
But $(A_i ,S_i) \forall i$ is weakly compatible. Therefore $S_iA_iu=A_iS_iu$ for every i and then $A_iA_iu=A_iS_iu=S_iA_iu=S_iS_iu$ for every i
Suppose $A_{iX} \subset T_iX$ for every i
=> There exists $v \in X$ such that $A_iu=T_iv \forall i$ $=>A_iu=S_iu=T_iv \forall i$
To prove that $T_iv=B_iv, \forall i$
Suppose $B_iv \neq T_iv, \forall i$, then $d(A_iu,B_iv) < Max\{d(S_iu,T_iv),d(A_iu,S_iu),d(B_iv,T_iv), [d(S_iu,B_iv) + d(A_iu,T_iv)]/2\} \forall i.=Max\{d(T_iv,T_iv),d(S_iu,S_iu),d(B_iv,T_iv), [d(T_iv,B_iv) + d(T_iv,T_iv)]/2\} \forall i$.
$=Max\{d(B_iv,T_iv), d(B_iv,T_iv)/2\} \forall i. = d(B_iv,T_iv) \forall i= d(B_iv,A_iu) \forall i$
Therefore $d(A_iuB_iv)< d(A_iu,B_iv) \forall i$ $= >< =$
Therefore $A_iu=B_iv \forall i$
$=> B_iv= A_iu=T_iv \forall i$
Therefore $B_iv= T_iv \forall i$
$=> A_iu= S_iu=T_iv =B_iv \forall i$
But $(B_k,T_i) \forall i$ is weakly compatible and for some k>1
$B_kT_iv= T_i B_kv$ for some k>1 $\forall i$
And $T_i T_iv= T_i B_kv = B_kT_iv = B_k B_kv$ for some k>1 and $\forall i$
We shall prove that A_iu is a common fixed point of A_i and S_i $\forall i$
Suppose $A_iu \neq A_iA_iu \forall i$
$d(A_iu, A_iA_iu)=d(A_iA_iu,B_iv) \forall i$ (Since,$A_iu=B_iv\forall i$)
$< Max\{d(S_iA_iu,T_iv),d(A_iA_iu,S_iA_iu),d(B_iv,T_iv), [d(S_iA_iu,B_iv) + d(A_iA_iu,T_iv)]/2\} \forall i$.
$=Max\{d(A_iA_iu,B_iv),0,0, ½[d(A_iA_iu,B_iv)+ d(A_iA_iu,B_iv)]\}$
$= d(A_iA_iu,B_iv) \forall i$
Therefore $d(A_iA_iu,B_iv)< d(A_iA_iu,B_iv) \forall i$
$= >< =$
Therefore $A_iA_iu=B_iv \forall i => A_i A_iu = A_iu= S_iA_iu \forall i$ (Since, $A_iA_iu =S_iA_iu \forall i$)
$=> A_iu$ is a common fixed point of A_i and S_i $\forall i$
This proves (I).
To prove that $B_kv=A_iu$ for some k>1, is a common fixed point of B_i and T_i $\forall i$
Suppose $B_kv \neq B_iB_kv$, $d(B_kv, B_iB_kv)=d(A_iu,B_iB_kv)< Max\{d(S_iu,T_iB_kv),d(A_iu,S_iu),d(B_iB_kv,T_iB_kv), [d(S_iu,B_iB_kv) +d(A_iu,T_iB_kv)]/2\} \forall i$
$= Max\{d(A_iu,B_iB_kv),0,d(B_iB_kv, B_iB_kv), [d(A_iu,B_iB_kv) + d(A_iu,B_iB_kv)]/2\} \forall i$
$= Max\{d(A_iu,B_iB_kv),0,0,d(A_iu, B_iB_kv) \} \forall i$
Therefore $d(B_kv, B_iB_kv)<d(A_iu,B_iB_kv) \forall i$
$= >< =$ (Since, $B_kv = A_iu$)
Therefore $A_iu=B_iB_kv$ That is, $B_kv= B_iB_kv=T_i B_kv$ (Since, $B_iv= T_iv$)
$=>B_kv$ is the common fixed point of B_i and T_i $\forall i$. This proves (II)
Now, A_iu is a common fixed point of A_i and S_i $\forall i$
$B_kv= A_iu$ is a common fixed point of B_i andT_i $\forall i$
Therefore, A_iu is the common fixed point of B_i , T_i and S_i for all i
The proof is similar when T_iX is assumed to be a complete subspace of X.
The cases in which A_iX or B_iX $\forall i$ is a complete subspace of X are similar to the cases in which S_iX or T_iX respectively is a complete space because $A_iX \subset T_iX$ and $B_iX \subset S_iX$ for every i.
Uniqueness: Suppose u,v are fixed points of A_i,B_i,T_i and S_i for every i.
Then $A_iu= S_iu=B_iu =T_iu =u \forall i$ and $A_iv= B_iv=T_iv =S_iv=v$ for every i.
$d(u,v)=d(A_iu,B_iv)$ for every i.
$< Max\{d(S_iu,T_iv),d(A_iu,S_iu), d(B_iv,T_iv),1/2[d(S_iu,B_iv)+d(A_iu,T_iv)]\}$
$=Max\{d(u,v),0,0,1/2[d(u,v)+d(u,v)]\}=Max\{d(u,v),d(u,v)\}$
$d(u,v)<d(u,v)$. $= >< =$ when $u \neq v$.
Therefore u = v. Hence Ai, B_i, S_i and T_i $\forall i$ have a unique common fixed point.

The following result due to U.Karuppiah [2] is a special case of the previous theorem 3.2.
Corollary3.2: Let $\{A_i\}$, S and T be self maps of a metric space (X, d) such that
(1) $A_1X \subset TX$ and $A_iX \subset SX$ for i > 1.
(2) (A_1, S) is weakly compatible.

(3) (A_1, S) or (A_i, T), $i > 1$ satisfies the property (E.A).
(4) $d(A_1x, A_iy) < \max\{d(S_x, T_y), d(A_1x, S_x), d(A_iy, T_y), d(A_1x, T_y), d(A_iy, S_x)\}$ for $i > 1$.
If the range of the one of the mappings $\{A_i\}$, S or T is a complete subspace of X, then
(I) A_1 and S have a common fixed point. (II) A_i, $i > 1$ and T have a common fixed point provided that (A_k, T) for some $k > 1$ is weakly compatible.
(III) Ai, S and T have a unique common fixed point provided that (I) and (II) are true.

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