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Counting the Subgroups of the One-Headed Group S_5 up to Automorphism

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Abstract: *In this paper, we aimed at determining all subgroups of the Symmetric group S_5 up to Automorphism class using Sylow's theorem and Lagrange's theorem. This is achieved by finding all subgroups of order m for which $m|O(S_5)$ and are subsets of S_5. It was vividly described and derived 156 subgroups of S_5 and their conjugacy class size and Isomorphism class. The Alternating group A_5 is the unique maximal normal subgroup of S_5. Further, the Symmetric group S_5 is centerless and every automorphism of it is inner. Also, every natural homomorphism to the automorphism group is an isomorphism. Hence, S_5 is complete. The derived subgroups can be used to determine the number of Fuzzy subgroups of the symmetric group S_5 for further research.*
Keywords: Symmetric group, Conjugacy class, Isomorphism, Automorphism, Complete group

I. Introduction

In mathematics, the notion of permutation is used with several slightly different meanings, all related to the act of permuting (rearranging in an ordered fashion) objects or values. Informally, a permutation of a set of values is an arrangement of those values into a particular order. Thus there are six permutations of the set {1,2,3}, namely, {[1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], and [3,2,1]}. In algebra and particularly in group theory, a permutation of a set S is defined as a bijection from S to itself (i.e., a map $f : S \rightarrow S$ for which every element of S occurs exactly once as image value). To such a map f is associated with the rearrangement of S in which each element s takes the place of its image $f(s)$.

Given any non empty set S, define A(S) to be the set of all bijections mapping of the set S onto itself. The set A(S) is a group with respect to composition of function. If the set S is finite with n elements, then the group A(S) is denoted by S_n. The order of S_n is n! And will be called Symmetric group. Any subset of S_n which is itself a group is called a subgroup of S_n. There are many references on subgroups of S_2, S_3 and S_4 ([2], [7], [8] and [10]). Our aim in this paper is to critically examine all subgroups of S_5 up to automorphism class and their conjugacy class size, which will aid our intention of counting the number of Fuzzy subgroups of S_5 in our next article. The set of all symmetry operations on all objects in the set S, can be modeled as a group action $g : G \times S \rightarrow S$, where the image of g in G and x in S is written as $g{\cdot}x$. If, for some g, $g{\cdot}x = y$ then x and y are said to be symmetrical to each other. For each object x, operations g for which $g{\cdot}x = x$ is the *symmetry group* of the object, a subgroup of G. If the symmetry group of x is the trivial group then x is said to be *asymmetric*, otherwise *symmetric*.

II. Preliminary

Definition 1: The symmetric group S_5 is defined in the following equivalent ways: It is the group of all permutations on a set of five elements, i.e., it is the Symmetric group of degree five. In particular, it is a symmetric group of prime degree and symmetric group of prime power degree. With this interpretation, it is denoted S_5 or Sym(5). Equivalently, it is the projective general linear group of degree two over the field of five elements, i.e. PGL(2,5) [5].

Definition 2: Let G be a group and let N be a proper normal subgroup of G. Then N is called maximal subgroup of G if there does not exists any proper normal subgroup M of G such that $N \leq M \leq G$ [12].

Definition 3: A homomorphism $\varphi: G \rightarrow K$ from a group G to a group K is a function with the property that $\varphi(g_1 * g_2) = \varphi(g_1) * \varphi(g_2)$ for all $g_1, g_2 \in G$, where $*$ denotes the group operation on G and on K [9].

Definition 4: An isomorphism $\varphi: G \rightarrow K$ between two groups G and K is a homomorphism that is also a bijection mapping G onto K. Two groups G and K are isomorphic if there exists an isomorphism mapping G onto K, written as $G \cong K$. While an automorphism is an isomorphism mapping a group onto itself [9].

Definition 5: A group is said to be complete if it satisfies the following equivalent conditions:

1 It is centerless and every automorphism of it is inner.
2 The natural homomorphism to the automorphism group, that sends each element to the conjugation via that element is an isomorphism.
3 Whenever it is embedded as a normal subgroup inside a bigger group, it is actually a direct factor

inside that bigger group.
Equivalently;
A group G is said to be complete if it satisfies the following equivalent conditions:
1 The center of G i.e. Z(G) is trivial and Inn(G)=Aut(G) (i.e. every automorphism of G is inner),
2 The natural homomorphism G→Aut(G) given by $g \to C_g$ (where $C_g = x \to gxg^{-1}$) is an isomorphism,
3 For any embedding of G as a normal subgroup of some group K, G is a direct factor of K [6].

Definition 6: A partial ordered on a nonempty set P is a binary relation ≤ on P that is reflexive, antisymmetric and transitive. The pair ⟨P, ≤⟩ is called a partially ordered set or poset. Poset ⟨P, ≤⟩ is totally ordered if every x, y ∈P are comparable, that is $x \le y$ or $y \le x$. A nonempty subset S of P is a chain in P if S is totally ordered by ≤ [11].

Definition 7: Let ⟨P, ≤⟩ be a poset and let S ⊆ P. An upper bound for S is an element $x \in$ P for which $s \le x$ ∀s ∈ S. The least upper bound of S is called the supremum or join of S. A lower bound for S is an element $x \in$ P for which $x \le s$ ∀s ∈ S. The greatest lower bound of S is called the infimum or meet of S. Poset ⟨P, ≤⟩ is called a lattice if every pair x, y of elements of P has a supremum and an infimum [11].

Note that the set of all subgroups of G under the "subgroup" relation is a lattice. This lattice is called the lattice subgroup of G.

Theorem 1: (Lagrange's Theorem) If G is a finite group and H is a subgroup of G, then order of H is a divisor of order of G [7].

Theorem 2: If G is a finite group and $x \in$G, then order of x is a divisor of order of G [7].

Theorem 3: (Cauchy's Theorem) Let G be a finite group and let p be a prime number that divides the order of G. Then G contains an element of order p [3].

Theorem 4: (The First Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where $n \ge 1$, p is a prime number and (p, m) = 1. Then G contains a subgroup of order p^k for each k where $1 \le k \le n$ [8].

Definition 8: Let G be a finite group and let $|G| = p^n m$ where n ≥ 1, p is a prime number and (p, m) = 1. The subgroup of G of order p^n is called the sylow p−subgroup of G [2].

Theorem 5: (Second Sylow Theorem) Let G be a finite group, and let p be a prime number dividing the order of G. Then all Sylow p-subgroups of G are conjugate, and any p-subgroup of G is contained in some Sylow p-subgroup of G. Moreover the number of Sylow p-subgroups in G divides the order of G and is congruent to 1 modulo p [8].

Theorem 6: (The Third Sylow Theorem) Let G be a finite group and let $|G| = p^n m$ where n ≥ 1, p is a prime number and (p, m) = 1. Then the number of Sylow p−subgroup is of the form (1+ kp), where k is a non-negative integer, and (1 + kp) divides the order of G [8].

Definition 9: A subgroup N of G is said to be a normal subgroup of G if for every g ∈ G and $n \in$ N, $gng^{-1} \in$ N [7].

Theorem 7: There is a unique Sylow p−subgroup of the finite group G if and only if it is normal [2].

Theorem 8: Let G be a group of order pq, where p and q are distinct primes and p < q. Then G has only one subgroup of order q. This subgroup of order q is normal in G [2].

Definition 10: A non-trivial group G is said to be simple if the only normal subgroups of G are the whole of G and the trivial subgroup $\{e\}$ whose only element is the identity element e of G [3].

III. The One-Headed Group S_5

The one-headed group (Symmetric group) S_5 is the group of permutations of the set S = {1, 2, 3, 4, 5}, i.e. if $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$, then the set of all bijections $f : S \to S$ defined by $\alpha(a_i) = a_j; i, j \le 5$ is called the Symmetric group S_5. The collection of all such permutations gives rise to a group of order 120 as follows:

$$S_5 = \{i, \alpha_1, \alpha_2, ..., \alpha_{10}, \sigma_1, \sigma_2, ..., \sigma_{20}, \tau_1, \tau_2, ..., \tau_{30}, \gamma_1, \gamma_2, ..., \gamma_{15}, \beta_1, \beta_2, ..., \beta_{24}, \delta_1, \delta_2, ..., \delta_{20}\}$$

Where;
i = (1) = the identity permutation.
α_1 = (4 5), α_2 = (3 5), α_3 = (3 4), α_4 = (2 5), α_5 = (2 3), α_6 = (2 4), α_7 = (1 5), α_8 = (1 4), α_9 = (1 3), α_{10} = (1 2).
σ_1 = (1 2 3), σ_2 = (1 3 2), σ_3 = (1 2 4), σ_4 = (1 4 2), σ_5 = (1 2 5), σ_6 = (1 5 2), σ_7 = (1 3 4), σ_8 = (1 4 3), σ_9 = (1 4 5), σ_{10} = (1 5 4), σ_{11} = (1 3 5), σ_{12} = (1 5 3), σ_{13} = (2 3 4), σ_{14} = (2 4 3), σ_{15} = (2 3 5), σ_{16} = (2 5 3), σ_{17} = (2 4 5), σ_{18} = (2 5 4), σ_{19} = (3 4 5), σ_{20} = (3 5 4).
τ_1 = (2 3 4 5), τ_2 = (2 5 4 3), τ_3 = (2 3 5 4), τ_4 = (2 4 5 3), τ_5 = (2 4 3 5), τ_6 = (2 5 3 4), τ_7 = (1 2 3 4), τ_8 = (1 4 3 2), τ_9 = (1 2 3 5), τ_{10} = (1 5 3 2), τ_{11} = (1 2 4 3), τ_{12} = (1 3 4 2), τ_{13} = (1 2 4 5), τ_{14} = (1 5 4 2), τ_{15} = (1 2 5 3), τ_{16} = (1 3 5 2), τ_{17} = (1 2 5 4), τ_{18} = (1 4 5 2), τ_{19} = (1 3 4 5), τ_{20} = (1 5 4 3), τ_{21} = (1 3 5 4), τ_{22} = (1 4 5 3), τ_{23} = (1 3 2 4), τ_{24} = (1 4 2 3), τ_{25} = (1 3 2 5), τ_{26} = (1 5 2 3), τ_{27} = (1 4 3 5), τ_{28} = (1 5 3 4), τ_{29} = (1 4 2 5), τ_{30} = (1 5 2 4).

γ_1 = (2 4)(3 5), γ_2 = (2 5)(3 4), γ_3 = (2 3)(4 5), γ_4 = (1 3)(2 4), γ_5 = (1 3)(2 5), γ_6 = (1 4)(2 3), γ_7 = (1 4)(2 5), γ_8 = (1 5)(2 3), γ_9 = (1 5)(2 4), γ_{10} = (1 4)(3 5), γ_{11} = (1 5)(3 4), γ_{12} = (1 2)(3 4), γ_{13} = (1 2)(3 5), γ_{14} = (1 3)(4 5), γ_{15} = (1 2)(4 5).
β_1 = (1 2 3 4 5), β_2 = (1 3 5 2 4), β_3 = (1 4 2 5 3), β_4 = (1 5 4 3 2), β_5 = (1 2 3 5 4), β_6 = (1 3 4 2 5), β_7 = (1 5 2 4 3), β_8 = (1 4 5 3 2), β_9 = (1 2 4 5 3), β_{10} = (1 4 3 2 5), β_{11} = (1 5 2 3 4), β_{12} = (1 3 5 4 2), β_{13} = (1 2 4 3 5), β_{14} = (1 4 5 2 3), β_{15} = (1 3 2 5 4), β_{16} = (1 5 3 4 2), β_{17} = (1 2 5 4 3), β_{18} = (1 5 3 2 4), β_{19} = (1 4 2 3 5), β_{20} = (1 3 4 5 2), β_{21} = (1 2 5 3 4), β_{22} = (1 5 4 2 3), β_{23} = (1 3 2 4 5), β_{24} = (1 4 3 5 2).
δ_1 = (1 2 3)(4 5), δ_2 = (1 3 2)(4 5), δ_3 = (1 2 4)(3 5), δ_4 = (1 4 2)(3 5), δ_5 = (1 2 5)(3 4), δ_6 = (1 5 2)(4 5), δ_7 = (1 3 4)(2 5), δ_8 = (1 4 3)(2 5), δ_9 = (1 4 5)(2 3), δ_{10} = (1 5 4)(2 3), δ_{11} = (1 3 5)(2 4), δ_{12} = (1 5 3)(2 4), δ_{13} = (1 5)(2 3 4), δ_{14} = (1 5)(2 4 3), δ_{15} = (1 4)(2 3 5), δ_{16} = (1 4)(2 5 3), δ_{17} = (1 3)(2 4 5), δ_{18} = (1 3)(2 5 4), δ_{19} = (1 2)(3 4 5), δ_{20} = (1 2)(3 5 4).
Now the order of an element x of a group G is the least positive integer n for which $x^n = e$, the identity element of the group G, where x^n represents $x \cdot x \cdot x \cdot \ldots \cdot x$ n-times. Then writing the elements of the group S_5 in the form x^n, we classify them according to their order, and the order of each element divides the order of S_5 (see Lagrange's theorem). The orders of these elements are given in the table below.

Table 1: Order of elements of S_5

Order	Elements	Formula Calculating Element Order
1	i	LCM{1}
2	$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}$	LCM{2,1}
3	$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}$	LCM{3,1}
4	$\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{23}, \tau_{24}, \tau_{25}, \tau_{26}, \tau_{27}, \tau_{28}, \tau_{29}, \tau_{30}$	LCM{4,1}
5	$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}$	LCM{5,1}
6	$\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{16}, \delta_{17}, \delta_{18}, \delta_{19}, \delta_{20}$	LCM{2,3}

IV. Main Results

According to the Lagrange's theorem, the order of any non-trivial subgroup of S_5 divides the order of S_5. Therefore we shall determine all subgroups of S_5 and their isomorphism class. Obviously, the only subgroup of S_5 of order 1 is the trivial subgroup $G_1 = \{i\}$.

4.1 Subgroups of order 2

Let H be arbitrary subgroup of S_5 of order 2. Since 2 is a prime number, H is cyclic. Hence, H is generated by an element of S_5 of order 2. Thus all subgroups of S_5 of order 2, isomorphic to the cyclic group Z_2 are:

$H_k = \{i, \alpha_j : 1 \leq j \leq 10\} = \langle\alpha_j\rangle$; $2 \leq k \leq 11$, (for each j, $H_k \equiv S_2$), and

$H_k = \{i, \gamma_j : 1 \leq j \leq 15\} = \langle\gamma_j\rangle$; $12 \leq k \leq 26$, (subgroup generated by double transposition in S_5)

4.2 Subgroups of order 3

Subgroups of S_5 of order 3 are generated by elements of S_5 of order 3. Thus, all subgroups of S_5 of order 3, isomorphic to the cyclic group Z_3 are

$L_k = \{i, \sigma_j, \sigma_{j+1} : \sigma_j^{-1} = \sigma_{j+1}; 1 \leq j \leq 19\} = \langle\sigma_j\rangle = \langle\sigma_{j+1}\rangle$; $27 \leq k \leq 36$.

Note that if $\sigma_j^{-1} = \sigma_{j+1}$, then j = j+2 for next k. L_k is cyclic since 3 is prime.

4.3 Subgroups of order 4

Let M be arbitrary subgroup of S_5 of order 4. Then by theorem 2, elements of M must have order 1, 2 or 4. Hence if M consists of elements of order 4, then M is generated by an element of order 4. Thus, we obtained

$M_k = \{i, \tau_j, \gamma_{(j+1)/2}, \tau_{j+1} : \tau_j^{-1} = \tau_{j+1}; j = 1, 3, ..., 29\} = \langle\tau_j\rangle = \langle\tau_{j+1}\rangle$; $37 \leq k \leq 51$.

We also have subgroups of S_5 of order 4 generated by pair of disjoint transpositions in S_5 as follows:
$M_{52} = \{i, \alpha_2, \alpha_6, \gamma_1\}$, $M_{53} = \{i, \alpha_3, \alpha_4, \gamma_2\}$, $M_{54} = \{i, \alpha_1, \alpha_5, \gamma_3\}$, $M_{55} = \{i, \alpha_6, \alpha_9, \gamma_4\}$, $M_{56} = \{i, \alpha_4, \alpha_9, \gamma_5\}$, $M_{57} = \{i, \alpha_5, \alpha_8, \gamma_6\}$, $M_{58} = \{i, \alpha_4, \alpha_8, \gamma_7\}$, $M_{59} = \{i, \alpha_5, \alpha_7, \gamma_8\}$, $M_{60} = \{i, \alpha_6, \alpha_7, \gamma_9\}$, $M_{61} = \{i, \alpha_2, \alpha_8, \gamma_{10}\}$, $M_{62} = \{i, \alpha_3, \alpha_7, \gamma_{11}\}$, $M_{63} = \{i, \alpha_3, \alpha_{10}, \gamma_{12}\}$, $M_{64} = \{i, \alpha_2, \alpha_{10}, \gamma_{13}\}$, $M_{65} = \{i, \alpha_1, \alpha_9, \gamma_{14}\}$, $M_{66} = \{i, \alpha_1, \alpha_{10}, \gamma_{15}\}$.
Furthermore, we have 5 other subgroups of S_5 of order 4 generated by double transpositions on four elements in S_5, i.e. M_k for $67 \leq k \leq 71$.

4.4 Subgroups of order 5

Let N be any arbitrary subgroup of S_5 of order 5. Since 5 is a prime number, the subgroup N is cyclic and is generated by an element of S_5 of order 5. Hence, we have 6 such subgroups given by

$N_k = \{i, \beta_j, \beta_{j+1}, \beta_{j+2}, \beta_{j+3}\} = \langle\beta_j\rangle = \langle\beta_{j+1}\rangle = \langle\beta_{j+2}\rangle = \langle\beta_{j+3}\rangle$; $72 \le k \le 77$.

4.5 Subgroups of order 6

If P is any arbitrary subgroup of S_5 of order 6, then we generates from the elements of S_5 of order 6 i.e. δ's, the following subgroups, isomorphic to the cyclic group Z_6;

$P_k = \{i, \delta_{2j-1}, \sigma_{2j}, \alpha_j, \sigma_{2j-1}, \delta_{2j} : \delta_{2j}^{-1} = \delta_{2j-1}, \sigma_{2j}^{-1} = \sigma_{2j-1}; 1 \le j \le 10\}$
$= \langle\delta_{2j}\rangle = \langle\delta_{2j-1}\rangle$; $78 \le k \le 87$.

Again by Sylow's theorem, since $6 = 2 * 3$, other subgroups of S_5 of order 6 can be generated from the product of the elements of S_5 of order 2 with those elements of order 3. i.e. α's and σ's given by
$\{i, \sigma_1, \alpha_1, \sigma_2, \delta_1, \delta_2\}$, $\{i, \sigma_3, \alpha_2, \sigma_4, \delta_3, \delta_4\}$, ..., $\{i, \sigma_{19}, \alpha_{10}, \sigma_{20}, \delta_{19}, \delta_{20}\}$. Hence, we have

$P_k = \{i, \sigma_{2j-1}, \alpha_j, \sigma_{2j}, \delta_{2j-1}, \delta_{2j} : \sigma_{2j}^{-1} = \sigma_{2j-1}; \delta_{2j}^{-1} = \delta_{2j-1}; 1 \le j \le 10\}$; $88 \le k \le 97$.

Also, S_3 is obviously a subset of S_5, i.e. we have subgroups generated by each of the following set of elements by permutation:
(1 2 3), (1 2 4), (1 2 5), (1 3 4), (1 3 5), (1 4 5), (2 3 4), (2 3 5), (2 4 5) and (3 4 5).
Hence we generate 10 such subgroups of S_5 of order 6, isomorphic to S_3 i.e. P_k such that $98 \le k \le 107$.

4.6 Subgroups of order 8

Since 8 is a multiple of 2 and 4, elements of the subgroup of order 8 must have orders 2 or 4 only. Consider the set of permutations

Q = {*i*, (2 3 4 5), (2 5 4 3), (2 4)(3 5), (2 4), (3 5), (2 3)(4 5), (2 5)(3 4)}, i.e.

$Q = \{i, \tau_1, \tau_2, \gamma_1, \alpha_6, \alpha_2, \gamma_3, \gamma_2\}$

Obviously, this is a subgroup of S_5 of order 8. To see this, let us construct a multiplication table of Q × Q as follows.

Table 2: Multiplication table of Q × Q.

*	*i*	τ1	τ2	γ1	α6	α2	γ3	γ2
i	*i*	τ1	τ2	γ1	α6	α2	γ3	γ2
τ1	τ1	γ1	*i*	τ2	γ3	γ2	α2	α6
τ2	τ2	*i*	γ1	τ1	γ2	γ3	α6	α2
γ1	γ1	τ2	τ1	*i*	α2	α6	γ2	γ3
α6	α6	γ2	γ3	α2	*i*	γ1	τ2	τ1
α2	α2	γ3	γ2	α6	γ1	*i*	τ1	τ2
γ3	γ3	α6	α2	γ2	τ1	τ2	*i*	γ1
γ2	γ2	α2	α6	γ3	τ2	τ1	γ1	*i*

Clearly, from table 2 above, the set Q is a subgroup of S_5 of order 8. By constructing such subgroups from the combinations of τ_j's, γ_j's and α_j's, we obtained 15 subgroups of S_5 of order 8, isomorphic to the Dihedral group D_8. i.e. Q_k such that $108 \le k \le 122$.

4.7 Subgroups of order 10

Let R be any arbitrary subgroup of S_5 of order 10. Now, consider the elements (1 2 3 4 5) of order 5 and the transposition (2 5)(3 4) of order 2 (since $10 = 2 * 5$). Then

$R_k = \langle(1\ 2\ 3\ 4\ 5), (2\ 5)(3\ 4)\rangle = \langle\beta_1, \gamma_2\rangle$
= {*i*, (1 2 3 4 5), (2 5)(3 4), (1 3 5 2 4), (1 4 2 5 3), (1 5)(2 4), (1 4)(2 3), (1 5 4 3 2), (1 3)(4 5), (1 2)(3 5)}. i.e.

$R_k = \{i, \beta_1, \gamma_2, \beta_2, \beta_3, \gamma_9, \gamma_6, \beta_4, \gamma_{14}, \gamma_{13}\}$

is a subgroup of S_5 of order 10. By constructing similar subgroups, we obtained 6 subgroups of S_5 of order 10, isomorphic to the Dihedral group D_{10}. i.e. R_k, $123 \le k \le 128$.

4.8 Subgroups of order 12

Since $12 = 2^2 * 3$, the direct product of S_2 and S_3 in S_5 is a subgroup of S_5. Hence, if T is any arbitrary subgroup of S_5 of order 12, then

$T = \{i, \alpha_{10}, \alpha_1, \sigma_1, \sigma_2, \alpha_5, \delta_1, \alpha_9, \gamma_{15}, \gamma_3, \delta_2, \gamma_{14}\}$

is a subgroup of S_5 of order 12. Hence, we obtained 10 such subgroups of order 12. i.e. T_k; $129 \le k \le 138$, isomorphic to the direct product of S_2 and S_3.

Similarly, A_4 is obviously a subgroup of S_5, and each of the elements (1 2 3 4), (1 2 3 5), (1 2 4 5), (1 3 4 5) and (2 3 4 5) generate A_4. Thus, we have 5 such subgroups i.e. T_k; $139 \le k \le 143$ isomorphic to A_4.

4.9 Subgroups of order 20

The composition of elements of S_5 of order 5 with those elements of order 4 formed subgroups of S_5 of order 20 ($20 = 5 * 4$). Hence, if U is an arbitrary subgroup of S_5 of order 20, then

$U = \langle(1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4)\rangle$

is a subgroup generated by two elements β_1 and τ_3. By considering similar elements, we obtained 6 such subgroups of order 20. i.e. U_k; $144 \le k \le 149$, isomorphic to the General Affine group GA(1,5).

4.10 Subgroups of order 24

Each of the following subset of S_5 consisting of four elements generates subgroup of S_5 of order 24. i.e. (1 2 3 4), (1 2 3 5), (1 2 4 5), (1 3 4 5) and (2 3 4 5). Hence, if V is any arbitrary subgroup of S_5 generated by any of the above elements, then V is a subgroup of order 24, i.e. V_k; $150 \le k \le 154$, isomorphic to S_4.

4.11 Subgroup of order 60

The only subgroup of S_5 of order 60 is the alternating group A_5, consisting of all the even permutations in S_5. Such subgroup is unique. Hence, we have

$A_5 = \langle(1\ 2\ 3\ 4\ 5), (1\ 2\ 3)\rangle = \langle\beta_1, \sigma_1\rangle$, i.e.

$A_5 = \{i, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6,$
$\gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18},$
$\beta_{19}, \beta_{20}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}\}$.

4.12 Subgroup of order 120

Every group is a subgroup of itself. Hence, the whole group S_5 is a subgroup of S_5 of order 120.

The table below gives the summary of all subgroups of S_5 and their Automorphism classes, Isomorphism classes, Order (in ascending order), Index, occurrence as normal subgroup, number of Conjugacy classes and size of each conjugacy class.

Table 3: Table classifying isomorphism types of subgroups

Automorphism class of subgroups	Isomorphism class	Order of subgroups	Index of subgroups	Occurrence as normal subgroup	Number of conjugacy classes	Size of each conjugacy class	Total number of subgroups
trivial subgroup	trivial group	1	120	1	1	1	1
S_2 in S_5	Cyclic group Z_2	2	60	0	1	10	10
Subgroup gen. by double trans. in S_5	Cyclic group Z_2	2	60	0	1	15	15
Z_3 in S_5	Cyclic group Z_3	3	40	0	1	10	10
Z_4 in S_5	Cyclic group Z_4	4	30	0	1	15	15
Subgroup gen. by pair of disjoint trans.	Klein-four group	4	30	0	1	15	15
Subgroup gen. by double trans. On 4 elements in S_5	Klein-four group	4	30	0	1	5	5
Z_5 in S_5	Cyclic group Z_5	5	24	0	1	6	6
Z_6 in S_5	Cyclic group Z_6	6	20	0	1	10	10
Twisted S_3 in S_5	Symmetric group S_3	6	20	0	1	10	10
S_3 in S_5	Symmetric group S_3	6	20	0	1	10	10
D_8 in S_5	Dihedral group D_8	8	15	0	1	15	15
D_{10} in S_5	Dihedral group D_{10}	10	12	0	1	6	6
Direct product of S_3 and S_2 in S_5	Direct product of S_3 and Z_2	12	10	0	1	10	10
A_4 in S_5	Alternating group A_4	12	10	0	1	5	5
GA(1,5) in S_5	General Affine group: GA(1,5)	20	6	0	1	6	6
S_4 in S_5	Symmetric group S_4	24	5	0	1	5	5

A_5 in S_5	Alternating Group A_5	60	2	1	1	1	1
whole group	symmetric group S5	120	1	1	1	1	1
Total (19 rows)				**3**	**19**		**156**

There are seven conjugacy classes corresponding to the unordered partitions of {1, 2, 3, 4, 5}. Now since cycle type determine conjugacy class and the length of a cycle is found to be the number of elements in that cycle, we notice that any conjugate of a *k* cycle is again a *k*-cycle [4]. This is also supported by the theorem:

Theorem 9: The conjugacy classes of any S_n are determined by cycle type. That is, if σ has cycle type $(k_1, k_2, \ldots, k_l)$, then any conjugate of σ has cycle type $(k_1, k_2, \ldots, k_l)$, and if γ is any other element of S_n with cycle type $(k_1, k_2, \ldots, k_l)$, then σ is conjugate to γ [1].

For the proof of the above theorem, see [4]. We therefore use this information to derive the following table, classifying the size of conjugacy class of elements of S_5.

Table 4: Size of Conjugacy classes of elements of S_5

Element	**Partition**	**Verbal description of cycle type**	**Representative element with the cycle type**	**Size of conjugacy class**	**Formula Calculating Size of Conjugacy Class**
i	1 + 1 + 1 + 1 + 1	five fixed points	(1) the identity element	1	$\frac{5!}{(1^5)(5!)}$
α_j	2 + 1 + 1 + 1	transposition: one 2-cycle, three fixed point	(1 2)	10	$\frac{5!}{[(2^1)(1!)][(1^3)(3!)]}$
γ_j	2 + 2 + 1	double transposition: two 2-cycles, one fixed point	(1 2)(3 4)	15	$\frac{5!}{[(2^2)(2!)][(1^1)(1!)]}$
σ_j	3 + 1 + 1	one 3-cycle, two fixed points	(1 2 3)	20	$\frac{5!}{[(3^1)(1!)][(1^2)(2!)]}$
δ_j	3 + 2	one 3-cycle, one 2-cycle	(1 2 3)(4 5)	20	$\frac{5!}{[(3^1)(1!)][(2^1)(1!)]}$
τ_j	4 + 1	one 4-cycle, one fixed point	(1 2 3 4)	30	$\frac{5!}{[(4^1)(1!)][(1^1)(1!)]}$
β_j	5	one 5-cycle	(1 2 3 4 5)	24	$\frac{5!}{(5^1)(1!)}$
Total				**120**	**5!**

The sum of the conjugacy classes is equal to the order of the group S_5. The center of a group G is defined to be the set of those elements that commute with every other element of G, given by $Z(G) = \{x : xg = gx \text{ for all } g \in G\}$. Observed that the center of S_5 is the trivial subgroup $\{ i \}$, consisting of the identity permutation. Hence, S_5 is centerless. S_5 is also almost simple group since it contains a centralizer-free simple normal subgroup, i.e. A_5. The Alternating group A_5 is simple. Hence, A_5 is the unique maximal normal subgroup of S_5. This is supported by the lemma given below.

Lemma 1: The alternating group A_5 is simple [3].

Proof: A_5 is given as the group of even permutations of the set {1, 2, 3, 4, 5}. There are 60 such permutations (see section 4.11) which is a combination of σ_j, γ_j, β_j and the identity permutation. Now, each σ_j in A_5 generates a Sylow 3-subgroup of order 3, and these subgroups are all conjugate to one another by the Second Sylow Theorem. It follows that any normal subgroup of A_5 that contains at least one σ_j must contain all σ_j; $1 \le j \le 20$, and thus its order must therefore be at least 21 (since it must also contain the identity permutation). Similarly each β_j in A_5 generates a Sylow 5-subgroup of order 5, and these subgroups are all conjugate to one another. Therefore any normal subgroup of A_5 that contains at least one β_j must contain all β_j; $1 \le j \le 24$, and thus its order must be at least 25.

Now if A_5 were to contain a subgroup of order 30, this subgroup would be the kernel of a non-constant homomorphism $\varphi: A_5 \to \{1, -1\}$ from A_5 to the multiplicative group consisting of the numbers 1 and -1. But any

σ_j or β_j would have to belong to the kernel of this homomorphism, and therefore this kernel would contain at least 45 elements, which is impossible. We conclude that A_5 cannot contain any subgroup of order 30. It follows from Lagrange's Theorem that any normal subgroup of A_5 that contains at least one σ_j or β_j must be the whole of A_5.

The group A_5 contains 5 Sylow 2-subgroups, which are of order 4. One of these consists of the identity permutation, together with the three permutations γ_{12}, γ_4 and γ_6. (Each of these permutations fixes the element 5). There are four other such Sylow 2-subgroups, and all of the Sylow 2-subgroups are conjugate to one another. It follows that A_5 does not contain any normal subgroup of order 4. Moreover A_5 cannot contain any normal subgroup of order 2, since any element of order 2 belongs to one of the five Sylow 2-subgroups of order 4, and is therefore conjugate to elements of order 2 in the other Sylow 2-subgroups.

Now any subgroup of A_5 whose order is divisible by 3 must contain σ_j by Cauchy's Theorem. Similarly any subgroup of A_5 whose order is divisible by 5 must contain β_j. It follows that the order of any proper normal subgroup of A_5 cannot be divisible by 3 or 5. But this order must divide 60. Therefore the order of any proper normal subgroup of A_5 must be at most 4. But we have seen that A_5 cannot contain any normal subgroup of order 4 or 2. Therefore any proper normal subgroup of A_5 is trivial, and therefore A_5 is simple.$*$

V. Conclusion

From abstract point of view, the symmetric group S_n is not a nilpotent group since it is centerless, i.e. has no central series. But it is a one-headed group since the Alternating group A_5 is its unique maximal normal subgroup. The generated subgroups of S_5 will be useful in the event of studying the Fuzzy subgroups of S_5 since we have analyzed and pointed out all the methods used in generating the elements of these subgroups. Though there are other properties such as isomorphism classes of Sylow subgroups and the corresponding Sylow numbers and fusion systems, extended automorphism group and the lattice structure of S_5 that are not treated in this article, but recommended for further studies. The Fuzzy subgroups of the One-headed group S_5 will be useful in the area of signal/image processing.

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