IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 8, Issue 5 (Nov. – Dec. 2013), PP 20-23
www.iosrjournals.org

Fixed points of self maps in d_p – complete topological spaces

V.Naga Raju[1] and V.Srinivas[2]
1,2 Department of Mathematics,College of Engineering,Osmania University,Hyderabad,India-500007

Abstract : *The purpose of this paper is to prove some fixed point theorems in d_p - complete topological spaces which generalize the results of Troy L Hicks and B.E.Rhoades[6].*
Keywords : *d_p - complete topological spaces, d-complete topological spaces, orbitally lower semi continuous and orbitally continuous maps.*

I. Introduction

In 1992, Troy L Hicks [5] introduced the notion of *d-complete* topological spaces as follows:

1.1 Definition: A topological space (*X*, *t*) is said to be *d- complete* if there is a mapping $d : X \times X \to [0,\infty)$ such that (i) $d(x, y) = 0 \Leftrightarrow x = y$ and (ii) $< x_n >$ is a sequence in *X* such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent implies that $< x_n >$ converges in (*X*, *t*).

Troy.L.Hicks and B.E.Rhoades[6] proved the following theorem in d – complete topological spaces .

1.2 Theorem : Let T be a selfmap of a topological space (X, t) and d : X x X → [0, ∞) such that $O_T(u)$ has a cluster point z ∈ X . If

a) G(x) = *d*(x, Tx) is T-orbitally continuous at z and Tz
b) T is orbitally continuous at z and
c) d(Tx, T^2x) < d(x, Tx) for all $x, Tx \in \overline{O_T(u)}$,then Tz = z.

In this paper we introduce d_2 - *complete* topological spaces as a generalization of *d*-complete topological spaces. In fact, we define d_p - complete topological spaces for any integer $p \geq 2$. For a non-empty set X, let X^p be its *p*-fold cartesian product.

1.3 Definition: A topological space (*X*, *t*) is said to be d_p - *complete* if there is a mapping $d_p : X^p \to [0, \infty)$ such that (i) $d_p(x_1, x_2, . . . , x_p) = 0 \Leftrightarrow x_1 = x_2 = . . . = x_p$ and (ii) $< x_n >$ is a sequence in *X* with $\lim_{n\to\infty} d_p(x_n, x_{n+1}, x_{n+2}, . . . , x_{n+p-1}) = 0$ implies that $< x_n >$ converges to some point in (*X*, *t*). A d_p - complete topological space is denoted by (*X*, *t*, d_p) .

1.4 Remark: The function *d* in the Definition 1.1 and the function d_2 (the case *p* = 2) in Definition 1.2 are both defined on $X \times X$ and satisfy condition (i) of the definitions which are identical. Since the convergence of an infinite series $\sum_{n=1}^{\infty} \alpha_n$ of real numbers implies that $\lim_{n\to\infty} \alpha_n = 0$, but not conversely; it follows that every *d*-complete topological space is d_2 - complete, but not conversely. Therefore the class of d_2 - complete topological spaces is wider than the class of *d*-complete spaces and hence a separate study of fixed point theorems of self-maps on d_2 - complete topological spaces is meaningful.

The purpose of this paper is to establish fixed point theorems of self-maps of d_p - complete topological spaces for $p \geq 2$.

II . Preliminaries

Let *X* be a non-empty set. A mapping $d_p : X^p \to [0, \infty)$ will be called a *p-non-negative* on *X* provided $d_p(x_1, x_2, . . . , x_p) = 0 \Leftrightarrow x_1 = x_2 = . . . = x_p$.

2.1 Definition: Suppose (X, t) is a topological space and d_p is a p-non negative on X. A sequence $<x_n>$ in X is said to be a d_p - *Cauchy sequence* if $d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) \to 0$ as $n \to \infty$.

In view of Definition 2.1, a topological space (X, t) is d_p - complete if there is a p-non- negative d_p on X such that every d_p - Cauchy sequence in X converges to some point in (X,t).

If T is a self map of a non-empty set X and $x \in X$, then the *orbit of* x, $O_T(x)$ is given by $O_T(x) = \{x, Tx, T^2x, \ldots\}$. If T is a self map of a topological space X, then a mapping $G : X \to [0,\infty)$ is said to be *T-orbitally lower semi-continuous* (resp. *T-orbitally continuous*) at $x^* \in X$ if $<x_n>$ is a sequence in $O_T(x)$ for some $x \in X$ with $x_n \to x^*$ as $n \to \infty$ then $G(x^*) \le \liminf_{n\to\infty} G(x_n)$ (resp. $G(x^*) = \lim_{n\to\infty} G(x_n)$). A self map T of topological space X is said to be *w-continuous* at $x \in X$ if $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

If d_p is a p-non-negative on a non-empty set X, and $T : X \to X$ then we write, for simplicity of notation, that

(2.2) $G_p(x) := d_p(x, Tx, T^2x, \ldots, T^{p-1}x)$ for $x \in X$

Clearly we have

(2.3) $G_p(x) = 0$ if and only if x is a fixed point of T.

III. Main results

3.1 Theorem: Suppose T is a self-map of a topological space (X, t) and d_p is a p-non-negative on X. Suppose that there is a $u \in X$ such that $O_T(u)$ has a cluster point $z \in X$. If

a) $G_p(x)$ is T-orbitally continuous at z and Tz

b) T is orbitally continuous at z, and

c) $G_p(Tx) < G_p(x)$ for all $x, Tx \in \overline{O_T(u)}$ (the closure of $O_T(u)$),

then $Tz = z$.

Proof: Let $a_i = G_p(T^i u)$ for $i \ge 1$. Then, by (c), we get $a_{i+1} < a_i$ and therefore $\lim_{n\to\infty} a_i = \alpha$ exists and in fact, $\alpha = \inf_i a_i$. Since z is a cluster point of $O_T(u)$, there is a sequence $<T^{i_k}u>$ in $O_T(u)$ such that $T^{i_k}u \to z$ as $k \to \infty$. Therefore, by (a), we have

(3.2) $G_p(z) = \lim_{k\to\infty} a_{i_k}$

Also, it follows from (b) that $T^{i_k+1}u = T(T^{i_k}u) \to Tz$ as $k \to \infty$ and since $G_p(x)$ is T-orbitally continuous at Tz we get

(3.3) $G_p(Tz) = \lim_{k\to\infty} a_{i_k+1}$

Now (3.2) and (3.3) imply that $G_p(Tz) = G_p(z)$ which forces $Tz = z$ (For if $Tz \ne z$, then (c) gives $G_p(Tz) < G_p(z)$).

IV. Consequences

To present certain consequences of the main result, we introduce some notations:

If d_p is a p-non-negative on a non-empty set X and T is a self-map of X, then for any $x, y \in X$ we write

(4.1) $H_p(x, y) = d_p(x, y, Ty, T^2y, \ldots, T^{p-2}y)$

(4.2) $E_p(x, y) = d_p(x, y, y, \ldots, y)$

Clearly

(4.3) $H_p(x, Tx) = G_p(x)$ and $E_p(x, x) = 0$.

4.4 Theorem: Let T be a self-map of a topological space (X, t) and d_p be a p-non- negative on X. Suppose that there is a $u \in X$ such that $O_T(u)$ has a cluster point $z \in X$. If

a) $G_p(x)$ is *T*-orbitally continuous at *z* and *T z*

b) *T* is orbitally continuous at *z*, and

c) $H_p(Tx, Ty) \le \dfrac{M_1(x)}{M_2(x)}$, where

$M_1(x) = \max\{H_p(x, y). H_p(x, Ty), H_p(x, y). E_p(y, Tx), G_p(x). H_p(x, Ty), G_p(y). E_p(y, Tx)\}$

and $M_2(x) = \max\{H_p(x, Ty), E_p(y, Tx)\}$ for all $x, y \in X$ with $x \ne Ty$ or $y \ne Tx$, then $Tz = z$.

Proof : Taking $y = T x$ in the inequality of the theorem and using (4.3), we get

$$G_p(Tx) \le \frac{\max\{G_p(x). H_p(x, T^2x), G_p(x). H_p(x, T^2x)\}}{H_p(x, T^2x)}$$

$$\Rightarrow G_p(Tx) \le \frac{G_p(x). H_p(x, T^2x)}{H_p(x, T^2x)}$$

$$\Rightarrow G_p(Tx) \le G_p(x)$$

and therefore the theorem follows from Theorem 3.1.

4.5 Remark: Note that, the result of Hicks and Rhoades ([6], Corollary 3, pp.849) is a particular case of Theorem 4.4 and the corresponding result for metric spaces has been proved by Achari ([1], Theorem 1).

4.6 Theorem: Let *T* be a self-map of a topological space (*X*, *t*) and d_p be a *p*-non- negative on *X*. Suppose that there is a $u \in X$ such that $O_T(u)$ has a cluster point $z \in X$. If

a) $G_p(x)$ is *T*-orbitally continuous at *z* and *T z*

b) *T* is orbitally continuous at *z*, and

c) $H_p(Tx, Ty) < \max\left\{H_p(x, y), \dfrac{G_p(x). G_p(y)}{H_p(x, y)}, A(x, y). H_p(x, y). E_p(y, Tx)\right\}$

for all $x, y \in X$ with $x \ne y$, where $A(x, y) = a(x, y, Ty, \ldots, T^{p-2}y)$ and $a : X^p \to [0, \infty)$. Then $Tz = z$.

Proof: Taking $y = T x$ in the inequality of the theorem and using (4.3), we get

$$G_p(Tx) < \max\left\{G_p(x), \frac{G_p(x). G_p(Tx)}{G_p(x)}\right\}$$

$$G_p(Tx) < \max\{G_p(x), G_p(Tx)\}$$

which implies $G_p(Tx) < G_p(x)$ and therefore the theorem follows from Theorem 3.1.

4.7 Remark: Note that, the result of Hicks and Rhoades ([6], Corollary 4, pp.849) is a particular case of Theorem 4.6 and the corresponding result for metric spaces has been first proved by L. B. Ciric ([2], Theorem 2).

4.8 Theorem: Let *T* be a self-map of a topological space (*X*, *t*) and d_p be a *p*-non-negative on *X*. Suppose that there is a $u \in X$ such that $O_T(u)$ has a cluster point $z \in X$. If

a) $G_p(x)$ is *T*-orbitally continuous at *z* and *T z*

b) *T* is orbitally continuous at *z* and

c) $H_p(Tx, Ty) H_p(x, y) < H_p(x, y)\{a_1 H_p(x, y) + a_2 G_p(x) + a_3 G_p(y) + a_4 E_p(y, Tx)\}$
$+ a_5 G_p(x) G_p(y).$

for all $x, y \in X$, where $a_i \ge 0$ and $\sum_{i=1}^{5} a_i = 1$. Then $Tz = z$ and *z* is unique.

Proof: Taking $y = T x$ in the inequality of the theorem and using (4.3), we get

$$G_p(Tx)\,G_p(x) < G_p(x)\{a_1 G_p(x) + a_2 G_p(x) + a_3 G_p(Tx)\} + a_5 G_p(x) G_p(Tx)$$
$$\Rightarrow G_p(Tx)\,G_p(x) < a_1\left[G_p(x)\right]^2 + a_2\left[G_p(x)\right]^2 + a_3 G_p(x)\,G_p(Tx) + a_5 G_p(x) G_p(Tx)$$
$$\Rightarrow G_p(Tx)\,G_p(x) < (a_1 + a_2)\left[G_p(x)\right]^2 + (a_3 + a_5) G_p(x)\,G_p(Tx)$$
$$\Rightarrow (1 - a_3 - a_5) G_p(Tx) < (a_1 + a_2) G_p(x)$$
$$\Rightarrow G_p(Tx) < \frac{a_1 + a_2}{1 - a_3 - a_5}.G_p(x)$$
$$\Rightarrow G_p(Tx) < \frac{a_1 + a_2}{a_1 + a_2 + a_4}.G_p(x)$$

which gives $G_p(Tx) < G_p(x)$ and therefore the theorem follows from Theorem 3.1.

4.9 Remark: Note that, the result of Hicks and Rhoades ([6], Corollary 5, pp.849) is a particular case of Theorem 4.8 and the corresponding result for metric spaces has been first proved by K.M. Ghosh ([4], Theorem 2).

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