Even-even gracefulness of some families of graphs

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Abstract: In this paper, we prove that the Dumbbell graph, Star graph, Cartesian product \( P_2 \times C_n \) and \( K_t + C_n \) are even-even graceful. The even-even graceful labeling of a graph \( G \) with \( q \) edges means that there is an injection \( f: E(G) \rightarrow \{2, 4, \ldots, 2q\} \) so that induced map \( f^*: V(G) \rightarrow \{0, 2, \ldots, (2k-2)\} \) defined by \( f^*(x) = \Sigma f(x, y) \) (mod 2k) where \( k = \max \{p, q\} \) makes all distinct and even.

Keywords: Even-even graceful labeling, Dumbbell graph, Star graph and wheel graph.

I. Introduction

Most graph labeling methods trace their origin to one introduced by Rosa\cite{1} in 1967, or the one given by Graham and Sloane\cite{2} in 1980. Rosa\cite{1} called a function \( f \) a \( \beta \)-valuation of a graph \( G \) with \( q \) edges if \( f \) is an injection from the vertices of \( G \) to the set \( \{0, 1, \ldots, q\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) − f(y)| \), the resulting edge labels are distinct. Golomb subsequently called such labeling graceful and this is now the popular term. For all terminology and notation Bondy\cite{3} has been followed. Solairaju and Chithra\cite{4} have introduced the concept of edge-odd graceful labeling. Gayathri and Duraisamy have introduced the concept of even edge-graceful labeling. A graph is even vertex-graceful if there exists an injective map \( \gamma: E(G) \rightarrow \{1, 2, \ldots, 2q\} \) so that the induced map \( \gamma^*: V(G) \rightarrow \{0, 2, 4, \ldots, 2k-2\} \) defined by \( \gamma^*(x) = \Sigma \gamma(xy) \) (mod 2k) where \( k = \max \{p, q\} \) makes all distinct. R. Sridevi, S. Navaneethakrishnan, A. Nagarajan and K. Nagarajan\cite{5} have introduced the concept of even-odd graceful labeling. They proved that some well known graphs namely \( P_n, P'_n, K_{1,n}, K_{1,2,n}, K_{n,n} \) and \( B_{n,n} \) are even-odd graceful. In this paper we introduce the definition even-even gracefulness and also prove that some well known graphs namely \( S_n, D(m,n) \) and \( P_2 \times C_n \) etc are even-even graceful.

Definition 1.1

The odd-even graceful labeling of a graph \( G \) with \( q \) edges is an injection \( f: V(G) \rightarrow \{1, 3, 5, \ldots, 2q + 1\} \) such that, when each edge \( uv \) is assigned the label \( |f(u)−f(v)| \), the resulting edge labels are \( \{2, 4, 6, \ldots, 2q\} \). A graph which admits an odd-even graceful labeling is called an odd-even graceful graph.

Definition 1.2

A graph is even vertex graceful if there exists an injective map \( f: E(G) \rightarrow \{1, 2, \ldots, 2q\} \) so that the induced map \( f^*: V(G) \rightarrow \{0, 2, 4, \ldots, 2k-2\} \) defined by \( f^*(x) = f(xy) \) (mod 2k) where \( k = \max \{p, q\} \) makes all distinct.

Definition 1.3

A graph is even-even graceful if there exists an injective map \( f: E(G) \rightarrow \{2, 4, \ldots, 2q\} \) so that the induced map \( f^*: V(G) \rightarrow \{0, 2, \ldots, (2k-2)\} \) defined by \( f^*(x) = \Sigma f(x, y) \) (mod 2k) where \( k = \max \{p, q\} \) makes all distinct and even.

II. Main Results

Definition 2.1

A star \( S_n \) is the complete bipartite graph \( K_{1,n} \). It is a tree with one internal node and \( n \) leaves.

Theorem 2.1

A star graph \( S_n \) is even-even graceful when \( n \) is even.

Proof: Let \( G \) be a star graph with \( n+1 \) vertices and \( n \) edges.

Let \( \{e_i, e_2, \ldots, e_n\} \) be the edge set of \( S_n \).

Define \( f: E(G) \rightarrow \{2, 4, \ldots, 2q\} \) such that (here \( q = n \)) \( f(e_i) = 2i; i = 1, 2, \ldots, n \).

The internal vertex of \( S_n \) has induced label

\[
2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + 3 + \ldots + n) = \frac{2n(n+1)}{2} = n(n+1) = n.k \text{ where } k = p = n+1
\]

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Hence, the induced label of internal vertex is '0' and other vertices have induced label from 2 to 2n.

Example 2.1 The star graph $S_n$ is even-even graceful.

\[ 2+4+6+...+2n \equiv 0 \pmod{2k} \text{ when } n \text{ is even} \]

Figure: 1

Definition 2.2 The Dumbbell graph $D(m,n)$ is formed by two disconnected cycles $C_m$ and $C_n$ joined by an edge.

Theorem 2.2 Dumbbell graph $D(m,n)$ is even-even graceful for $m = n$.

Proof: For any $n \geq 3$, the Dumbbell graph $D(m,n)$ has $2n$ vertices and $2n+1$ edges. Let $\{e_1,e_2,\ldots,e_n\}$ be the edge set of the first cycle $C_m$. ‘$e_{m+1}$’ be a connecting edge. $\{e_{m+2},e_{m+3},\ldots,e_{2n+1}\}$ be the edge set of the second cycle $C_n$. We begin with the first cycle $C_m$ by labeling 2 to 2n to each edge anticlockwise consecutively from one side of the connected vertex. Then we label $2n+2$ to the connected edge. Finally we label $2n+4$ to $4n+2$ to each edge of the second cycle $C_n$ clockwise from one side of the connecting vertex.

Hence, the vertices of first cycle $C_m$ has induced labels $f(v_i) = 4i-2 ; i = 1,2,\ldots,n$ and the vertices of second cycle $C_n$ has induced labels $f(v_i) = 4i ; i = 1,2,3,\ldots,n$.

Example: 2.2 The Dumbbell graph with even-even graceful labeling.

Figure: 2

Theorem 2.3 The ladder graph $P_2 \times C_n$ is even-even graceful.

Proof:

The graph $P_2 \times C_n$ has 2n vertices and 3n edges. First we consider $e_1$ and $e_{3n}$, the two outer edges of $P_2 \times C_n$.

Let $\{e_2,e_3,\ldots,e_n\}$ be the edge set of one of the long sides of the ladder and $\{e_{2n+1},e_{2n+2},\ldots,e_{3n-1}\}$ be the edge set of the other long side of ladder. Finally let $\{e_{n+1},e_{n+2},\ldots,e_{2n}\}$ be the edge set of rungs of ladder.

Define $f : E(G) \rightarrow \{2,4,\ldots,2q\}$ such that

\[
 f(e_1) = 2 ; f(e_{3n}) = 6n \text{ and } f(e_i) = 2i ; \quad i = 2,3,\ldots,3n-1.
\]

From the above labeling, the induced vertex labels of the two paths $P_n$ are

\[
 f'(v_i) = 4(n+1)+2i \text{ for } i = 1,2,\ldots,n-3;
\]

\[
 f'(v_{n-3}) = 0 ; f'(v_{n-2}) = 2 \text{ & } f'(v_{n-1}) = 4(n+1);
\]

\[
 f'(v_{n}) = 2(n+1);
\]

\[
 f'(v_i) = 2i \text{ for } i = 2,3,\ldots,n.
\]

Hence the graph $P_2 \times C_n$ is an even-even graceful.
Example 2.3 The following figure shows that the graph $P_2 \times C_5$ is even-even graceful.

**Definition 2.3**

The wheel, $W_n$, is the graph obtained by joining every vertex of the cycle $C_n$ to exactly one isolated vertex called the center. The edges incident to the center are called spokes.

**Theorem 2.4** The wheel $W_n$ is even-even graceful when $n \equiv 0 \pmod{4}$

**Proof:**

The graph $W_n$ has $n+1$ vertices and $2n$ edges. Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be the edge set of the spokes and $\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ be the edge set of consecutive cycle. Let $v_0$ be a center vertex and $v_1, v_2, \ldots, v_n$ be the consecutive cycle vertices.

Define $f: E(G) \rightarrow \{2, 4, \ldots, 2q\}$ such that

- $f(v_0v_i) = 2i$ for $i = 1, 2, \ldots, n$
- $f(v_i v_n) = 4n - 2(i-1)$ for $i = 1, 2, \ldots, n$

Hence the induced mapping are $f'(v_0) = n$;

- $f'(v_1) = 2n+4$;
- $f'(v_2) = 2$;
- $f'(v_i) = 0$ and
- $f'(v_i) = 4n-2i+6$ for $i = 4, 5, \ldots, n$. 

Figure: 3

Figure: 4
**Example 2.4** The following figure shows that the graph $W_8$ is an even-even graceful.

![Diagram of $W_8$](image)

**Definition 2.4** The join of graphs $K_1$ and $C_n$, $K_1 + C_n$, is obtained by joining every vertex of $K_1$ with every vertex of $C_n$ with an edge.

**Theorem 2.5** The graph $K_1 + C_n$ is even-even graceful if $n$ is a multiple of 4.

**Proof:** The graph $K_1 + C_n$ has $n+1$ vertices and $2n$ edges. Let ‘v’ be vertex of $K_1$ and $v_1,v_2,...,v_n$ be vertices of the cycle. Start at the first edge which are incident to the $K_1$ with 2 and continue in strictly increasing order by 2. The smallest edge label is 2 and largest edge label is $2n$.

Similarly, label the edges of $C_n$, start from right hand side with $2n+2$ and continue in strictly increasing order by 2. So the smallest edge label of $C_n$ is $2n+2$ and largest edge label is $4n$.

Hence the induced labels of vertices are,

$f(v) = n$ ; $f(v_1) = 0$ ; $f(v_i) = 2$ and $f(v_i) = 4n-2i+2$ if $n = 2,4,...,n-1$
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Example: 2.5 The graph $K_1 + C_{12}$ and its even-even graceful labeling are shown in the following Figure.

III. Conclusion

In this paper we have introduced the definition for ‘even-even graceful labeling’. We have proved that the Dumbbell graph, Star graph, Cartesian product $P_2 \times C_n$ and $K_1 + C_n$ are all even-even graceful. We have also proved that the wheel $W_n$ is even-even graceful when $n \equiv 0(\text{mod}4)$.

References

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