Common Fixed Point Theorems For Weakly Compatible Mappings In Generalisation Of Symmetric Spaces.

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I. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. Hicks [5] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set X is a nonnegative real valued function d on X × X such that (i) d(x, y) = 0 if, and only if, x = y, and (ii) d(x, y) = d(y, x). Let d be a symmetric on a set X and for r > 0 and any x ∈ X, let B(x, r) = {y ∈ X: d(x, y) < r}. A topology τd on X is given by U ∈ τd if, and only if, for each x ∈ U, B(x, r) ⊂ U for some r > 0. A symmetric d is a semi-metric if for each x ∈ X and each r > 0, B(x, r) is a neighbourhood of x in the topology τd. Note that limn→∞ d(xn, x) = 0 if and only if xn → x in the topology τd.

II. Preliminaries

Before proving our results, we need the following definitions and known results in this sequel.

Definition 2.1 ([4]) Let (X, d) be a symmetric space. (W.3) Given {xn}, x and y in X, \limn→∞ d(xn, x) = 0 and \limn→∞ d(xn, y) = 0 imply x = y. (W.4) Given {xn}, {yn} and x in X \limn→∞ d(xn, x) = 0 and \limn→∞ d(xn, yn) = 0 imply that \limn→∞ d(yn, x) = 0.

Definition 2.2 ([12]) Two self mappings A and B of a metric space (X, d) are said to be weakly commuting if d(ABx, BAx) ≤ d(Ax, Bx), ∀x ∈ X.

Definition 2.3 ([6]) Let A and B be two self mappings of a metric space (X, d). A and B are said to be compatible if \limn→∞ d(ABx, BAx) = 0, whenever (xn) is a sequence in X such that \limn→∞ Axn = limn→∞ Bxn = t for some t ∈ X.

Remark 2.4. Two weakly commuting mappings are compatibles but the converse is not true as is shown in [6].

Definition 2.5 ([7]) Two self mapping T and S of a metric space X are said to be Weakly Compatible if they commute at there coincidence points, i.e., if Tu = Su for some u ∈ X, then TSu = STu.

Note 2.6. Two compatible maps are weakly compatible. M. Aamri [2] introduced the concept property (E.A) in the following way.

Definition 2.7 ([2]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence {xn} such that \limn→∞ Txn = limn→∞ Sxn = t for some t ∈ X.

Definition 2.8 ([2]). Two self mappings S and T of a metric space (X, d) will be non-compatible if there exists at least one sequence (xn) in X such that \limn→∞ d(STxn, TSxn) is either nonzero or non-existent.

Remark 2.9. Two noncompatible self mappings of a metric space (X, d) satisfy the property (E.A). In the sequel, we need a function \phi : R+ → R+ satisfying the condition 0 < \phi (t) < t for each t > 0.

Definition 2.10. Let A and B be two self mappings of a symmetric space (X, d). A and B are said to be compatible if \limn→∞ d(ABx, BAx) = 0 whenever (xn) is a sequence in X such that \limn→∞ d(Axn, t) = \limn→∞ d(Bxn, t) = 0 for some t ∈ X.

Definition 2.11. Two self mappings A and B of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.12. Let A and B be two self mappings of a symmetric space (X, d). We say that A and B satisfy the property (E.A) if there exists a sequence (xn) such that \limn→∞ d(Axn, t) = \limn→∞ d(Bxn, t) = 0 for some t ∈ X.
Remark 2.13. It is clear from the above Definition 2.10, that two self mappings S and T of a symmetric space (X, d) will be noncompatible if there exists at least one sequence \((x_n)\) in X such that \(\lim_{n \to \infty} d(Sx_n, t) = \lim_{n \to \infty} d(Tx_n, t) = 0\) for some \(t \in X\), but \(\lim_{n \to \infty} d(STx_n, TSx_n)\) is either non-zero or does not exist. Therefore, two noncompatible self mappings of a symmetric space (X, d) satisfy the property (E.A).

Definition 2.14. Let \((X, d)\) be a symmetric space. We say that \((X, d)\) satisfies the property \((H)\) if given \(\{x_n\}, \{y_n\}\) and \(x \in X\), and \(\lim_{n \to \infty} d(x_n, x) = 0\ and \lim_{n \to \infty} d(y_n, x) = 0\ imply \lim_{n \to \infty} d(y_n, x_n) = 0\.

Note that \((X, d)\) is not a metric space.

### III. Implicit Relation

Implicit relations on metric spaces have been used in many articles. (See [4], [10], [13].

Let \(R_+\) denote the non-negative real numbers and let \(F\) be the set of all Continuous functions \(F: R_+ \to R_+\) satisfying the following conditions:

\(F_1: \) there exists an upper semi-continuous and non-decreasing function \(F: R_+ \to R_+\), \(f(0) = 0\), \(f(t) \leq t\) for \(t > 0\), such that for \(u \geq v\), \(F(u, v, v, 0) \leq 0\ or\ F(u, v, 0, v) \leq 0\ or\ F(u, 0, v, v) \leq 0\ implies\ u \leq f(v)\).

\(F_2: \) \(F(u, 0, 0, 0) > 0\ and\ F(u, u, u, 0) > 0\ , \forall u > 0\).

Example 3.1. \(F(t_1, t_2, t_3, t_4) = t_1 - \max\{t_2, t_3, t_4\}\), where \(0 < \alpha < 1\).

\(F_1: \) Let \(u \geq 0\ and\ F(u, v, v, 0) = u + \alpha v \leq 0\,\text{then}\,u \leq \alpha v\). Similarly, let \(u > 0\ and\ F(u, v, 0, v) \leq 0\,\text{then}\,u \leq \alpha v\) and again let \(u > 0\ and\ F(u, 0, v, v) \leq 0\,\text{then}\,u \leq \alpha v\). If \(u = 0\ then\ u \leq \alpha v\). Thus \(F_1\) is satisfied with \(f(t) = \alpha t\).

\(F_2: \) \(F(u, 0, 0, 0) = u > 0\,\forall u > 0\ and\ F(u, u, u, 0) = u(1 - u) > 0\ , \forall v > 0\).

Thus \(F \in F\).

Example 3.2. \(F(t_1, t_2, t_3, t_4) = t_1 - \psi(\max\{t_2, t_3, t_4\})\), where \(R_+ \to R_+\) is upper semi-continuous, non-decreasing and \(\psi(0) = 0\,\psi(t) < t\) for \(t > 0\).

\(F_1: \) Let \(u > 0\ and\ F(u, v, v, 0) = u - \psi(v) \leq 0\,\text{then}\,u \leq \psi(v)\). Similarly, let \(u > 0\ and\ F(u, v, 0, v) \leq 0\,\text{then}\,u \leq \psi(v)\) and again let \(u > 0\ and\ F(u, 0, v, v) \leq 0\,\text{then}\,u \leq \psi(v)\). If \(u = 0\ then\ u \leq \psi(v)\). Thus \(F_1\) is satisfied with \(f = \psi\).

\(F_2: \) \(F(u, 0, 0, 0) = u > 0\,\forall u > 0\ and\ F(u, u, u, 0) = u - \psi(u) > 0\,\forall u > 0\).

Thus \(F \in F\).

### IV. Main Result

Theorem 4.1: Let \(d\) be a symmetric for \(X\) that satisfies \((W.3), (W.4)\) and \((H_0)\). Let \(\{A_i\}, \{A_j\}\) \((i \neq j)\) and S be self mappings of \((X, d)\) such that

\[(1) F\left(\int_0^1 \left(\int_0^{d(Ax, Ay)} \phi(t) dt\right) d(SX, S) \phi(t) dt, \int_0^1 \left(\int_0^{d(Sx, Sy)} \phi(t) dt\right) d(SX, S) \phi(t) dt, \int_0^1 \left(\int_0^{d(Sy, Sx)} \phi(t) dt\right) d(SX, S) \phi(t) dt\right) \leq 0.

for all \((x, y) \in X^2, (i \neq j)\) where \(F \in F\) and \(\phi: R_+ \to R_+\) is a Lebesque-integrable mapping which is summable, non-negative and such that \((2) \phi(t) dt > 0\) for all \(t > 0\).

Suppose that \(AX \subseteq SX\) and \(AX \subseteq SX\), \((i \neq j)\) \((A_i)\) \((A_j)\) and \((A_i)\) \((A_j)\) \((i \neq j)\) are weakly compatible and \((A_i)\) \((A_j)\) \((i \neq j)\) is closed subspace of \(X\), then \(\{A_i\}, \{A_j\}\) \((i \neq j)\) have a unique common fixed point in \(X\).

Proof: Suppose that \(\{A_j\}\) and \(T, \forall j\) satisfy property \((E.A)\). Then, there exists a sequence \(\{x_n\}\) in \(X\) such that

\[\lim_{n \to \infty} d(A_jx_n, 2) = \lim_{n \to \infty} d(Sx_n, 2) = 0\ \text{for some} \ z \in X, \forall j \]

Therefore, by \((H_0)\) we have \(\lim_{n \to \infty} d(A_jx_n, Sx_n) = 0, \forall j\).

Since \(A_j(X) \subseteq S(X)\), \(\forall j\), there exists in \(X\) a sequence \(\{y_n\}\) such that \(A_jy_n = Sy_n, \forall j\).

Hence, \(\lim_{n \to \infty} d(Sy_n, z) = 0\).

Let us show that \(\lim_{n \to \infty} d(A_jy_n, 2) = 0, \forall i\).

Suppose that \(\lim_{n \to \infty} d(A_jy_n, A_jx_n) > 0\). Then, using \((1)\), we have

\[\int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt, \int_0^1 \left(\int_0^{d(SX, SX)} \phi(t) dt\right) d(SX, SX) \phi(t) dt, \int_0^1 \left(\int_0^{d(SX, SX)} \phi(t) dt\right) d(SX, SX) \phi(t) dt\right) \leq 0, (i \neq j)\]

We have,

\[\int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt, \lim_{n \to \infty} \int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt, \lim_{n \to \infty} \int_0^1 \left(\int_0^{d(SX, SX)} \phi(t) dt\right) d(SX, SX) \phi(t) dt\right) \leq 0, (i \neq j)\]

From \(F_1\), there exists an upper semi-continuous and non-decreasing function \(f: R_+ \to R_+, f(0) = 0, f(t) < t\) for \(t > 0\), such that \(\lim_{n \to \infty} d(A_jy_n, A_jx_n) \phi(t) dt \leq \left(\lim_{n \to \infty} \int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt, (i \neq j)\right)\)

Therefore \(\lim_{n \to \infty} \int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt > 0\), which is a contradiction. Then we have

\[\lim_{n \to \infty} \int_0^1 \left(\int_0^{d(A_jy_n, A_jx_n)} \phi(t) dt\right) d(SX, SX) \phi(t) dt = 0, \forall i\].

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Suppose that SX is a closed subspace of X. Then \( z = Su \) for some \( u \in X \). Consequently, we have \( \lim_{n \to \infty} d(A_j x_n, A_j x_n) = \lim_{n \to \infty} d(A_j x_n, Su) = \lim_{n \to \infty} d(Sx_n, Su) = \lim_{n \to \infty} d(Sy_n, Su) = 0 \).

We claim that \( Au = Su \). Using (1),

\[
F_0^d(A u, A_j x_n) \phi(t) dt, f_0^d(Su, x_n) \phi(t) dt, c_0^d(Su, A_j x_n) \phi(t) dt, d_0^d(A u, A_j x_n) \phi(t) dt \geq 0,
\]

and letting \( n \to \infty \), we have \( F_0^d(A u, A_j x_n) \phi(t) dt, 0, 0, 0 \geq 0, \forall i, j(i \neq j) \).

which is a contradiction with \( F_2 \), if \( \lim_{n \to \infty} f_0^d(A u, A_j x_n) \phi(t) dt = 0 \).

Thus we obtain \( \lim_{n \to \infty} f_0^d(A u, A_j x_n) \phi(t) dt = 0 \) and (2) implies that \( \lim_{n \to \infty} f_0^d(A u, A_j x_n) = 0, \forall i, j(i \neq j) \).

By (W.3) we have \( z = Au = Su \). \( \forall i \) The weak compatibility of \( \{A_j\} \) and \( S \forall i \) implies that \( A_S u = S A u \forall i \); i.e., \( A_z = S z \). \( \forall i \) On the other hand, since \( A_X \subseteq SX \), \( \forall i \) there exists \( v \in X \) such that \( Au = Sv \). \( \forall i \) We claim that \( A v = Sv \). \( \forall i \) If not, condition (1) gives

\[
F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j).
\]

And we have, \( F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, A_j x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j) \).

From \( F_2, f_0^d(Su, A_j x_p) \phi(t) dt = f_0^d(A u, A_j x_p) \phi(t) dt \leq f(f_0^d(Su, A_j x_p) \phi(t) dt) \).

Which is a contradiction since \( f_0^d(Su, A_j x_p) \phi(t) dt > 0 \), by (2)

Hence, \( z = Au = Su = A v = Sv \). \( \forall i \), the weak compatibility of \( \{A_j\} \) and \( S \forall j \) implies that \( A_S u = S A v \).

i.e., \( A_z = S z \).

Let us show that \( z \) is a common fixed point of \( \{A_j\}, \{A_j\}, \) and \( S \).

If \( z \neq A z \), \( \forall i \) using (1), we get

\[
F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j).
\]

And we have, \( F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, A_j x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j) \).

Which is a contradiction with \( F_2 \), since \( f_0^d(A u, A_j x_p) \phi(t) dt > 0 \) by (2)

Thus \( z = A z = A z \).

If \( z \neq A z \) using (1) we get

\[
F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j).
\]

And we have \( F_0^d(A u, A_j x_p) \phi(t) dt, f_0^d(Su, A_j x_p) \phi(t) dt, c_0^d(Su, A_j x_p) \phi(t) dt, d_0^d(A u, A_j x_p) \phi(t) dt \leq 0. \ (i \neq j) \).

which is a contradiction with \( F_2 \) since \( f_0^d(A u, A_j x_p) \phi(t) dt > 0 \) by (2).

Thus \( z = A z = S z = A z \).

The cases in which \( A X \) or \( A X \) is a closed subspace of \( X \) are similar to the cases in which \( SX \) is closed since \( A X \subseteq SX \) and \( A X \subseteq SX \).

Uniqueness.

For the uniqueness of \( z \), suppose that \( w \neq z \) is another common fixed point of \( \{A_j\}, \{A_j\} \) and \( S \).

Using (1), we obtain, \( F_0^d(A z, A_j w) \phi(t) dt, f_0^d(S z, x_p) \phi(t) dt, c_0^d(S z, A_j w) \phi(t) dt, d_0^d(A z, A_j w) \phi(t) dt \leq 0. \ (i \neq j) \).

And we have \( F_0^d(A z, A_j w) \phi(t) dt, f_0^d(S z, A_j w) \phi(t) dt, c_0^d(S z, A_j w) \phi(t) dt, d_0^d(A z, A_j w) \phi(t) dt \leq 0. \ (i \neq j) \).

which is a contradiction with \( F_2 \) since \( f_0^d(A z, A_j w) \phi(t) dt > 0 \) by (2). Thus \( z = w \), and the common fixed point is unique. This completes the proof of the theorem.

**Corollary 4.2:** Let \( d \) be a symmetric for \( F \) that satisfies (W.3),(W.4) and (H.4). Let \( A, B \) and \( S \) be self mappings of \( (X,d) \) such that (1)\( f_0^d(A x, z) \phi(t) dt, f_0^d(S x, y) \phi(t) dt, c_0^d(S x, A_j z) \phi(t) dt, d_0^d(A x, S y) \phi(t) dt \leq 0. \ (i \neq j) \).

for all \((x,y) \in X^2 \), where \( F \in F \) and \( \phi : R \rightarrow R \) is a Lebesque-integrable mapping which is summable, non-negative and such that (2) \( \int_0^\infty \phi(t) dt > 0 \) for all \( c > 0 \).

Suppose that \( A X \subseteq SX \) and \( B X \subseteq SX \), \( (A, S) \) and \( (B, S) \) are weakly compatible and (A, S) or (B, S) satisfies property (E.A). If the range of one of the mappings \( A, B \) or \( S \) is a closed subspace of \( X \), then \( A, B \) and \( S \) have a unique common fixed point in \( X \).

**Proof.** The proof of Corollary 4.2 follows from theorem 4.1 by putting \( A_1 = A; A_2 = B. \ (i \neq j) \).
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Theorem 4.3. Let d be a symmetric for X that satisfies (W.3), (W.4) and (H3). Let \{A_i, B\} \forall i be self mappings of \(X,d\) such that (1) \(\int_{0}^{d(A_i,x,y)} \phi(t)dt + \int_{0}^{d(B,x,y)} \phi(t)dt \leq 0\) for all \((x,y) \in X^2\), where \(F \subseteq F\) and \(\phi: R \rightarrow R\) is a Lebesque-integrable mapping which is summable, non-negative and such that (2) \(\int_{0}^{c} \phi(t)dt > 0\) for all \(c > 0\). Suppose that \(A_X \subseteq BX \forall i\), \((A_i, B) \forall i\) is weakly compatible and \((A_i, B) \forall i\) satisfy the property (E.A). If the range of one of the mappings \(\{A_i\}\), or B is a closed subspace of X, then \(\{A_i\}\) and \(B \forall i\) have a unique common fixed point in X.

Proof. Suppose that \(A_i\) and B \(\forall i\) satisfy property (E.A). Then there exists a sequence \(\{x_n\}\) in X such that that \(\lim_{n \rightarrow \infty} d(A_i x_n, z) = \lim_{n \rightarrow \infty} d(B x_n, z) = 0\) for some \(z \in X\).

Therefore, by (H3) we have \(\lim_{n \rightarrow \infty} d(A_i x_n, B x_n) = 0. \forall i\).

Since \(A_X \subseteq BX \forall i\), there exists in X a sequence \(\{y_n\}\) such that \(A x_n = B y_n\).

Hence, \(\lim_{n \rightarrow \infty} d(B y_n, z) = 0\).

Let us show that \(\lim_{n \rightarrow \infty} d(A y_n, z) = 0. \forall i\).

Suppose that \(\lim_{n \rightarrow \infty} d(A y_n, A x_n) > 0\). Then, using (1), we have

\(\int_{0}^{d(A y_n, A x_n)} \phi(t)dt + \int_{0}^{d(B y_n, B x_n)} \phi(t)dt \leq 0\) for all \(i\).

And we have, \(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt, \lim_{n \rightarrow \infty} \int_{0}^{d(B y_n, B x_n)} \phi(t)dt, \lim_{n \rightarrow \infty} \int_{0}^{d(A x_n, B x_n)} \phi(t)dt \leq 0. \forall i\).

From (1), there exists an upper semi-continuous and non-decreasing function \(f: R \rightarrow R, f(0) = 0, f(t) < t\) for \(t > 0\) such that \(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt \leq 0\).

Therefore \(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt, \lim_{n \rightarrow \infty} \int_{0}^{d(B y_n, B x_n)} \phi(t)dt, \lim_{n \rightarrow \infty} \int_{0}^{d(A x_n, B x_n)} \phi(t)dt \leq 0. \forall i\).

which is a contradiction. Then we have that

\(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt = 0. (2)\) implies that \(\lim_{n \rightarrow \infty} d(A_i y_n, A_i x_n) = 0\)

By (W.4), we deduce that \(\lim_{n \rightarrow \infty} d(A y_n, z) = 0. \forall i\).

Suppose that BX is a closed subspace of X. Then \(z = Bu\) for some \(u \in X\). Consequently, we have \(\lim_{n \rightarrow \infty} d(A y_n, A x_n) = \lim_{n \rightarrow \infty} d(B x_n, Bu) = \lim_{n \rightarrow \infty} d(A x_n, Bu) = 0\).

We claim that \(Au = Bu\) Using (1), \(f(0) = 0, f(t) < t\) for \(t > 0\) such that \(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt \leq 0. \forall i\).

and letting \(n \rightarrow \infty\), we have \(\lim_{n \rightarrow \infty} \int_{0}^{d(A y_n, A x_n)} \phi(t)dt, 0, 0, 0) \leq 0. \forall i\).

which is a contradiction with \(F_i\) if \(\lim_{n \rightarrow \infty} \int_{0}^{d(A x_n, B x_n)} \phi(t)dt = 0\). Thus we obtain \(\lim_{n \rightarrow \infty} \int_{0}^{d(A u, B v)} \phi(t)dt = 0\)

and (2) implies that \(\lim_{n \rightarrow \infty} d(A_i u, B x_n) = 0. \forall i\).

By (W.3) we have \(z = Au = Bu\). \(\forall i\) The weak compatibility of \(\{A_i\}\) and B \(\forall i\), implies that \(A_B A = B A U \forall i ; i.e., A_B A = B = B\). \(\forall i\)

The proof is similar when \(A X \forall i\) is assumed to be a closed subspace of X, since \(A X \subseteq BX \forall i\).

Uniqueness.

If \(A \subseteq Bu = u\) and \(A v = v\) \(\forall i\) and \(u \neq v\) then (1) given,

\(\int_{0}^{d(A u, A v)} \phi(t)dt, \int_{0}^{d(B u, B v)} \phi(t)dt, \int_{0}^{d(A u, A v)} \phi(t)dt, \int_{0}^{d(B u, B v)} \phi(t)dt \leq 0. \forall i\).

And we have

\(\int_{0}^{d(u, v)} \phi(t)dt, \int_{0}^{d(u, v)} \phi(t)dt, \int_{0}^{d(u, v)} \phi(t)dt, 0 = 0. \forall i\).

which is a contradiction with \(F_i\) since \(\int_{0}^{d(u, v)} \phi(t)dt > 0\) by (2). Thus \(u = v\) and the common fixed point is unique. This completes the proof of the theorem.

Corollary 4.4: Let d be a symmetric for X that satisfies (W.3), (W.4) and (H3). Let A, B be self mappings of \(X,d\) such that (1) \(\int_{0}^{d(A x, y)} \phi(t)dt, \int_{0}^{d(B x, y)} \phi(t)dt, \int_{0}^{d(A x, y)} \phi(t)dt, \int_{0}^{d(B y, y)} \phi(t)dt \leq 0. \forall i\).

for all \((x, y) \in X^2\), where \(F \subseteq F\) and \(\phi: R \rightarrow R\) is a Lebesque-integrable mapping which is summable, non-negative and such that (2) \(\int_{0}^{c} \phi(t)dt > 0\) for all \(c > 0\).

Suppose that \(A X \subseteq B X, (A, B)\) is weakly compatible and \((A, B)\) satisfy the property (E.A). If the range of one of the mappings A or B is a closed subspace of X, then A, B have a unique common fixed point in X.

Proof. The proof of Corollary 4.4 follows from theorem 4.3 by putting \(A_i = A \forall i\).

If \(\phi(t) = 1, A_i = A, \forall i\) and in Corollary (4.4), we obtain Theorem 2.1 of [1].
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References.


