Contra $R^*$- Continuous And Almost Contra $R^*$- Continuous Functions

Renu Thomas* and C.Janaki**

*Department of Mathematics, Sree Narayana Guru College, Coimbatore (TN) INDIA.
* Department of Mathematics, L.R.G. Govt.Arts college for Women,Tirupur (TN) INDIA

Abstract: In this paper we present and study a new class of functions as a new generalization of contra continuity. Furthermore we obtain some of their basic properties and relationship with $R^*$-regular graphs.

Keywords: Contra $R^*$-continuous function, almost contra $R^*$-continuous functions, $R^*$-regular graphs, $R^*$-locally indiscrte.

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I. Introduction

In 1996,Dontchev [5] introduced the notion of contra continuity. Several new generalizations to this class were added by Dontchev and Noiri [6] as contra-continuous functions and S-closed spaces, contra semi continuous,contra $\delta$ -precontinuous functions etc. C.W.Baker [2] introduced and investigated the notion of contra $\beta$ continuity, Jafari and Noiri [11] studied the contra precontinuous and contra $\alpha$ continuous functions.

Almost contra pre continuous function was introduced by Ekici [7].In this direction we will introduce the concept of almost contra $R^*$-continuous functions. We include the properties of contra $R^*$-continuous functions and the $R^*$-regular graphs.

Throughout this paper, the spaces $X$ and $Y$ always mean the topological spaces $(X, \tau)$ and $(Y, \sigma)$ respectively. For $A \subseteq X$, the closure and the interior of $A$ in $X$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively. Also the collection of all $R^*$-open subsets of $X$ containing a fixed point $x$ is denoted by $R^*-O(X,x)$.

II. Preliminaries

Definition: 2.1. A subset $A$ of a topological space $(X, \tau)$ is called (1) a regular open [17] if $A = \text{int} (\text{cl}(A))$ and regular closed [17] if $A = \text{cl}(\text{int}(A))$.

The intersection of all regular closed subset of $(X, \tau)$ containing $A$ is called the regular closure of $A$ and is denoted by $rcl(A)$.

Definition: 2.2. [4] A subset $A$ of a space $(X, \tau)$ is called regular semi open set if there is a regular open set $U$ such that $U \subseteq A \subseteq \text{cl}(U)$. The family of all regular semi open sets of $X$ is denoted by $\text{RSO}(X)$.

Lemma: 2.3. [5] In a space $(X, \tau)$, the regular closed sets, regular open sets and clopen sets are regular semiopen.

Definition: 2.4. A subset of a topological space $(X, \tau)$ is called

1. a regular generalized (briefly rg-closed) [17] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
2. a generalized pre regular closed (briefly gpr-closed) [10] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
3. a regular weakly generalized closed (briefly rwg-closed) [15] if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
4. a generalized regular closed (briefly gr-closed) [14] if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
5. a regular generalized weak closed set (briefly rgw-closed) [19] if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semi open in $X$.

The complements of the above mentioned closed sets are their respectively open sets.

Definition: 2.5 [12] A subset $A$ of a space $(X, \tau)$ is called $R^*$-closed if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semiopen in $(X, \tau)$. We denote the set of all $R^*$-closed sets in $(X, \tau)$ by $R^*\text{C}(X)$.

Definition: 2.6 [12] A function $f : X \rightarrow Y$ is called $R^*$-continuous if $f^{-1}(V)$ is $R^*$-closed in $X$ for every closed set $V$ of $Y$.

Definition: 2.7 [5] A function $f : X \rightarrow Y$ is called contra continuous if $f^{-1}(V)$ is closed in $X$ for every open set $V$ of $Y$. 
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Definition: 2.8 A function \( f : X \to Y \) is called

1. contra rg- continuous, if \( f^{-1}(V) \) is rg- closed in \( X \) for each open set \( V \) of \( Y \).
2. contra gpr- continuous, if \( f^{-1}(V) \) is gpr- closed in \( X \) for each open set \( V \) of \( Y \).
3. contra rwg-continuous, if \( f^{-1}(V) \) is rwg- closed in \( X \) for each open set \( V \) of \( Y \).
4. contra gr- continuous, if \( f^{-1}(V) \) is gr- closed in \( X \) for each open set \( V \) of \( Y \).
5. contra rgw-continuous, if \( f^{-1}(V) \) is grw- closed in \( X \) for each open set \( V \) of \( Y \).
6. an R-map [8] if \( f^{-1}(V) \) is regular closed in \( X \) for each regular closed set \( V \) of \( Y \).
7. perfectly continuous if \[1, 7\] \( f^{-1}(V) \) is clopen in \( X \) for each open set \( V \) in \( Y \).
8. almost continuous if [20] \( f^{-1}(V) \) is open in \( X \) for each regular open set \( V \) in \( Y \).
9. regular set connected if [9] \( f^{-1}(V) \) is clopen in \( X \) for each regular open set \( V \) in \( Y \).
10. RC-continuous [8] if \( f^{-1}(V) \) is regular closed in \( X \) for each open set \( V \) in \( Y \).

Definition: 2.9 [21] A space is said to be weakly Hausdorff if each element of \( X \) is an intersection of regular closed sets.

Definition: 2.10 [22] A space is said to be Ultra Hausdorff if for every pair of distinct points \( x \) and \( y \) in \( X \), there exist disjoint clopen sets \( U \) and \( V \) containing \( x \) and \( y \) respectively.

Definition: 2.11 [22] A topological space \( X \) is called a Ultra normal space, if each pair of disjoint closed sets can be separated by disjoint clopen sets.

Definition: 2.12 [23] A topological space \( X \) is said to be hyperconnected if every open set is dense.

III. Contra R*-continuous functions

Definition: 3.1 A space \( X \) is called locally R*-indiscrete if every R*-open subset of \( X \) is closed.

Definition: 3.2 A function \( f : X \to Y \) is called contra R*-continuous if \( f^{-1}(V) \) is R*-closed in \( X \) for every open set \( V \) of \( Y \).

Definition: 3.3 A function \( f : X \to Y \) is strongly R*-open if the image of every R*-open set of \( X \) is R*-open in \( Y \).

Definition: 3.4 A function \( f : X \to Y \) is almost R*-continuous if \( f^{-1}(V) \) is R*-open in \( X \) for each regular open set \( V \) of \( Y \).

Theorem: 3.5 Every contra R*-continuous function is contra rg-continuous, contra gpr-continuous, contra rwg-continuous, contra gr-continuous, contra rgw-continuous but not conversely.

Proof: Obvious from definitions.

Example 3.6: Let \( X = \{a, b, c, d\} \equiv Y, \tau = \{X, \phi, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\} \quad \sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}\)

Define a mapping \( f : X \to Y \) as the identity mapping. Here the function \( f \) is contra rg-continuous, contra gpr-continuous and contra rwg-continuous but not contra R*-continuous since \( f^{-1}(a) = \text{aand} f^{-1}(c) = c \) are not R*-closed.

Example 3.7: Let \( X = \{a, b, c, d\} \equiv Y, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \quad \sigma = \{Y, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}\).

Define a mapping \( f : X \to Y \) as \( f(a) = c, f(b) = a, f(c) = d, f(d) = b \), the function \( f \) is contra gr-continuous but not contra R*-continuous.

Example 3.8: \( X = \{a, b, c, d\} \equiv Y, \tau = \{X, \phi, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}\} \quad \sigma = \{Y, \phi, \{c, d\}, \{a, c, d\}\}\).

Define a mapping \( f : X \to Y \) as \( f(a) = b, f(b) = a, f(c) = d, f(d) = c \), the function \( f \) is contra rgw-continuous but not contra R*-continuous.

Remark: 3.9 Contra continuous and contra R*-continuous are independent concepts.
Example 3.10
Let \( X = Y = \{a, b, c\} \) \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) \( \sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}\} \). Define \( f : X \to Y \) as the identity mapping. Here \( f \) is contra \( R^* \)-continuous but not contra continuous since \( f^{-1} \{b\} = \{b\} \) is not closed in \( X \).

Example 3.11: Let \( X = Y = \{a, b, c, d\} \) \( \tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\} \) \( \sigma = \{Y, \phi, \{d\}\} \). Define \( f : X \to Y \) as the identity mapping. \( f^{-1} \{d\} = \{d\} \) is not \( R^* \)-closed and hence is not contra \( R^* \)-continuous but contra continuous.

Theorem 3.12: Every RC continuous function is contra \( R^* \)-continuous but not conversely.
Proof: Straight forward.

Example 3.13: Let \( X = Y = \{a, b, c\} \) \( \tau = \{\{X, \phi, \{a\}, \{b\}, \{a, b\}\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\} \) \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, c\}\} \).

Define \( f : X \to Y \) as \( f(a) = b, f(b) = c, f(c) = a \).
Here \( f \) is contra \( R^* \)-continuous but not RC-continuous.

Remark 3.14: The composition of two contra \( R^* \)-continuous functions need not be contra \( R^* \)-continuous as seen in the following example.

Example 3.15: Let \( X = Y = Z = \{a, b, c\} \) \( \tau = \{X, \phi, \{a\}, \{c\}\}, \{a, c\}\} \) \( \sigma = \{Y, \phi, \{a\}, \{b\}\} \).

Define \( f : X \to Y \) by \( f(a) = a, f(b) = c, f(c) = a \). Then \( f \) and \( g \) are contra \( R^* \)-continuous but \( g \circ f : X \to Z \) is not contra \( R^* \)-continuous since \( (g \circ f)^{-1} \{b\} = f^{-1} (g^{-1} \{b\}) = f^{-1} \{a\} = \{c\} \) is not \( R^* \)-closed.

Theorem 3.16: If \( f : (X, \tau) \to (Y, \sigma) \) is a contra \( R^* \)-continuous function and \( g : (Y, \sigma) \to (Z, \eta) \) is a continuous function, then the function \( g \circ f : X \to Z \) is contra \( R^* \)-continuous.

Proof: Let \( V \) be open in \( Z \). Since \( g \) is continuous, \( g^{-1}(V) \) is open in \( Y \). \( f \) is contra \( R^* \)-continuous, so \( f^{-1}(g^{-1}(V)) \) is \( R^* \)-closed in \( X \). Hence \( g \circ f \) is contra \( R^* \)-continuous.

Theorem 3.17: If \( f : (X, \tau) \to (Y, \sigma) \) is a contra \( R^* \)-continuous map and \( g : (Y, \sigma) \to (Z, \eta) \) is regular set connected function, then \( g \circ f : X \to Z \) is \( R^* \)-continuous and almost \( R^* \)-continuous.

Proof: Let \( V \) be regular open in \( Z \). Since \( g \) is regular set connected, \( g^{-1}(V) \) is clopen in \( Y \). Since \( f \) is a contra \( R^* \)-continuous \( f^{-1}(g^{-1}(V)) \) is \( R^* \)-closed in \( X \) hence \( g \circ f \) is almost \( R^* \)-continuous.

Theorem 3.18: If \( f : (X, \tau) \to (Y, \sigma) \) is \( R^* \)-irresolute and \( g : (Y, \sigma) \to (Z, \eta) \) is a contra \( R^* \)-continuous function, then \( g \circ f : X \to Z \) is contra \( R^* \)-continuous.

Proof: Let \( V \) be open in \( Z \). Since \( g \) is contra \( R^* \)-continuous, \( g^{-1}(V) \) is \( R^* \)-closed in \( Y \). Since \( f \) is a contra \( R^* \)-irresolute, \( f^{-1}(g^{-1}(V)) \) is \( R^* \)-closed in \( X \). Hence \( g \circ f \) is contra \( R^* \)-continuous.

Theorem 3.19: If \( f : (X, \tau) \to (Y, \sigma) \) is \( R^* \)-continuous and the space \( X \) is \( R^* \)-locally indiscrete, then \( f \) is contra continuous.

Proof: Let \( V \) be an open set in \( Y \). Since \( f \) is \( R^* \)-continuous \( f^{-1}(V) \) is \( R^* \)-open in \( X \). And since \( X \) is locally \( R^* \)-indiscrete, \( f^{-1}(V) \) is closed in \( X \). Hence \( f \) is contra continuous.

Theorem 3.20: If \( f : (X, \tau) \to (Y, \sigma) \) is contra \( R^* \)-continuous, \( X \) is \( R^*-T_{1/2} \) space, then \( f \) is RC-continuous.

Proof: Let \( V \) be open in \( Y \). Since \( f \) is contra \( R^* \)-continuous, \( f^{-1}(V) \) is \( R^* \)-closed in \( X \). And \( X \) is \( R^*-T_{1/2} \) space, hence \( f^{-1}(V) \) is regular closed in \( X \). Thus for every open set \( V \) of \( Y \), \( f^{-1}(V) \) is regular closed in \( X \). Hence \( f \) is RC-continuous.

Theorem 3.21: Suppose \( R^*-O(X) \) is closed under arbitrary unions, then the following are equivalent for a function \( f : (X, \tau) \to (Y, \sigma) \).

(i) \( f \) is contra \( R^* \)-continuous

(ii) for each closed subset \( V \) of \( Y \), \( f^{-1}(V) \in R^*-O(X) \).

(iii) for each \( x \in X \) and each \( V \subseteq C(Y, f(x)) \) there exists a set \( U \subseteq R^*O(X, x) \) such that \( f(U) \subseteq V \).
Proof: (i) \( \Rightarrow \) (ii): Let \( f \) be contra R*-continuous. Then \( f^{-1}(V) \) is R*-closed in X for every open set \( V \) of Y. i.e \( f^{-1}(V) \) is R*-open in X for every closed set \( V \) of Y. Hence \( f^{-1}(V) \in R^*-O(X) \).

(ii) \( \Rightarrow \) (iii): For every closed subset \( V \) of Y, \( f^{-1}(V) \in R^*O(X) \). Then for each \( x \in X \) and each \( V \in C(Y,f(x)) \), there exists a set \( U_x \in R^*O(X,x) \) such that \( f(U_x) \subset V \). So there exists \( U \in R^*O(X) \) such that \( f^{-1}(V) = \cup \{ U_x : x \in f^{-1}(V) \} \) and hence \( f^{-1}(V) \) is R*-open.

Definition 3.22: [7] For a function \( f: X \rightarrow Y \), the subset \( \{(x,f(x)) : x \in X\} \subset X \times Y \) is called the graph of \( f \) and is denoted by \( G(f) \).

Lemma 3.23: [3] Let \( G(f) \) be the graph of \( f \), for any subset \( A \subset X \) and \( B \subset Y \), we have \( f(A) \cap B = \emptyset \) if and only if \( (A \times B) \cap G(f) = \emptyset \).

Definition 3.24: The graph \( G(f) \) of a function \( f: X \rightarrow Y \) is said to be contra R*-closed if for each \( (x,y) \in (X,Y) - G(f) \) there exists \( U \in R^*O(X,x) \) and \( V \in C(Y,y) \) such that \( (U \times V) \cap G(f) = \emptyset \).

Lemma 3.25: The graph \( G(f) \) of a function \( f: X \rightarrow Y \) is said to be contra R*-closed if for each \( (x,y) \in (X,Y) - G(f) \) there exists \( U \in R^*O(X,x) \) and \( V \in C(Y,y) \) such that \( f(U) \cap V = \emptyset \).

Proof: The proof is a direct consequence of definition 3.24 and lemma 3.23.

IV. Almost contra R*-continuous function

Definition 4.1: A function \( f: X \rightarrow Y \) is said to be almost contra R*-continuous if \( f^{-1} \) is R*-closed in X for each regular open set \( V \) in Y.

Theorem 4.2: If a function \( f: X \rightarrow Y \) is almost contra R*-continuous and X is locally R*-indiscrete space, then \( f \) is almost continuous.

Proof: Let \( U \) be a regular open set in Y. Since \( f \) is almost contra R*-continuous \( f^{-1}(U) \) is R*-closed set in X and X is locally R*-indiscrete space, which implies \( f^{-1}(U) \) is an open set in X. Therefore \( f \) is almost continuous.

Theorem 4.3: If a function \( f: X \rightarrow Y \) is contra R*-continuous, then it is almost contra R*-continuous.

Proof: Obvious because every regular open set is open set.

Remark 4.4: The converse of the theorem need not be true in general as seen from the following example.

\[ X = \{a,b,c \} \subset Y, \quad \tau = \{ X, \emptyset, \{a\}, \{b\}, \{a,b\} \} \quad \sigma = \{ X, \emptyset, \{a\}, \{b\}, \{a\} \}. \]

R*-C(X) = \{ X, \emptyset, \{a\}, \{b\}, \{a\} \}. Define \( f(c) = a, \ f(a) = b, \ f(b) = c, \ f^{-1}((b)) = \{a\} \) which is not R*-closed in X.

Theorem 4.5: The following are equivalent for a function \( f: X \rightarrow Y \)

1. \( f \) is almost contra R*-continuous
2. for every regular closed set \( F \) of Y, \( f^{-1}(F) \) is R*-open set of X.

Proof: (i) Let \( F \) be a regular closed set in Y, then \( Y-F \) is a regular open set in Y. By (i) \( f^{-1}(Y-F) = X-f^{-1}(F) \) is R*-closed in X. Therefore (ii) holds.

(ii) \( \Rightarrow \) (i). Let \( G \) be a regular open set in Y. Then \( Y-G \) is regular closed in Y. By (ii) \( f^{-1}(Y-G) \) is an R*-open set in X. This implies \( X-f^{-1}(G) \) is R*-open which implies \( f^{-1}(G) \) is R*-closed set in X. Therefore (i) holds.

Theorem 4.6: The following are equivalent for a function \( f: X \rightarrow Y \)

1. \( f \) is almost contra R*-continuous.
2. \( f^{-1}(int(cl(G))) \) is a R*-closed set in X for every open set \( G \) of Y.
3. \( f^{-1}(cl(int(F))) \) is a R*-open set in X for every open subset \( F \) of Y.

Proof: (i) \( \Rightarrow \) (ii). Let \( G \) be an open set in Y. Then \( int(cl(G)) \) is regular open set in Y. By (i) \( f^{-1}(int(cl(G))) \in \text{R*-C}(X) \).

(ii) \( \Rightarrow \) (i). Proof is obvious.
(i) \( \Rightarrow \) (iii). Let \( F \) be a closed set in \( Y \). Then \( cl(int(F)) \) is a regular closed set in \( Y \). By (i) \( f^{-1}(cl(int(F))) \in R^*(O(X)) \).

(iii) \( \Rightarrow \) (i). Proof is obvious.

**Definition 4.7:** A space \( X \) is said to be
1. \( R^*-T_{1\delta} \)-space [13] if every \( R^* \)-closed set is regular closed.
2. \( R^*-T_0 \) if for each pair of distinct points in \( X \), there is an \( R^* \)-open set of \( X \) containing one point but not the other.
3. \( R^*-T_1 \) if for every pair of distinct points \( x \) and \( y \), there exists \( R^* \)-open sets \( G \) and \( H \) such that \( x \in G \), \( y \notin G \) and \( x \notin H \).
4. \( R^*-T_2 \) if for every pair of distinct points \( x \) and \( y \), there exists disjoint \( R^* \)-open sets \( G \) and \( H \) such that \( x \in G \) and \( y \in H \).

**Theorem 4.8:** If \( f: X \rightarrow Y \) is an almost contra \( R^* \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( R^*-T_1 \).

Proof: Suppose \( Y \) is weakly Hausdorff, for any distinct points \( x \) and \( y \) in \( X \), there exists \( V \) and \( W \) regular closed sets in \( Y \) such that \( f(x) \in V \), \( f(y) \notin V \) and \( V \cap W \neq \emptyset \). Since \( f \) is an injective function, there exists \( U \) and \( W \) in \( Y \) such that \( f(x) \in U \) and \( f(y) \notin U \). This shows that \( X \) is \( R^*-T_1 \).

**Corollary 4.9:** If \( f: (X,T) \rightarrow (Y,O) \) is a contra \( R^* \)-continuous injection and \( Y \) is weakly Hausdorff, then \( X \) is \( R^*-T_2 \).

Proof: Every contra \( R^* \)-continuous is almost contra \( R^* \)-continuous and by the above theorem [4.8] the result follows.

**Theorem 4.10** If \( f:X \rightarrow Y \) is an almost contra \( R^* \)-continuous injective function from a space \( X \) into the Ultra Hausdorff space \( Y \), then \( Y \) is an \( R^*-T_2 \).

Proof: Let \( x \) and \( y \) be any two distinct points in \( X \). Since \( f \) is an injective function such that \( f(x) \neq f(y) \) and \( Y \) is Ultra Hausdorff space, there exists disjoint clopen sets \( U \) and \( V \) containing \( f(x) \) and \( f(y) \) respectively. Then \( x \in f^{-1}(U) \) and \( y \in f^{-1}(V) \), were \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( R^* \)-open sets in \( X \).

Therefore \( Y \) is \( R^*-T_2 \).

**Definition 4.11** A topological space \( X \) is called a \( R^* \)-normal space, if each pair of disjoint closed sets can be separated by disjoint \( R^* \)-open sets.

**Theorem 4.12** If \( f:X \rightarrow Y \) is an almost contra \( R^* \)-continuous, closed, injective function and \( Y \) is Ultra Normal, then \( X \) is \( R^* \)-normal.

Proof: Let \( E \) and \( F \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective \( f(E) \subseteq Y \) and \( f(F) \subseteq Y \). This implies \( E \subseteq f^{-1}(U) \) and \( F \subseteq f^{-1}(V) \). Since \( f \) is an almost contra \( R^* \)-continuous injection \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( R^* \)-open sets in \( X \). This shows that \( X \) is \( R^* \)-normal.

**Theorem 4.13** For two functions \( f:X \rightarrow Y \) and \( k:Y \rightarrow Z \), let \( k \circ f : X \rightarrow Z \) is a composition function. Then the following holds:

1. If \( f \) is almost contra \( R^* \)-continuous and \( k \) is an R-map, then \( k \circ f \) is almost contra \( R^* \)-continuous.
2. If \( f \) is almost contra \( R^* \)-continuous and \( k \) is perfectly continuous, then \( k \circ f \) is \( R^* \)-continuous and contra \( R^* \)-continuous.
3. If \( f \) is almost contra \( R^* \)-continuous and \( k \) is almost continuous, then \( k \circ f \) is almost contra \( R^* \)-continuous.

Proof: (1) Let \( V \) be any regular open set in \( X \). Since \( k \) is an R-map, \( k^{-1}(V) \) is regular open in \( Y \). Since \( f \) is almost contra \( R^* \)-continuous, \( f^{-1}(k^{-1}(V)) = (k \circ f)^{-1}(V) \) is \( R^* \)-closed in \( X \). Therefore \( k \circ f \) is almost contra \( R^* \)-continuous.

(2) Let \( V \) be an open set in \( Z \). Since \( k \) is perfectly continuous, \( k^{-1}(V) \) is clopen in \( Y \). Since \( f \) is an almost contra \( R^* \)-continuous, \( f^{-1}(k^{-1}(V)) = (k \circ f)^{-1}(V) \) is \( R^* \)-open and \( R^*-closed \) set in \( X \). Therefore \( k \circ f \) is \( R^* \)-continuous and contra \( R^* \)-continuous.

(3) Let \( V \) be a regular open set in \( Z \). Since \( k \) is almost continuous, \( k^{-1}(V) \) is open in \( Y \). Since \( f \) is contra \( R^* \)-continuous, \( f^{-1}(k^{-1}(V)) = (k \circ f)^{-1}(V) \) is \( R^*-closed \) in \( X \). Therefore \( k \circ f \) is almost contra \( R^* \)-continuous.

**Theorem 4.14** Let \( f: X \rightarrow Y \) is a contra \( R^* \)-continuous function and \( g: Y \rightarrow Z \) is \( R^* \)-continuous. If \( Y \) is \( R^*-T_{1/2} \), then \( g \circ f : X \rightarrow Z \) is an almost contra \( R^* \)-continuous function.
Proof: Let \( V \) be regular open and hence open set in \( Z \). Since \( g \) is \( R^* \)-continuous, \( g^{-1}(V) \) is \( R^* \)-open in \( Y \) and \( Y \) is \( T_{1/2} \)-space implies \( g^{-1}(V) \) is regular open in \( Y \). Since \( f \) is almost contra \( R^* \)-continuous, \( f^{-1}\left(g^{-1}(V)\right) = (g \circ f)^{-1}(V) \) is \( R^* \)-closed set in \( X \). Therefore \( g \circ f \) is almost contra \( R^* \)-continuous.

**Theorem 4.15** If \( f: X \to Y \) is surjective, strongly \( R^* \)-open (or strongly \( R^* \)-closed) and \( g: Y \to Z \) is a function such that \( g \circ f : X \to Z \) is almost contra \( R^* \)-continuous, then \( g \) is almost contra continuous.

Proof: Let \( V \) be any regular closed set (resp regular open) set in \( Z \). Since \( g \circ f \) is almost contra \( R^* \)-continuous \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is \( R^* \)-open (resp \( R^* \)-closed) in \( X \). Since \( f \) is surjective and strongly \( R^* \)-open (or strongly \( R^* \)-closed), \( f\left(f^{-1}\left(g^{-1}(V)\right)\right) = g^{-1}(V) \) is \( R^* \)-open (resp \( R^* \)-closed). Therefore \( g \) is almost contra \( R^* \)-continuous.

**Definition 4.16** A topological space \( X \) is said to be \( R^* \)-ultra connected if every two non empty \( R^* \)-closed subsets of \( X \) intersect.

**Theorem 4.17** If \( X \) is \( R^* \)-ultra connected and \( f: X \to Y \) is an almost contra \( R^* \)-continuous surjection, then \( Y \) is hyperconnected.

Proof: Let \( X \) be \( R^* \)-ultra connected and \( f: X \to Y \) is an almost contra \( R^* \)-continuous surjection. Suppose \( Y \) is not hyperconnected. Then there is an open set \( V \) such that \( V \) is not dense in \( Y \). Therefore there exists non empty regular open subsets \( B_1 = \text{int}(V) \) and \( B_2 = Y - \text{cl}(V) \) in \( Y \). Since \( f \) is an almost contra \( R^* \)-continuous surjection, \( f^{-1}(B_1) \) and \( f^{-1}(B_2) \) are disjoint \( R^* \)-closed sets in \( X \). This is contrary to the fact that \( X \) is \( R^* \)-ultra connected. Therefore \( Y \) is hyperconnected.

**V. \( R^* \)-Regular graphs**

**Definition 5.1** A graph \( G(f) \) of a function \( f: X \to Y \) is said to be \( R^* \)-regular if for each \((x,y) \in (X,Y) - G(f)\) there exists \( U \in R^* C(X,x) \) and \( V \in RO(Y,y) \) such that \((U \times V) \cap G(f) = \emptyset\).

**Lemma 5.2** The graph \( G(f) \) of a function \( f: X \to Y \) is \( R^* \)-regular (resp strong contra \( R^* \)-closed) in \( X \times Y \) if and only if for each \((x,y) \in (X,Y) - G(f)\), there is an \( R^* \)-closed (resp \( R^* \)-open) set \( U \) in \( X \) containing \( x \) and \( V \in RO(Y,y) \) (resp \( V \in RC(Y,y) \)) such that \( f(U) \cap V = \emptyset \).

Proof: Obvious.

**Theorem 5.3** If a function \( f: X \to Y \) is \( R^* \)-continuous and \( Y \) is \( T_2 \), then \( G(f) \) is \( R^* \)-regular in \( X \times Y \).

Proof: Let \((x,y) \in (X,Y) - G(f)\). Then \( y \neq f(x) \). Since \( Y \) is \( T_2 \), there exists open set \( V \) in \( Y \) such that \( f(x) \in V \), \( y \in \text{Wand} V \cap W = \emptyset \). Then \( \text{int}(cl(V)) \cap \text{int}(cl(V)) = \emptyset \). Since \( f \) is almost \( R^* \)-continuous is \( f^{-1}(\text{int}(cl(V))) \) is \( R^* \)-closed set in \( X \) containing \( x \). Set \( U = f^{-1}(\text{int}(cl(V))) \), then \( f(U) \subset \text{int}(cl(V)) \). Therefore \( f(U) \cap \text{int}(cl(V)) = \emptyset \). Hence \( G(f) \) is \( R^* \)-regular in \( X \times Y \).

**Theorem 5.4** Let \( f: X \to Y \) be a function and let \( g: X \to X \times Y \) be the graph function of \( f \), defined by \( g(x) = (x,f(x)) \) for every \( x \in X \). If \( g \) is almost contra \( R^* \)-continuous function, then \( f \) is an almost contra \( R^* \)-continuous.

Proof: Let \( V \in RC(Y) \), then
\[ X \times V = X \times \text{Cl}(\text{int}(V)) = \text{Cl}(\text{int}(X)) \times \text{Cl}(\text{int}(V)) = \text{Cl}(\text{int}(X \times V)). \]
Therefore, \( X \times V \in RC(X \times Y) \). Since \( g \) is almost contra \( R^* \)-continuous, \( f^{-1}(V) = g^{-1}(X \times V) \in R^* O(X) \). Thus \( f \) is almost contra \( R^* \)-continuous.

**Theorem 5.5** Let \( f: X \to Y \) have a \( R^* \)-regular \( G(f) \). If \( f \) is injective, then \( X \) is \( R^* - T_0 \).

Proof: Let \( x \) and \( y \) be two distinct points of \( X \). Then \((x,f(y)) \in (X,Y) - G(f) \). Since \( G(f) \) is \( R^* \)-regular, there exists \( R^* \)-closed set \( U \) in \( X \) containing \( x \) and \( V \in RO(Y,f(y)) \) such that \( f(U) \cap V = \emptyset \) by lemma 5.2.
and hence \( U \cap f^{-1}(V) = \emptyset \). Therefore \( y \not\in U \). Thus \( y \in X - U \) and \( x \not\in X - U \) and \( X - U \) is \( R^* \)-open set in \( X \). This implies that \( X \) is \( R^* \)-T_0.

**Theorem 5.6** Let \( f : X \to Y \) be a \( R^* \)-regular \( G(f) \). If \( f \) is surjective then \( Y \) is weakly \( T_2 \).

**Proof:** Let \( y_1 \) and \( y_2 \) be two distinct points of \( Y \). Since \( f \) is surjective \( f(x) = y_1 \) for some \( x \in X \) and \( x, y_2 \in X \times Y - G(f) \). By lemma 5.2, there exists a \( R^* \)-closed set \( U \) of \( X \) and \( F \in \mathcal{RO}(Y) \) such that \( (x, y_2) \in U \times F \) and \( f(U) \cap F = \emptyset \). Hence \( Y_1 \not\in \mathcal{OF} \) Then \( Y_2 \not\in \mathcal{OF} - F \in \mathcal{RC}(Y) \) and \( Y_1 \in Y - F \). This implies that \( Y \) is weakly \( T_2 \).

**References**