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Solution of Dirichlet Boundary Value Problem by Mellin Transform.

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Abstract: *An infinite slab subject to temperature variation is analyzed, the problem is formulated using conformal mapping and solved by mellin transform and method of residue. A closed form solution for the temperature distribution is obtained .A detailed verification of the solution is carried out and find to satisfying the Laplace equation.*

$$r^2\frac{\partial^2 w}{\partial r^2}+r\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial \theta^2}=0 \qquad r\geq 0, 0\leq\theta\leq\pi$$

I. Introduction

To illustrate the use of the Mellin transform in the solution of a physical problem, L.N. Sneddon [1] showed how the distribution of potential in an infinite wedge without a crack can be computed. Analysis of an elastic wedge under out of plane stress is carried out by G.N Emenogu and J.N Nnadi.[2] .In this paper, we investigated an infinite slab A,B,C,D,E,F of thickness a, subjected to temperature $\emptyset(x,a)=T\ along$ BC and to $\emptyset(x,0)=2T$ along DE, $\emptyset(x,a)=0$ along AB and $\emptyset(x,0)=0$ along EF .the problem is to find the temperature distribution within the entire slap

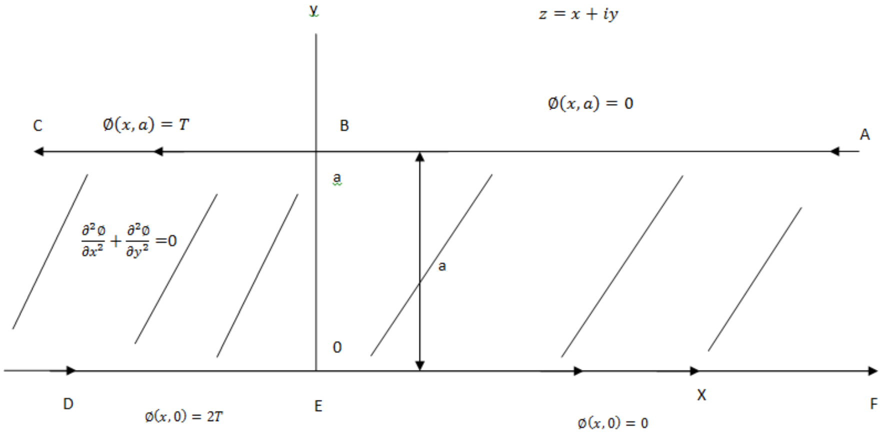


Fig1 Geometry of the plane that forms the slab

The problem in mathematical terms is that of finding a formula for the temperature distribution $\emptyset(x,y)$ so that the following boundary value problem for the Laplace equation is solved

$$\frac{\partial^2\emptyset}{\partial x^2}+\frac{\partial^2\emptyset}{\partial y^2}=0, \qquad -\infty< x<\infty,\ 0\leq y\leq a \qquad 1.1$$

$$\emptyset(x,a)=T, \qquad -\infty< x\leq 0 \qquad 1.2$$
$$=0, \qquad x>0$$

$$\emptyset(x,0)=2T, \qquad -\infty< x\leq 0 \qquad 1.3$$
$$=0, \qquad x>0$$

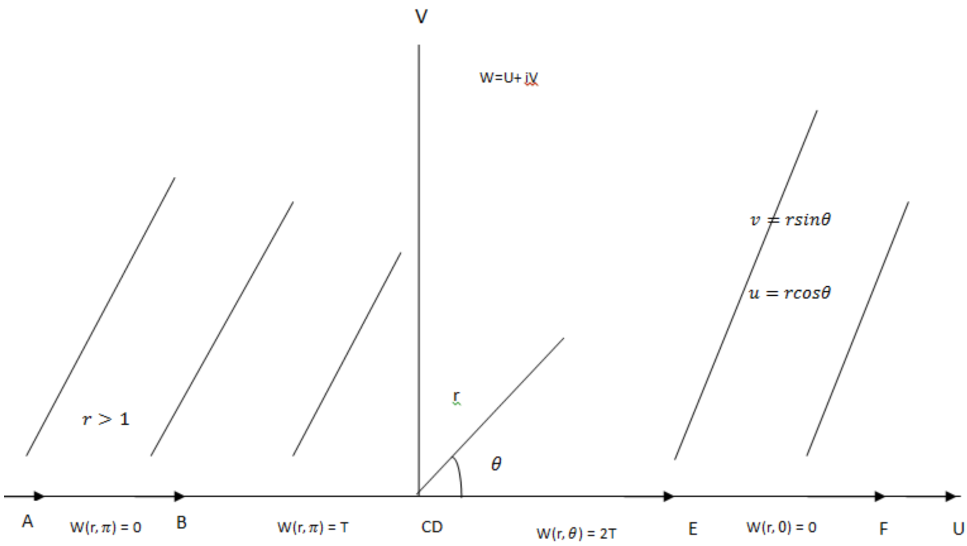
II. A transformation of the plane representing the slab

Because of the nature of the boundary conditions, elementary methods of solution such as method of separation of variables cannot be applied. Therefore, the region of analysis is transformed to a region where integral transform can be applied

2.1 The Conformal Transformation

$$w(z) = e^{\frac{\pi z}{a}}, \qquad z = x + iy$$

Maps the shaded plane region in fig 1 onto the upper half w-plane of Fig 2



2.2 transformation of the problem

The conformal mapping which transforms the infinite slab into the upper half plane also changes the boundary value problem (1.1), (1.2) and (1.3) such that.

$$w(u, v) = \emptyset(x, y) \qquad 2.3\ (a)$$

That is

$$w\big(u(x, y), v(x, y)\big) = \emptyset(x, y) \qquad 2.3\ (b)$$

Where

$$-\infty \le u(x, y) \le \infty, \qquad v(x, y) \ge 0$$
$$-\infty \le x \le \infty, \qquad 0 \le y \le a$$

Following 2.3, the boundary conditions in terms of w becomes

$$\left.\begin{aligned} w\big(u(x, a), v(x, a)\big) = \emptyset(x, a) = T \\ w\big(u(x, 0), v(x, 0)\big) = \emptyset(x, 0) = 2T \end{aligned}\right\}$$

The function w is written as a complex variable in the form

$$w(u, v) = u + iv$$

W is also written in polar form as

$$w(r, \theta) = re^{i\theta}, where$$
$$u(r, \theta) = rcos\theta \ \ and \ \ v(r, \theta) = rsin\theta$$

To link u and v to $w(x, y)$, we note that

$$\begin{aligned} w(x, y) &= e^{\frac{\pi}{a}(x+iy)} \\ &= e^{\left(\frac{\pi}{a}x + i\frac{\pi}{a}y\right)} \\ &= e^{\frac{\pi}{a}x} . e^{i\frac{\pi}{a}y} \\ &= e^{\frac{\pi}{a}x}(cos\frac{\pi}{a}y + isin\frac{\pi}{a}y) \\ &= e^{\frac{\pi}{a}x cos\frac{\pi}{a}y} + ie^{\frac{\pi}{a}x sin\frac{\pi}{a}y} \\ &= u + iv \end{aligned}$$

Hence

$$u(x, y) = e^{\frac{\pi}{a}x} cos\frac{\pi}{a}y \ \text{ And } v(x, y) = e^{\frac{\pi}{a}x} sin\frac{\pi}{a}y$$
$$u(x, 0) = e^{\frac{\pi}{a}x} cos0 \ \ \text{ And } v(x, 0)=0$$
$$u(x, 0) = e^{\frac{\pi}{a}x} \quad and \ \ v(x, 0) = 0$$

On the other hand

$$w(r, \theta) = rcos\theta + isin\theta$$

Implies

$$u(x, y) = e^{\frac{\pi}{a}x} cos\frac{\pi}{a}y = rcos\theta,$$

$$v(x,y) = e^{\frac{\pi}{a}x} sin\frac{\pi}{a}y = rsin\theta,$$

Hence

$$(e^{\frac{\pi}{a}x} cos\frac{\pi}{a}y)^2 = r^2cos^2\theta$$
$$(e^{\frac{\pi}{a}x} sin\frac{\pi}{a}y)^2 = r^2sin^2\theta$$

Then

$$e^{2\frac{\pi}{a}x}cos^2\frac{\pi}{a}y + e^{2\frac{\pi}{a}x}sin^2\frac{\pi}{a}y = r^2cos^2\theta + r^2sin^2\theta$$

Or

$$e^{2\frac{\pi}{a}x}\left(cos^2\frac{\pi}{a}y + sin^2\frac{\pi}{a}y\right) = r^2(cos^2\theta + sin^2\theta)$$

Similarly,

$$\frac{e^{\frac{\pi}{a}x} sin\frac{\pi}{a}y}{e^{\frac{\pi}{a}x} cos\frac{\pi}{a}y} = \frac{rsin\theta}{rcos\theta} = Tan\theta$$

This is

Tan $\theta = tan\frac{\pi}{a}y,$ $\quad thus$

$$\theta = \frac{\pi}{a}y.$$

This means that

$$w(r,\theta) = w(e^{\frac{\pi}{a}x}, e^{\frac{\pi}{a}y})$$

2.3 Solution of the transformed problem

We now solve the transformed problem

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0 \qquad r \geq 0, 0 \leq \theta \leq \pi \qquad 2.5$$

$$w(r,\pi) = T, \quad 0 \leq r \leq 1 \qquad 2.6$$
$$= 0 \qquad r > 1$$

$$w(r,0) = 2\pi \quad 0 \leq r \leq 1 \qquad 2.7$$
$$= 0 \qquad r > 1$$

The asymptotic behavior of $w(r,0) \approx r^{\circ}$ $as\ r \to 0\ and\ w(r,\pi) \approx r^{-\frac{1}{2}}\ as\ r \to \infty$

By use of the mellin integral transform defined by

$$\overline{w}(s,\theta) = \int_0^\infty r^{s-1}\, w(r,\theta)dr \qquad 0 < Res < \frac{1}{2}$$

Equation (2.5)-(2.7) are transformed into

$$\frac{d^2\overline{w}}{d\theta^2} + s^2\overline{w} = 0 \qquad 0 < Res < \frac{1}{2} \qquad 2.8$$

$$\overline{w}(s,\pi) = {}^{T}\!/_{s} \qquad 2.9$$

$$\overline{w}(s,0) = {}^{2T}\!/_{s} \qquad 2.10$$

The solution of the ordinary differential equation (2.8) is considered as

$$\overline{w}(s,\theta) = Asins\theta + Bcoss\theta \qquad 2.11$$

The constants A and B are determined from the boundary conditions, now from (2.11) and (2.10) we get

$$\overline{w}(s,0) = B = {}^{2T}\!/_{s}$$

And

$$\overline{w}(s,\pi) = Asin\,\pi s + Bcos\,\pi s = {}^{T}\!/_{s}$$

That is

$$Asin\,\pi s = {}^{T}\!/_{s} - {}^{2T}\!/_{s}\ cos\,\pi s$$
$$A = \frac{T}{s}\left(\frac{1 - 2cos\,\pi s}{Sin\pi s}\right)$$

Hence (2.11) becomes

$$\overline{w}(s,\theta) = T\left\{\frac{1 - 2cos\pi s}{sin\pi s}\right\}\frac{sins\theta}{s} + \frac{2\pi}{s}cosS\theta \qquad (2.12)$$

2.4 solution of the original problem

We next apply the inverse formula to get the potential sought for in the form

$$\overline{w}(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{w}(s,\theta)\, r^{-s} ds, \;\; 0 < c < \frac{1}{2} \qquad (2.13)$$

Where $\overline{w}(s,0)$ is given in (2.12)
The contour integral in (2.13) can be evaluated by method of Cauchy integral (or residue) theorem [3].
The poles of $\overline{w}(s,\theta)$are all simple, these simple poles are located at $s = \pm n,\;\; n = 1,2,3, \ldots$ and at $s = 0$
At the poles $s = n,\;\; n = 1,2,3,4, \ldots.$, the residues are
Residue

$$\begin{aligned}
\overline{w}(r,\theta)r^{-s} &= \lim_{s\to n}(s-n)\;\overline{w}(s,\theta)r^{-s} \\
&= \lim_{s\to n}\left\{T\left(\frac{s-n}{sin\pi s}\right)\left(\frac{1-2cos\pi s}{s}\right)sins\theta + \frac{2T(s-n)}{s}\cos s\theta\right\}r^{-s} \\
&= \frac{T}{\pi}\lim_{s-n}\left\{\frac{1}{cos\pi s}\left(\frac{1-2cos\pi s}{s}\right)sins\theta + 0\right\}r^{-s} \\
&= \frac{T}{\pi}\left\{\frac{1}{cosn\pi}\left(\frac{1-2cosn\pi}{s}\right)sinn\theta + 0\right\}r^{-n} \\
&= \frac{T}{\pi}(-1)^n(1-2(-1)^n)\;\frac{sinn\theta}{n}r^{-n} \\
&= \frac{T}{\pi}((-1)^n-2)\frac{sinn\theta}{n}r^{-n}
\end{aligned}$$

Hence for $n = 1,2,3,4, \ldots.(s = n)$

$$\overline{w}(r,\theta) = \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-2}{n}\right]r^{-n}sinn\theta$$

At the poles $s = -n,\;\; n = 1,2,3, \ldots,$ the residues are

$$\begin{aligned}
\lim_{s\to n}(s+n)\overline{w}(s,\theta)r^{-s} &= T\lim_{s\to n}\frac{(s+n)}{sin\pi s}\left(\frac{1-2cos\pi s}{s}\right)sins\theta r^{-s} \\
&= \frac{T}{\pi}\lim_{s\to n}\left\{\frac{1}{cos\pi s}\left(\frac{1-2cos\pi s}{s}\right)sins\theta r^{-s}\right\} \\
&= \frac{T}{\pi}(-1)^n\left(\frac{1-2(-1)^n}{-n}\right)\sin(-n)\theta r^n \\
&= \frac{T}{\pi}(-1)^n\left(\frac{1-2(-1)^n}{n}\right)r^n sinn\theta \\
&= \frac{T}{\pi}\left(\frac{(-1)^n-2}{n}\right)r^n sinn\theta
\end{aligned}$$

Hence for $n = 1,2,3 \ldots..(s = -n)$

$$\overline{w}_2(r,\theta) = \frac{T}{\pi}\sum_{n-1}^{\infty}\left[\frac{(-1)^n}{n} - 2\right]r^n sinn\theta$$

Residue at $s = 0$ is derived in the following way

$$\begin{aligned}
\lim_{s\to 0}(s-0)\overline{w}(s,0)r^{-s} &= lim\left\{T\frac{s}{sin\pi s}(1-2cos\pi s)\frac{sins\theta}{s} + \frac{2Ts}{s}coss\theta\right\} \\
&= lim\left\{\frac{T}{\pi}\frac{\pi s}{sin\pi s}(1-2cos\pi s)\theta\frac{sins\theta}{\theta s} + 2Tcos\theta\right\} \\
&= \frac{T}{\pi}\cdot 1\cdot\theta\cdot 1 + 2T \\
&= 2T - \frac{\theta T}{\pi}
\end{aligned}$$

Hence the solution-sought for is

$$\begin{aligned}
w(r,\theta) &= 2T - \theta\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-2}{n}\right]r^n sinn\theta, \quad r < 1 \qquad 2.14 \\
&= \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n-2}{n}\right]r^{-n}sinn\theta, \quad r > 1
\end{aligned}$$

Thus solution is closed.

2.5 VERIFICATION OF RESULT

In the section, we as certain that (2.14) is indeed the solution of (2.5)-(2.7)

2.5(a) SATISFACTION OF THE LAPLACE EQAUTION

Since

$$w(r,\theta) = 2T - \theta\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^n sinn\theta, \quad r < 1$$

$$0 \le \theta \le \pi$$

$$= \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^{-n} sinn\theta, \qquad r > 1$$

It follows that:

1. For $r < 1$

$$\frac{\partial W}{\partial r} = \frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{n-1} sinn\theta$$

$$\frac{\partial^2 W}{\partial r^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n-1)[(-1)^n - 2]\, r^{n-2} sinn\theta,$$

$$\frac{\partial W}{\partial \theta} = -\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^n cosn\theta,$$

$$\frac{\partial^2 W}{\partial \theta^2} = -\frac{T}{\pi}\sum_{n=1}^{\infty} n[(-1)^n - 2]\, r^n sinn\theta,$$

Therefore,

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n-1)[(-1)^n - 2]\, r^{n-2} sinn\theta$$

$$+\frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{n-2} sinn\theta$$

$$-\frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{n-1} sinn\theta = \theta$$

ii. for $r > 1$

$$\frac{\partial W}{\partial r} = -\frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{-n-1} sinn\theta$$

$$\frac{\partial^2 W}{\partial r^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n-1)[(-1)^n - 2]\, r^{-n-2} sinn\theta,$$

$$\frac{\partial W}{\partial \theta} = \frac{T}{\pi}\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{-n} cosn\theta,$$

$$\frac{\partial^2 W}{\partial \theta^2} = -\frac{T}{\pi}\sum_{n=1}^{\infty} n[(-1)^n - 2]\, r^{-n} sinn\theta,$$

Therefore

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = \frac{T}{\pi}\sum_{n=1}^{\infty}(n+1)[(-1)^n - 2]\, r^{-n-2} sinn\theta$$

$$-\sum_{n=1}^{\infty}[(-1)^n - 2]\, r^{-n-2} sinn\theta$$

$$-\sum_{n=1}^{\infty} n[(-1)^n - 2]\, r^{n-1} sinn\theta = \theta$$

Thus the Laplace equation (3.1.1) is satisfied for $r \ge 0$

2.5(b) SATISFACTION OF BOUNDARY CONDITIONS

i. For $r < 1$

$$W(\mathrm{r},\pi) = 2T - \pi\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^n sinn\pi$$

$$= 2T - T + 0$$

$$= T$$

For $r > 1$

$$W(\mathrm{r}, \pi) = \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^{-n} sinn\pi$$

ii. For $r < 1$

$$W(\mathrm{r}, 0) = 2T - 0\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^{-n} sin0n$$
$$=2T$$

For $r > 1$

$$W(\mathrm{r}, 0) = \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] r^{-n} sin0n$$
$$= 0$$

Therefore the boundary conditions are satisfied for both r <1 and r >1

2.6 CONCLUSION

From

$$w(r, \theta) = w\left(e^{\frac{\pi}{a}x}, e^{\frac{\pi}{a}y}\right) = W(u, v)$$

Together with the harmonicity condition

$$\emptyset(x, y) = W(u, v)$$

We get the original potential with

$$r = e^{\frac{\pi}{a}x} \; and \; \theta = \frac{\pi}{a}y,$$

As

$$\emptyset(x, y) = 2T - \frac{\pi}{a}y\frac{T}{\pi} + \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] e^{n\frac{\pi}{a}x} sinn\frac{\pi}{a}y, \qquad e^{\frac{\pi}{a}x} < 1$$

$$0 \le \frac{\pi}{a}y \le \pi$$

$$= \frac{T}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 2}{n}\right] e^{-n\frac{\pi}{a}x} sinn\frac{\pi}{a}y, \qquad e^{\frac{\pi}{a}x} > 1$$

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