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A New Theorem on Product Summability of Infinite Series

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Abstract: *A given theorem is a some advanced proof in product summability of infinite series.Many other results some known and unknown are derived .*
Key Words And Phrases*: summability, absolute summability, product summability.*

I. Introduction :

Let $\sum a_n$ be a given infinite series with partial sum S_n . Let u_n^α denote the n^{th} Cesàro mean of order $\alpha > -1$ of the sequence $\{s_n\}$.

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k < \infty \tag{1.1}$$

Let $\{p_n\}$ be a sequence of positive real constants, such that

$$P_n = p_0 + p_1 + ...p_n \quad \text{as} , \ n \to \infty \qquad (P_{-1} = p_{-1} = 0)$$

The (N,p) transform of ϕ_n of $\{s_n\}$ generated by $\{p_n\}$ is defined by –

$$\phi_n = \frac{1}{P_n} \sum_{v=0}^{m} p_{n-v} s_v \tag{1.2}$$

The sequence-to-sequence transformation-

$$\phi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence $\{\phi_n\}$ of $(\bar{N},p_n)$ transform of $\{s_n\}$ generated by $\{p_n\}$,

The series $\sum a_n$ is said to be summable $|R,p_n|_k$, $k \geq 1$, if –

$$\sum_{n=1}^{\infty} n^{k-1} \left| \phi_n - \phi_{n-1} \right|^k < \infty \tag{1.4}$$

In special case when $p_n = 1$, $\forall$ n $|R,p_n|_k$ summability reduces to $|C,1|_k$ summability.

The series $\sum a_n$ is said to be summable $|(\bar{N},p)(N,q)|$ when the (N,p) transform of the (N,q) transform of $\{s_n\}$ is a sequence of bounded variation.

Let $\{T_n\}$ defines the sequence of $(\bar{N},q_n)$ transform of the $(\bar{N},p_n)$ transform of $\{s_n\}$ generated by the sequence $\{q_n\}$ and $\{p_n\}$ respectively .

The series $\sum a_n$ is said to be summable $|(R,q_n)(R,p_n)|_k$, $k \geq 1$, if

$$\mathcal{A}_k = \left\{ (T_n); \sum_{n=1}^{\infty} n^{k-1} |a_n|^k < \infty;\ a_n = T_n - T_{n-1} \right\} \quad (1.5)$$

and

$$\mathcal{A}_k^{\delta} = \left\{ (T_n); \sum_{n=1}^{\infty} n^{\delta k+k-1} |a_n|^k < \infty; a_n = T_n - T_{n-1} \right\} \quad (1.6)$$

We may assume ,

$$Q_n = q_0 + q_1 + q_2 + ...q_n \quad ; \quad n \to \infty$$
$$R_n = r_0 + r_1 + r_2 + ...r_n \quad ; \quad n \to \infty$$

In 2008, SULAIMAN [3] proved the theorem.The main objective of this paper is to generalize the theorem of SULAIMAN[3]. However our theorem is as follows.

II. Theorem 1

Let $k \geq 1$ and $\delta \geq 0$, $\{\lambda_n\}$ be a sequence of constants,

$$f_v = \sum_{r=v}^{n} \frac{q_r}{P_r} \quad , \qquad F_v = \sum_{r=v}^{n} p_r f_r \quad (2.1)$$

Let $$p_n Q_n = O(P_n) \quad (2.2)$$

$$\sum_{n=v+1}^{\infty} \frac{n^{\delta k+k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left\{ \frac{v^{\delta k+k-1} q_v^{k-1}}{Q_v^k} \right\} \quad (2.3)$$

Then, sufficient conditions for implications-

$\sum a_n$ is summable $|R, r_n, \delta|_k \Rightarrow \sum a_n \lambda_n$ is summable

$$\left|(R, q_n)(R, p_n), \delta\right|_k \quad (2.4)$$

are,

$$|\lambda_v| F_v = O(Q_v) \quad (2.5)$$
$$|\lambda_n| = O(Q_n) \quad (2.6)$$
$$p_v R_v |\lambda_v| = O(Q_v) \quad (2.7)$$
$$p_v q_v R_v |\lambda_v| = O(Q_v Q_{v-1} r_v) \quad (2.8)$$
$$p_v q_v R_n |\lambda_n| = O(P_n Q_n r_n) \quad (2.9)$$
$$R_{v-1} |\Delta\lambda_v| F_{v+1} = O(Q_v r_v) \quad (2.10)$$
$$R_{v-1} |\Delta\lambda_v| = O(Q_v r_v) \quad (2.11)$$

THEOREM 2

Let (2.3) be satisfied and

$$P_v = O(p_v Q_v) \quad (2.12)$$
$$Q_n = O(n q_n) \quad (2.13)$$

then, necessary conditions for implication (2.4) to be satisfied are ,

$$|\lambda_n| = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right),$$

$$|\Delta\lambda_v| = O\left(\frac{v^{(1+\delta)-1/k} r_v Q_v}{(1+F_{v+1}) R_v}\right) \tag{2.14}$$

III. Proof Of The Theorem 1:

Let $\{S_n\}$ be the sequence of partial sums of $\sum a_n\lambda_n$. Let v_n and V_n be the $(\overline{N}, r_n)$, $(\overline{N}, q_n)$, $(\overline{N}, p_n)$ transforms of the sequences $\{s_n\}$, $\{S_n\}$ respectively,

We write,

$$t_n = v_n - v_{n-1}, \quad T_n = V_n - V_{n-1}$$

Therefore ,

$$t_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^{n} R_{v-1} a_v \tag{3.1}$$

$$V_n = \frac{1}{Q_n}\sum_{r=0}^{n} q_r \frac{1}{P_r}\sum_{v=0}^{r} p_v S_v$$

$$= \frac{1}{Q_n}\sum_{v=0}^{n} p_v S_v \sum_{r=v}^{n} \frac{q_r}{P_r} \tag{3.2}$$

$$= \frac{1}{Q_n}\sum_{v=0}^{n} p_v S_v f_v$$

Also,

$$T_n = V_n - V_{n-1}$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{r=0}^{n} p_r S_r f_r + \frac{p_n S_n f_n}{Q_{n-1}}$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{r=0}^{v} p_r f_r \sum_{v=0}^{r} a_v\lambda_v + \frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=0}^{n} a_v\lambda_v$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{v=0}^{n} a_v\lambda_v \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=0}^{n} a_v\lambda_v$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}}\sum_{r=v}^{n} p_r f_r +$$

$$+ \frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=1}^{n} R_{v-1} a_v \frac{\lambda_v}{R_{v-1}}$$

$$= \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v} R_{r-1} a_r\right)\Delta_v\left(\frac{\lambda_v}{R_{v-1}}\sum_{r=v}^{n} p_r f_r\right) + \left(\sum_{v=1}^{n} R_{v-1} a_v\right)\frac{p_n f_n \lambda_n}{R_{n-1}} +$$

$$+\frac{p_nq_n}{P_nQ_{n-1}}\left(\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}R_{r-1}a_r\right)\Delta\left(\frac{\lambda_v}{R_{v-1}}\right)\right)+\sum_{v=1}^{n}R_{v-1}a_v\left(\frac{\lambda_n}{R_{n-1}}\right)$$

$$=\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(t_v\lambda_vf_v+\frac{R_{v-1}}{r_v}p_vt_v\lambda_vf_v+\frac{R_{v-1}}{r_v}t_v\Delta\lambda_vF_{v+1}\right)+$$

$$+\frac{p_nq_nR_n}{Q_nQ_{n-1}r_n}t_nf_n\lambda_n+$$

$$+\frac{p_nq_n}{P_nQ_{n-1}}\left(\sum_{v=1}^{n-1}\left(t_v\lambda_v+\frac{R_{v-1}}{r_v}t_v\Delta\lambda_v\right)\right)+\frac{p_nq_nR_n}{P_nQ_{n-1}r_n}t_n\lambda_n$$

$$=\sum_{j=1}^{7}T_{nj} \qquad (3.3)$$

In order , to complete the proof, it is sufficient to show that-

$$\sum_{n=1}^{\infty}n^{\delta k+k-1}\left|T_{n,j}\right|^{k}<\infty, \qquad j=1,2,3,4,5,6,7 \qquad (3.4)$$

Now appling Hölder's inequality,

$$\sum_{n=2}^{m+1}n^{\delta k+k-1}\left|T_{n,1}\right|^{k}=\sum_{n=2}^{m+1}n^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(t_v\lambda_vf_v\right)\right|^{k}$$

$$\le\sum_{n=2}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\frac{1}{q_v^{k-1}}|t_v|^k|\lambda_v|^kF_v^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1}$$

$$=\square(1)\sum_{v=1}^{m}\frac{1}{q_v^{k-1}}|t_v|^k|\lambda_v|^kF_v^k\sum_{n=v+1}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}$$

$$=\square(1)\sum_{v=1}^{m}v^{\delta k+k-1}|t_v|^k|\lambda_v|^k\frac{F_v^k}{Q_v^k} \qquad \{\text{using }(2.5)\}$$

$$=\square(1)\sum_{n=2}^{m+1}n^{\delta k+k-1}\left|T_{n,2}\right|^{k}=\sum_{n=2}^{m+1}n^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{R_{v-1}p_v}{r_v}t_v\lambda_vf_v\right)\right|^{k}$$

$$\le\sum_{n=2}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{R_v^kp_v^k}{q_v^{k-1}}|t_v|^k|\lambda_v|^kf_v^k\right)\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1}$$

$$=\square(1)\sum_{v=1}^{m}\left(\frac{R_v^kp_v^k}{q_v^{k-1}}|t_v|^k|\lambda_v|^kf_v^k\right)\sum_{n=v+1}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}$$

$$=\square(1)\sum_{v=1}^{m}\left(v^{\delta k+k-1}\frac{R_v^kp_v^k}{Q_v^k}|t_v|^k|\lambda_v|^kf_v^k\right)$$

$$=\square(1)\sum_{v=1}^{m}\left(v^{\delta k+k-1}|t_v|^k\right) \qquad \{\text{using }(2.7)\}$$

$$=\square(1)$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_{n,3}\right|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{R_{v-1}}{r_v}t_v\Delta\lambda_v F_{v+1}\right)\right|^k$$

$$\le \sum_{n=1}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}\sum_{v=1}^{n-1}\left(\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k F_{v+1}^k\right)\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right\}^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k F_{v+1}^k\right)\sum_{n=v+1}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(v^{\delta k+k-1}\frac{R_{v-1}^k}{Q_v^{k-1}r_v^k}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k F_{v+1}^k\right)$$

{using (2.10)}

$$= O(1)$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_{n,4}\right|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{p_nq_nR_n}{Q_nQ_{n-1}r_n}t_n\lambda_n f_n\right|^k$$

$$= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}\frac{p_n^kq_n^kR_n^k}{Q_n^kQ_{n-1}^kr_n^k}\left|\lambda_n\right|^k\left|t_n\right|^k \quad \text{\{using (2.8)\}}$$

$$= O(1)$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_{n,5}\right|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{p_nq_n}{P_nQ_{n-1}}\sum_{v=1}^{n-1}t_v\lambda_v\right|^k$$

$$\le \sum_{n=1}^{m+1} n^{\delta k+k-1}\frac{p_n^kq_n^k}{P_n^kQ_{n-1}}\sum_{v=1}^{n-1}\left|t_v\right|^k\left|\lambda_v\right|^k\frac{1}{q_v^{k-1}}\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{v-1}}\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left|t_v\right|^k\left|\lambda_v\right|^k\frac{1}{q_v^{k-1}}\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\frac{p_n^kq_n^k}{P_n^kQ_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left|t_v\right|^k\left|\lambda_v\right|^k\frac{1}{q_v^{k-1}}\sum_{n=v+1}^{m+1}\frac{n^{\delta k+k-1}q_n^k}{Q_n^kQ_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}\left|t_v\right|^k\left|\lambda_v\right|^k\frac{1}{Q_v^k} \quad \text{\{using (2.3)\}}$$

$$= O(1) \quad \text{\{using (2.6)\}}$$

$$\sum_{n=2}^{m+1} n^{\delta k+k-1}\left|T_{n,6}\right|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\frac{p_nq_n}{P_nQ_{n-1}}\sum_{v=1}^{n-1}\frac{R_{v-1}}{r_v}t_v\Delta\lambda_v\right|^k$$

$$\le \sum_{n=1}^{m+1} n^{\delta k+k-1}\frac{p_n^kq_n^k}{P_n^kQ_{n-1}}\sum_{v=1}^{n-1}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\left\{\sum_{v=1}^{n-1}\frac{q_v}{Q_{v-1}}\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left|t_v\right|^k\left|\Delta\lambda_v\right|^k\frac{R_{v-1}^k}{q_v^{k-1}r_v^k}\sum_{n=v+1}^{m+1} n^{\delta k+k-1}\frac{p_n^kq_n^k}{P_n^kQ_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{\delta k+k-1}|t_v|^k |\Delta\lambda_v|^k \frac{R_{v-1}^k}{Q_v^{k-1} r_v^k}$$

$$= O(1) \qquad \{\text{Using } (2.11)\} \qquad (3.5)$$

Finally,

$$\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n,7}|^k = \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{p_n q_n R_n}{P_n Q_{n-1} r_n} t_n \lambda_n\right|^k$$

$$= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|t_n|^k |\lambda_n|^k \left(\frac{p_n q_n R_n}{P_n Q_n r_n}\right)^k \quad \{\text{using } 2.9\}$$

$$(3.6)$$

$$= O(1)$$

This completes the proof of the THEOREM 1.

PROOF OF THE THEOREM 2 :

For $k \geq 1$, $\delta \geq 0$, define-

$$A^* = \{(a_j);\ \sum a_j \text{ is summable } |R, r_n, \delta|_k\}$$

$$B^* = \{(b_j);\ \sum b_j\lambda_j \text{ is summable } |(R, q_n)(R, p_n), \delta|_k\} \qquad (3.7)$$

from (3.3), we have-

$$T_n = \sum_{v=1}^{n}\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right) a_v \lambda_v \qquad (3.8)$$

with t_n and T_n as defined by (2.12) and (3.8), the space A^* and B^* BK-spaces with norms defined by-

$$\|c\|_1 = \left\{|t_0|^k + \sum_{n=1}^{\infty} n^{\delta k+k-1}|t_n|^k\right\}^{1/k}$$

$$\|c\|_2 = \left\{|T_0|^k + \sum_{n=1}^{\infty} n^{\delta k+k-1}|T_n|^k\right\}^{1/k} \qquad (3.9)$$

respectively, by the hypothesis of the theorem,

$$\|c\|_1 < \infty \Rightarrow \|c\|_2 < \infty \qquad (3.10)$$

The inclusion map i; $A^* \to B^*$ defined by i(a)=a is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exists a constant, K>0, such that-

$$\|c\|_2 \leq K\|c\|_1 \qquad (3.11)$$

Let e_n denote the n^{th} coordinate vector, from (3.4) and (3.8) with $\{a_n\}$ defined by $a_n = e_n - e_{n+1}$, n=v, a_n =0, otherwise , we have

$$t_n = \begin{cases} 0, & n<v \\ \dfrac{r_v}{R_v}, & n=v \\ -\dfrac{r_n r_v}{R_n R_{n-1}}, & n>v \end{cases}$$

$$T_n = \begin{cases} 0, n < v \\ \left(\dfrac{q_v F_v}{Q_v Q_{v-1}} + \dfrac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v, n = v \\ \Delta_v\left(\dfrac{q_n F_v}{Q_n Q_{n-1}} + \dfrac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v, n > v \end{cases} \tag{3.12}$$

From (3.9), we have – $\|c\|_1 = \left\{ v^{\delta k+k-1}\left(\dfrac{q_v}{Q_v}\right)^k + \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\dfrac{q_n q_v}{Q_n Q_{n-1}}\right)^k \right\}^{1/k}$

$$\|c\|_2 = \left\{ v^{\delta k+k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \right.$$

$$\left. + \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left|\Delta_v\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right|^k \right\}^{1/k} \tag{3.13}$$

Applying (3.11), we obtain-

$$\left\{ v^{\delta k+k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \right.$$

$$\left. + \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left|\Delta_v\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right|^k \right\}$$

$$= O(1)\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k + \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{r_n r_v}{R_n R_{n-1}}\right)^k \right\} \tag{3.14}$$

As the right hand side of (3.13), by (2.3) is

$$= O(1)\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k + \frac{r_v^k}{R_v^{k-1}}\sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{r_n^k}{R_n^k R_{n-1}}\right) \right\}$$

$$= O(1)\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k + \left(\frac{r_v}{R_v}\right)^{k-1} n^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k \right\} \qquad (3.15)$$

$$= O(1)\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k \right\}$$

And the fact that each term of the left hand side of (3.13) is

$$O(1)\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k \right\}, \text{ we obtain-}$$

$$\left\{ v^{\delta k+k-1}\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)^k |\lambda_v|^k \right\} = O\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k \right\} \qquad (3.16)$$

Which implies by (3.3)-

$$\left(\frac{q_v}{Q_v Q_{v-1}}\right)^k (1+F_v)^k |\lambda_v|^k = O\left(\frac{r_v}{R_v}\right)^k \qquad (3.17)$$

that is –

$$|\lambda_v| = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right) \qquad (3.18)$$

Also, we have by (3.13)

$$\sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left|\left(\frac{q_n p_v f_v}{Q_n Q_{n-1}}\right)\lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\Delta\lambda_v\right|^k$$
$$= O\left\{ v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k \right\} \qquad (3.19)$$

The above, via the linear independence of λ_v and $\Delta\lambda_v$ implies-

$$\sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)^k |\Delta\lambda_v|^k = O\left\{ v^{\delta k+k-1}\left(\frac{q_v}{Q_v}\right)^k \right\} \qquad (3.20)$$

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left\{ v^{\delta k+k-1}\left(\frac{q_v}{Q_v}\right)^k \right\}$$

{using (3.3)}

As by (3.4), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty} \Delta\left(\frac{1}{Q_{n-1}^k}\right) = O(1) \sum_{n=v+1}^{\infty} \frac{\left|\Delta Q_{n-1}^k\right|}{Q_{n-1}^k Q_n^k} = O(1) \sum_{n=v+1}^{\infty} \frac{Q_{n-1}^{k-1} q_n}{Q_{n-1}^k Q_n^k}$$

$$= O(1) \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{q_n}{Q_{n-1}Q_n}\right)^k \tag{3.21}$$

Then,

$$\left|\Delta\lambda_v\right|^k (1+F_{v+1})^k \left(\frac{1}{Q_v^k}\right) = O\left\{v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k\right\} \tag{3.22}$$

Which implies

$$\lambda_v = O\left\{\frac{v^{(\delta+1)-1/k} r_v Q_v}{(1+F_{v+1})R_v}\right\} \tag{3.23}$$

Also by (3.14)

$$\sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left|\left(\frac{q_n p_v f_v}{Q_n Q_{n-1}}\lambda_v\right)\right|^k = O\left\{v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k\right\}$$

$$p_v^k f_v^k \left|\lambda_v\right|^k \sum_{n=v+1}^{\infty} n^{\delta k+k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left\{v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k\right\} \tag{3.24}$$

$$p_v^k f_v^k \left|\lambda_v\right|^k \left(\frac{1}{Q_v^k}\right) = O\left\{v^{\delta k+k-1}\left(\frac{r_v}{R_v}\right)^k\right\}$$

Which implies

$$\lambda_v = O\left\{v^{(\delta+1)-1/k}\left(\frac{r_v Q_v}{p_v f_v R_v}\right)\right\} \tag{3.25}$$

This completes the proof of the THEOREM 2.

IV. Corollaries

Cor. 1

Let $k \geq 1$, define-

$$f_v = \sum_{r=v}^{n} \frac{q_r}{r}, \qquad F_v = \sum_{r=v}^{n} f_r \tag{4.1}$$

Let $n = O(Q_n)$ (4.2)

Then sufficient conditions for the implication,

$\sum a_n$ is summable $\left|C,1,\delta\right|_k \Rightarrow \sum a_n\lambda_n$ is summable $\left|(R,q_n)(C,1),\delta\right|_k$ (4.3)

are (2.5), (2.6) and the following-

$$v\left|\lambda_v\right| = O(Q_V) \tag{4.4}$$

$$vq_v\left|\lambda_v\right| = O(Q_v Q_{v-1}) \tag{4.5}$$

$$nq_n\left|\lambda_n\right| = O(nQ_n) \tag{4.6}$$

$$v\left|\Delta\lambda_v\right|F_{v+1} = O\left(Q_v\right) \quad (4.7)$$

$$\left|\Delta\lambda_v\right| = O\left(q_v\right) \quad (4.8)$$

$$v\left|\Delta\lambda_v\right| = O\left(Q_v\right) \quad (4.9)$$

Proof: The proof follows from Theorem 1, by putting

$p_n = r_n = 1$ for all n.

Cor. 2 If $\delta = 0$, then above corollary reduces to Cor. (2.1) of SULAIMAN[3].

Cor. 3 Let $k \geq 1$, define-

$$f_v = \sum_{r=v}^{n} \frac{1}{P_r}, \qquad F_v = \sum_{r=v}^{n} p_r f_r \quad (4.10)$$

Let (2.2) be satisfied, then sufficient conditions for the implication,

$$\sum a_n \text{ is summable } \left|C,1,\delta\right|_k \Rightarrow \sum a_n\lambda_n \text{ is summable } \left|\left(C,1\right)\left(R,p_n\right),\delta\right|_k \quad (4.11)$$

are,

$$\left|\lambda_v\right|F_v = O\left(v\right) \quad (4.12)$$

$$\left|\lambda_v\right| = O\left(n\right) \quad (4.13)$$

$$p_v\lambda_v = O\left(1\right) \quad (4.14)$$

$$\left|\Delta\lambda_v\right|F_{v+1} = O\left(1\right) \quad (4.15)$$

$$\left|\Delta\lambda_v\right| = O\left(1\right) \quad (4.16)$$

Proof:The proof follows from Theorem 1 by putting $q_n = r_n = 1$, for all n, noticing that (2.3) is satisfied as-

$$\sum_{n=v+1}^{\infty} \frac{1}{n\left(n-1\right)} = \sum_{n=v+1}^{\infty}\left[\frac{1}{\left(n-1\right)} - \frac{1}{n}\right] = \frac{1}{v} \quad (4.17)$$

Cor. 4

If $\delta = 0$, then above corollary reduces to Cor. (2.2) of SULAIMAN[3].

Cor. 5

Let f_v, F_v be as defined in (4.1), let (2.3) and (4.2) be satisfied, then sufficient conditions for implications

$$\sum a_n \text{ is summable } \left|R,r_n,\delta\right|_k \Rightarrow \sum a_n\lambda_n \text{ is summable } \left|\left(R,q_n\right)\left(C,1\right),\delta\right|_k \quad (4.18)$$

are,
(2.5), (2.6), (2.10), (2.11) and the followings-

$$R_v\left|\lambda_v\right| = O\left(Q_v\right) \quad (4.19)$$

$$q_vR_v\left|\lambda_v\right| = O\left(Q_vQ_{v-1}r_v\right) \quad (4.20)$$

$$q_nR_n\left|\lambda_n\right| = O\left(nQ_nr_n\right) \quad (4.21)$$

Proof: The proof follows from Theorem 1

by putting $p_n = 1 \ \forall\ n$.

Cor. 6 If we put $\delta = 0$, then above corollary reduces to Cor. (2.3) of SULAIMAN[3].

Cor. 7 Let f_v and F_v be defined in (4.1), let (4.3), (2.12) be satisfied and

$$v = O\left(Q_v\right) \quad (4.22)$$

then, necessary conditions for the implication (4.3) are,

$$\lambda_v = O\left(\frac{Q_vQ_{v-1}}{\left(1+F_v\right)vq_v}\right), \qquad \lambda_v = O\left(\frac{Q_v}{v^{(1/k+\delta)}f_v}\right)$$

$$\Delta\lambda_v = \square\left(\frac{Q_v}{v^{(1/k+\delta)}\left(1+F_{v+1}\right)}\right) \tag{4.23}$$

Proof: The proof follows from Theorem 2, by putting $p_n = r_n = 1$ for all n.

Cor. 8

By putting $\delta = 0$, corollary.7 reduces to Cor. 2.4 of SULAIMAN[3] .

Cor. 9

Let f_v and F_v be as defined in (4.5) ,

let $$P_v = \square\left(vp_v\right) \tag{4.24}$$

then, necessary conditions for the implication (4.10) to be satisfied are,

$$\lambda_v = \square\left(\frac{v}{\left(1+F_v\right)}\right), \qquad \lambda_v = \square\left(\frac{v^{(1+\delta-1/k)}}{f_v p_v}\right), \qquad \Delta\lambda_v = \square\left(\frac{v^{(1+\delta-1/k)}}{1+F_{v+1}}\right) \tag{4.25}$$

Proof: The proof follows from Theorem 2 by putting $q_n = r_n = 1$, for all n, noticing that (2.3) is satisfied as in the case of (4.17).

Cor. 10

By putting $\delta = 0$, corollary.9 reduces to Cor. 2.5 of SULAIMAN[3].

Cor. 11

Let f_v and F_v be as defined in (4.1) , Let (2.3), (2.13) and (4.2) be all satisfied, then, necessary conditions for the implication (4.18) to be satisfied are,

$$\lambda_v = \square\left(\frac{Q_v Q_{v-1} r_v}{\left(1+F_v\right)R_v q_v}\right), \qquad \lambda_v = \square\left(\frac{v^{(1+\delta)-1/k} r_v Q_v}{f_v R_v}\right), \qquad \Delta\lambda_v = \square\left(\frac{v^{(1+\delta)-1/k} r_v Q_v}{\left(1+F_{v+1}\right)R_v}\right) \tag{4.26}$$

Proof: The proof follows from Theorem 2, by putting $p_n = 1 \ \forall\ n$.

Cor. 12

When $\delta = 0$, corollary. 11 reduces to Cor. 2.6 of SULAIMAN[3].

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