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Some Properties of semi-symmetric non-metric connection in LP-Sasakian manifold

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Abstract: *S.K. Chaubey and R.H. Ojha [4] introduced a semi-symmetric non-metric connection in almost contact manifold and also studied the connection in Sasakian manifold. The present paper deals with some propertied of semi-symmetric non-metric connection in LP-Sasakian manifold.*

Key words: *Lorentzian almost paracontact manifold, LP-Sasakian manifold and semi-symmetric non metric connection.*

I. Introduction

An n-dimensional differentiable manifold M, is called a Lorentzian almost paracontact manifold (briefly LAP- Sasakian manifold) [2],[3] if it admits (1, 1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2(X) = X + \eta(X)\xi \tag{1}$$

$$\eta(\xi) = -1 \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{3}$$

$$g(X, \xi) = \eta(X) \tag{4}$$

$$\nabla_X \xi = \phi(X) \tag{5}$$

$$(\nabla_X \phi)(Y) = g(X, Y) + \eta(Y)X + 2\,\eta(X)\eta(Y) \tag{6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LAP- Sasakian manifold, the following relations holds

$$\text{(a)} \quad \phi\xi = 0, \qquad \text{(b)} \quad \eta(\phi X) = 0 \tag{7}$$

$$\text{rank}\ \phi = n - 1 \tag{8}$$

In an LAP manifold, if we put

$$`\phi(X, Y) = g(X, \phi Y) \tag{9}$$

for any vector fields X and Y, then the tensor field $`\phi(X, Y)$ is a symmetric (0, 2) tensor field [2], that is

$$`\phi(X, Y) = `\phi(Y, X) \tag{10}$$

An LAP- Sasakian manifold satisfying the relation [2]

$$(\nabla_z `\phi)(X, Y) = g(Y, Z)\eta(X) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z) \tag{11}$$

is called a normal Lorentzian paracontact manifold or Lorentzian para-sasakian manifold (briefly LP-Sasakian) manifold.

Also, since the vector field η is closed in an LP- Sasakian manifold, we have [2] , [5].

$$(\nabla_X \eta)(Y) = `\phi(X, Y) \tag{12}$$

for any vector field X and Y.

In an n–dimensional LP- Sasakian manifold, the following relations holds [1], [5]

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{13}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{14}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{15}$$

$$R(\xi, X)\xi = X - \eta(X)\xi \tag{16}$$

$$S(X, \xi) = (n-1)\eta(X) \tag{17}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{18}$$

for any vector fields X, Y and Z , where R is the Riemannian curvature tensor and S is the Ricci tensor.

II. Semi-symmetric non-metric connection

A linear connection B on (M_n, g) defined as

$$B_XY = \nabla_XY - \eta(Y)X - g(X, Y)T \quad (19)$$

for arbitrary vector fields X and Y, is said to be a semi-symmetric non-metric connection [4].
Now, if we put (19) as

$$B_XY = \nabla_XY + H(X, Y) \quad (20)$$

where $H(X, Y) = -\eta(Y)X - g(X, Y)T$ (22)

Let us define

(a) $S(X, Y, Z) = g(S(X,Y), Z)$ (b) $H(X, Y, Z) = g(H(X,Y), Z)$ (23)

then we can write

(a) $S(X, Y, Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z)$ (24)

(b) $H(X, Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y)$ (25)

Theorem 1. Let B be a semi-symmetric non-metric connection in a LP–Sasakian manifold with a Riemannian connection ∇, then we have

(a) $(B_X`\Box)(Y, \bar{Z}) = \eta(Y)[(\nabla_X\eta)(Z) + (\nabla_X\eta)(\bar{Z})]$ (26)

(b) $(B_X`\Box)(\bar{Y}, Z) = \eta(Z)[(\nabla_X\eta)(Y) + (\nabla_X\eta)(\bar{Y})]$ (27)

Proof. We have,

$$X(`\Box(Y, Z)) = (\nabla_X`\Box)(Y, Z) + `\Box(\nabla_XY, Z) + `\Box(Y, \nabla_XZ)$$

$$= (B_X`\Box)(Y, Z) + `\Box(B_XY, Z) + `\Box(Y, B_XZ)$$

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) + `\Box(\nabla_XY - B_XY, Z) + `\Box(Y, \nabla_XZ - B_XZ)$$

With the help of equation (20), the above equation becomes

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) - `\Box(H(X, Y), Z) - `\Box(Y, H(X, Z))$$

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) - `\Box(H(X, Y), Z) - `\Box(H(X, Z), Y) \quad (28)$$

Using equation (9) in this equation, we obtain

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) - g(H(X, Y), \bar{Z}) - g(H(X, Z), \bar{Y}) \quad (29)$$

From equation (23) (b) and equation (29), we obtain

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) - H(X, Y, \bar{Z}) - H(X, Z, \bar{Y}) \quad (30)$$

Now from equation (25) above becomes

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) + \eta(Y)g(X, \bar{Z}) + \eta(\bar{Z})g(X, Y)$$
$$+ \eta(Z)g(X, \bar{Y}) + \eta(\bar{Y})g(X,Z)$$

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) + \eta(Y)g(X, \bar{Z}) + \eta(Z)g(X, \bar{Y})$$

$$(B_X`\Box)(Y, Z) = (\nabla_X`\Box)(Y, Z) + \eta(Y)(\nabla_X\eta)(Z) + \eta(Z)(\nabla_X\eta)(Y) \quad (31)$$

From equation (9) and (31), we obtain

$$(B_X`\Box)(Y, Z) = g(Z, X)\eta(Y) + g(Y, X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)$$
$$+ \eta(Y)(\nabla_X\eta)(Z) + \eta(Z)(\nabla_X\eta)(Y) \quad (32)$$

Replace Z by $\bar{Z}$ in equation (32) it becomes

$$(B_X`\Box)(Y, \bar{Z}) = g(\bar{Z}, X)\eta(Y) + g(Y, X)\eta(\bar{Z}) + 2\eta(X)\eta(Y)\eta(\bar{Z})$$
$$+ \eta(Y)(\nabla_X\eta)(\bar{Z}) + \eta(\bar{Z})(\nabla_X\eta)(Y)$$

$$(B_X`\Box)(Y, \bar{Z}) = g(\bar{Z}, X)\eta(Y) + \eta(Y)(\nabla_X\eta)(\bar{Z})$$

$$(B_X`\Box)(Y, \bar{Z}) = (\nabla_X\eta)(Z)\eta(Y) + \eta(Y)(\nabla_X\eta)(\bar{Z})$$

$$(B_X`\Box)(Y, \bar{Z}) = \eta(Y)[(\nabla_X\eta)(Z) + (\nabla_X\eta)(\bar{Z})]$$

Similarly replace Y by $\bar{Y}$ in equation (32) we get equation (27).

Theorem 2. Let B be a semi-symmetric non-metric connection in LP–Sasakian manifold with a Riemannian connection ∇, then we have

$$(B_{\bar{X}}`\Box)(Y, Z) - (B_Y`\Box)(Z, \bar{X}) - (B_Z`\Box)(\bar{X}, Y) = 0$$

Proof:- Barring X in equation (31), we get.

$$(B_{\bar{X}}`\Box)(Y, Z) = (\nabla_{\bar{X}}`\Box)(Y, Z) + \eta(Y)(\nabla_{\bar{X}}\eta)(Z) + \eta(Z)(\nabla_{\bar{X}}\eta)(Y) \quad (33)$$

From equation (11), the above equation becomes.

$$(B_{\bar{X}}`\Box)(Y, Z) = g(Z, \bar{X})\eta(Y) + g(Y, \bar{X})\eta(Z) + 2\eta(\bar{X})\eta(Y)\eta(Z)$$

$$+ \eta(Y)(\nabla_{\overline{X}} \eta)(Z) + \eta(Z)(\nabla_{\overline{X}} \eta)(Y)$$

$$= `\square(Z, X)\,\eta(Y) + `\square(Y, X)\eta(Z) + \eta(Y)`\square(\overline{X}, Z)$$

$$+ \eta(Z)`\square(\overline{X}, Y) \qquad (34)$$

Similarly we get another two equations by taking cyclic of above equation.

$$(B_Y`\square)(Z, \overline{X}) = `\square(Y, X)\eta(Z) + \eta(Z)`\square(\overline{X}, Y) \qquad (35)$$

$$(B_Z`\square)(Z, \overline{X}) = `\square(Z, X)\,\eta(Y) + \eta(Y)`\square(\overline{X}, Z) \qquad (36)$$

From equations (34), (35) and (36), we get the result.

Theorem 3. Let B be a semi-symmetric non-metric connection in LP–Sasakian manifold with a Riemannian connection ∇, then we have

$$(B_{\overline{Y}}`\square)(Z, X) - (B_Z`\square)(X, \overline{Y}) - (B_X`\square)(\overline{Y}, Z) = 0$$

Proof:- Barring Y in equation (31), we get.

$$(B_X`\square)(\overline{Y}, Z) = (\nabla_X`\square)(\overline{Y}, Z) + \eta(\overline{Y})(\nabla_X\eta)(Z) + \eta(Z)(\nabla_{\overline{X}}\eta)(Y) \qquad (37)$$

From equation (11), the above equation becomes.

$$(B_X`\square)(\overline{Y}, Z) = g(Z, X)\eta(\overline{Y}) + g(\overline{Y}, X)\eta(Z) + 2\eta(X)\eta(\overline{Y})\eta(Z) + \eta(Z)(\nabla_X\eta)(\overline{Y})$$

$$= g(\overline{Y}, X)\eta(Z) + \eta(Z)(\nabla_X\eta)(\overline{Y})$$

$$= `\square(X, Y)\eta(Z) + \eta(Z)`\square(X, \overline{Y}) \qquad (38)$$

Similarly we get another two equations by taking cyclic of above equation.

$$(B_{\overline{Y}}`\square)(Z, X) = `\square(X, \overline{Y})\eta(Z) + `\square(Z, Y)\,\eta(X) + \eta(Z)`\square(X, \overline{Y}) + \eta(X)`\square(\overline{Y}, Z) \qquad (39)$$

$$(B_Z`\square)(Z, \overline{X}) = \eta(X)`\square(\overline{Y}, Z) + `\square(Z, Y)\,\eta(X) \qquad (40)$$

From equations (38), (39) and (40), we get the required result.

Theorem 4. Let B be a semi-symmetric non-metric connection in LP–Sasakian manifold with a Riemannian connection ∇, then we have

$$(B_{\overline{Y}}`\square)(Z, X) - (B_Z`\square)(X, \overline{Y}) - (B_{\overline{X}}`\square)(\overline{Y}, Z) = 0$$

Proof:- Barring Y in equation (31), we get.

$$(B_X`\square)(Y, \overline{Z}) = (\nabla_X`\square)(Y, \overline{Z}) + \eta(Y)(\nabla_X\eta)(\overline{Z}) + \eta(\overline{Z})(\nabla_X\eta)(Y) \qquad (41)$$

$$(B_X`\square)(Y, \overline{Z}) = (\nabla_X`\square)(Y, \overline{Z}) + \eta(Y)(\nabla_X\eta)(\overline{Z})$$

From equation (11), the above equation becomes.

$$(B_X`\square)(Y, \overline{Z}) = g(\overline{Z}, X)\eta(Y) + g(Y, X)\eta(\overline{Z}) + 2\eta(X)\eta(Y)\eta(\overline{Z}) + \eta(Y)(\nabla_X\eta)(\overline{Z})$$

$$= g(\overline{Z}, X)\eta(Y) + \eta(Y)(\nabla_X\eta)(\overline{Z})$$

$$= `\square(X, Z)\eta(Y) + \eta(Y)`\square(X, \overline{Z}) \qquad (42)$$

Similarly we get another two equations by taking cyclic of above equation.

$$(B_Y`\square)(\overline{Z}, X) = `\square(Y, Z)\eta(X) + \eta(X)`\square(Y, \overline{Z}) \qquad (43)$$

$$(B_Z`\square)(X, Y) = `\square(Y, Z)\,\eta(X) + `\square(X, Z)\,\eta(Y) + \eta(X)`\square(Y, \overline{Z}) + \eta(Y)`\square(X, \overline{Z}) \qquad (44)$$

From equations (42), (43) and (44), we get the required result.

Refrences

[1] I. Mihai, A. A. Shaikh and U. C. De. , On Lorentzian Para Sasakian manifolds, Korean J. Math. Sciences, 6 (1999) , 1-13.

[2] K. Matsumoto, On Lorentzian Para contact manifolds, Bull. of yamagata Univ., Nat. Sci., 12 (1989), 151- 156.

[3] K. Matsumoto and I. Mihai, On a certain transformation in Lorentzian Para contact manifold , Tensor N.S., 47, (1989) , 189-197.

[4] S. K. Chaubey and H. Ojha, On a semi-symmetric non-metric and quarter-symmetric metric connections, Tensor N.S., 70, No. 2 (2008), 202- 213.

[5] U. C. De , K. Matsumoto and A. A. Shaikh , Lorentzian Para Sasakian manifolds , Rendicontidel Seminario Matematico di Messina, Series II , Supplemento No., 3 (1999), 149-158.