IOSR Journal of Mathematics (IOSRJM)
ISSN: 2278-5728 Volume 2, Issue 4 (Sep-Oct 2012), PP 29-32
www.iosrjournals.org

Some Results Related to the Lattice of Fuzzy Topologies on a Fixed Set

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Abstract: *The set of all L - fuzzy topologies on a fixed set X is a complete lattice denoted by LFT(X,L). In this paper, we determine some classes of automorphisms of this lattice when X is a nonempty set and L is an F-lattice.*

***Mathematics Subject Classification**: 54A40*
***Keywords**: fuzzy topological space, lattice automorphism, t- homomorphism, pseudo-complement, F-lattice, order reversing involution.*

I. Introduction

In 1958, Juris Hartmanis [2] determined the automorphisms of the lattice LT(X) of all topologies on a fixed set X as follows : for $p \in S(X)$ and $\tau \in LT(X)$, define the mapping A_p by $A_p(\tau) = \{ p(U) : U \in \tau\}$. Then $A_p(\tau)$ is a topology on X and A_p is an automorphism of LT(X). If X is infinite or X contains atmost two elements, the set of all automorphisms of LT(X) is precisely $\{A_p : p \in S(X)\}$. Otherwise, the set of all automorphisms of LT(X) is $\{A_p : p \in S(X)\} \cup \{B_p : p \in S(X)\}$ where $B_p : LT(X) \to LT(X)$ is defined by $B_p(\tau) = \{ X\text{-}p(U): U \in \tau\}$ for $\tau \in LT(X)$. From this result, we can conclude that, if X is an infinite set and P is any topological property, then the set of topologies in LT (X) possessing the property P may be identified simply from the lattice structure of LT(X), since the only automorphisms of LT(X) for infinite X are those which simply permute elements of X. Therefore any automorphism of LT(X) must map all the topologies in LT(X) onto their homeomorphic images. Thus the topological properties of elements of LT(X) must be determined by the position of the topologies in LT(X). In this paper, we determine some classes of automorphisms of lattice LFT(X,L) where L is a complete, distributive and pseudo complemented lattice (or an F - lattice).

II. Preliminaries

Let X be any nonempty set and L be a complete and distributive lattice.

Definition. 1

Let X be a nonempty set, L a complete lattice. An L-fuzzy subset A of X is a mapping $A: X \to L$. The family of all L-fuzzy subsets of X is denoted by L^X. For brevity, we call an L-fuzzy subset of X as a fuzzy subset of X.

Definition. 2

Define the partial order $\leq$ in L^X by : For all $A,B \in L^X$, $A \leq B \Leftrightarrow A(x) \leq B(x)$ for all $x \in X$. With this partial order, L^X is a complete lattice. The smallest and the greatest elements of $(L^X, \leq)$ are the constant functions taking the values 0 and 1 respectively and are denoted by $\underline{0}$ and $\underline{1}$.

Definition. 3

Let L be a lattice. A mapping $' : L \to L$ is called order-reversing if, for all $a,b \in L$, $a \leq b \Rightarrow a' \geq b'$; called an involution on L if, $a'' = a$ for all $a \in L$. It is obvious that an involution is always a bijection.

Definition. 4

A complete and distributive lattice L is called an F- lattice, if L has an order reversing involution $' : L \to L$. Let X be a non-empty ordinary set, L an F- lattice, $'$ the order reversing involution on L. For all $A \in L^X$ use the order reversing involution $'$ on L^X by $A'(x) = (A(x))'$ for all $x \in X$. Call $' : L^X \to L^X$ the pseudo-complementary operation on L^X, A' the pseudo-complementary set of A, (or the pseudo-complement of A) in L^X.

Lemma : 1

Let X a nonempty ordinary set, L an F-lattice, then the pseudo-complementary operation $' : L^X \to L^X$ is an order reversing involution.

Definition. 5

Let X be a nonempty ordinary set, L an F- lattice, $\delta \subseteq L^X$. Then δ is called a fuzzy topology on X and (X,δ) or (L^X,δ) is called a fuzzy topological space, if δ satisfies the following three conditions :

(i) $\underline{0}, \underline{1} \in \delta$

(ii) For all $A \subseteq \delta$, $\vee A \in \delta$.

(iii) For all $A,B \in \delta$, $A \wedge B \in \delta$.

Definition. 6

An element of δ is called an open set in L^X. A pseudo-complement of an open set is called a closed set in L^X

Definition. 7

Let X be a nonempty set and δ_1, δ_2 be two fuzzy topologies on X. We say δ_1 is " coarser than " δ_2 (or δ_2 is finer than δ_1) if $\delta_1 \leq \delta_2$.

Remark. 1

The relation " coarser than " denoted by $\leq$ is a partial order relation on the set of all fuzzy topologies on X. The set of all fuzzy topologies on X denoted by LFT(X,L) is a complete lattice under the relation $\leq$ defined above. The smallest element of LFT(X,L) is the indiscrete fuzzy topology $\delta = \{ \underline{0}, \underline{1} \}$ and the greatest element is the discrete fuzzy topology $\delta = L^X$.

Definition. 8 [1]

A t-homomorphism from a lattice L into a lattice M is a function $f : L \to M$ such that

(i) h is a homomorphism

(ii) $h(\underline{0}) = \underline{0}$ and $h(\underline{1}) = \underline{1}$

(iii) $h(\vee k_i) = \vee h(k_i)$ where $\{k_i : i \in I\}$ is an arbitrary subset of L.

Remark. 2

Obviously every t-homomorphism is a homomorphism. But the converse need not be true.

III. Main Results

Let X be any non-empty set and L be an F- lattice. Let $p: X \to X$ be a bijection and $g: L \to L$ be a t-homomorphism. For $c \in L^X$, define $H_{p,g}$ by $H_{p,g}(c)(x) = g(c(p^{-I}(x)))$; $c \in L^X$, $x \in X$.

Lemma. 1

$H_{p,g}$ is a bijection.

Proof:

Suppose $c,d \in L^X$ such that $H_{p,g}(c) = H_{p,g}(d)$. This implies that for each $x \in X$, $H_{p,g}(c)(x) = H_{p,g}(d)(x) \Rightarrow g(c(p^{-I}(x))) = g(d(p^{-I}(x)))$. Since g is one to one, this implies $c(p^{-I}(x)) = d(p^{-I}(x))$. Since p is a bijection this implies $c = d$. Hence $H_{p,g}$ is one to one. Let $a \in L^X$. Since g is onto, for all $x \in X$, $g^{-I}(a(x))$ exists. Define $d: X \to L$ as follows : For $y \in X$, $d(y) = g^{-I}(a(p(y)))$. Clearly $d \in L^X$. For, $x \in X$, $H_{p,g}(d)(x) = g(d(p^{-I}(x))) = g(g^{-I}(a(p(p^{-I}(x))))) = a(x)$. That is, $H_{p,g}(d) = a$. Hence $H_{p,g}$ is onto.

Lemma. 2

$H_{p,g}$ is a t- homomorphism.

Proof:

Let $c,d \in L^X$. Then for each $x \in X$, $H_{p,g}(c \vee d)(x) = g((c \vee d)(p^{-I}(x)) = g(c(p^{-I}(x))) \vee g(d(p^{-I}(x))) = H_{p,g}(c)(x) \vee H_{p,g}(d)(x) = (H_{p,g}(c) \vee H_{p,g}(d))(x)$. Thus $H_{p,g}(c \vee d) = H_{p,g}(c) \vee H_{p,g}(d)$. Similarly $H_{p,g}(c \wedge d) = H_{p,g}(c) \wedge H_{p,g}(d)$. Thus $H_{p,g}$ is a homomorphism. Further, since g is a t-homomorphism, for each $x \in X$, $H_{p,g}(\underline{0})(x) = g(\underline{0}(p^{-I}(x))) = g(\underline{0}) = \underline{0}$ and $H_{p,g}(\underline{1})(x) = g(\underline{1}(p^{-I}(x))) = g(\underline{1}) = \underline{1}$. Hence $H_{p,g}(\underline{0}) = \underline{0}$ and $H_{p,g}(\underline{1}) = \underline{1}$.

Also for d_i, $i \in I$ in L^X and for all $x \in X$, $H_{p,g}(\vee(d_i))(x) = g((\vee d_i)(p^{-I}(x))) = \vee g(d_i(p^{-I}(x))) = \vee H_{p,g}(d_i)(x)$. That is, $H_{p,g}(\vee(d_i)) = \vee H_{p,g}(d_i)$. Hence $H_{p,g}$ is a t- homomorphism.

Lemma. 3

$H_{p,g}$ is a t- isomorphism.

Proof:

Follows from the lemma 1 and lemma 2 .

Lemma. 4

If δ is a fuzzy topology, then the collection $H_{p,g}^*(\delta) = \{H_{p,g}(a) : a \in \delta\}$ is also a fuzzy topology.

Proof:

Let $H_{p,g}^*(\delta) = \{H_{p,g}(a) : a \in \delta\}$. Then

(1) $\underline{0} \in \delta \Rightarrow H_{p,g}(\underline{0}) \in H_{p,g}^*(\delta) \Rightarrow \underline{0} \in H_{p,g}^*(\delta)$ and $\underline{1} \in \delta \Rightarrow H_{p,g}(\underline{1}) \in H_{p,g}^*(\delta) \Rightarrow \underline{1} \in H_{p,g}^*(\delta)$

(2) Let $f_1, f_2 \in H_{p,g}^*(\delta)$. Then $f_1 = H_{p,g}(a)$ and $f_2 = H_{p,g}(b)$ for some $a,b \in \delta$. We have $a,b \in \delta \Rightarrow a \wedge b \in \delta$. Now $f_1 \wedge f_2 = H_{p,g}(a) \wedge H_{p,g}(b) = H_{p,g}(a \wedge b) \in H_{p,g}^*(\delta)$, since $a \wedge b \in \delta$.

(3) Let f_i, $i \in I$ belongs to $H_{p,g}^*(\delta)$. Then $f_i = H_{p,g}(a_i)$ for some $a_i \in \delta$.

We have $a,b \in \delta \Rightarrow \vee a_i \in \delta$. Now $\vee f_i = \vee H_{p,g}(a_i) = H_{p,g}(\vee a_i) \in H_{p,g}^*(\delta)$, since $\vee a_i \in \delta$.

Hence $H_{p,g}^*(\delta)$ is a fuzzy topology.

Lemma. 5

For $\delta \in$ LFT(X,L), define $H_{p,g}{}^*(\delta) = \{H_{p,g}(a) : a \in \delta\}$. Then $H_{p,g}{}^*(\delta)$ is a fuzzy topology on X and $H_{p,g}{}^*$ is an automorphism of LFT(X,L).

Proof:

From Lemma 4, it follows that $H_{p,g}{}^*(\delta)$ is a fuzzy topology. For $\delta_1, \delta_2 \in$ LFT(X,L), let $H_{p,g}{}^*(\delta_1) = H_{p,g}{}^*(\delta_2)$. This implies

$\{ H_{p,g}(a) : a \in \delta_1 \} = \{H_{p,g}(b) : b \in \delta_2 \} \Rightarrow \{ a : a \in \delta_1 \} =\{ b : b \in \delta_2 \}$

$\Rightarrow \delta_1 = \delta_2$.

Therefore $H_{p,g}{}^*$ is one to one.

Let $\tau \in$ LFT(X,L). Consider the collection $\delta = \{H_{p,g}{}^{-I}(a) : a \in \tau\}$. Then δ is fuzzy topology and $H_{p,g}{}^*(\delta) = \{ H_{p,g}{}^{-I}(a) : a \in \tau\} = \{a : a \in \tau\} = \tau$. Therefore $H_{p,g}{}^*$ is onto.

Further $\delta_1 \subseteq \delta_2 \Leftrightarrow \{a : a \in \delta_1 \} \subseteq \{b : b \in \delta_2 \} \Leftrightarrow \{ H_{p,g}(a) : a \in \delta_1 \} \subseteq \{H_{p,g}(b) : b \in \delta_2 \} \Leftrightarrow H_{p,g}{}^*(\delta_1) \subseteq H_{p,g}{}^*(\delta_2)$.

Hence $H_{p,g}{}^*$ is an automorphism of LFT(X,L).

Theorem. 1

Let X be any non- empty set and L be an F- lattice. For a bijection p on X and a t- homomorphism g on L, define $H_{p,g}$ by $H_{p,g}(a)(x) = g(a(p^{-I}(x)))$; $a \in L^X$, $x \in X$. Then $H_{p,g}$ is an automorphism on L^X . Further, for $\delta \in$ LFT(X,L), let $H_{p,g}{}^*(\delta) = \{H_{p,g}(a) : a \in \delta\}$. Then $H_{p,g}{}^*$ is an automorphism of LFT(X,L).

Proof:

Follows from lemma 5.

Theorem. 2

Let X be a finite set and L be a finite F-lattice. For bijections p: X→ X and g: L→ L , define $F_{p,g}{}^*$ by $F_{p,g}{}^*(\delta) = \{comp(H_{p,g}(a)) : a \in \delta\}$ where $comp(H_{p,g}(a))$ denotes the pseudo-complement of $H_{p,g}(a)$ in L^X. Then $F_{p,g}{}^*$ is an automorphism of LFT(X,L).

Proof:

We have $\underline{0} = comp(\underline{1}) = comp(H_{p,g}(\underline{1}))$ and $\underline{1} = comp(\underline{0}) = comp(H_{p,g}(\underline{0}))$. Since $\underline{0}, \underline{1}$ are in δ, it follows that $\underline{0}, \underline{1}$ are in $F_{p,g}{}^*(\delta)$. Let $f_1, f_2 \in F_{p,g}{}^*(\delta)$. Then $f_1 = comp(H_{p,g}(a))$ and $f_2 = comp(H_{p,g}(b))$ for some $a, b \in \delta$. We have $a, b \in \delta \Rightarrow a \wedge b \in \delta$. Now,

$f_1 \vee f_2 = comp(H_{p,g}(a)) \vee comp(H_{p,g}(b))$

$= comb\{ H_{p,g}(a) \wedge H_{p,g}(b)\}$

$= comb\{ H_{p,g}(a \wedge b)\} \in F_{p,g}{}^*(\delta)$

Similarly, $f_1 \wedge f_2 \in F_{p,g}{}^*(\delta)$. Thus $F_{p,g}{}^*(\delta)$ is a fuzzy topology on X.

For $\delta_1, \delta_2 \in$ LFT(X,L), let $F_{p,g}{}^*(\delta_1) = F_{p,g}{}^*(\delta_2)$. This implies,

$\{comp(H_{p,g}(a)) : a \in \delta_1\} = comp(H_{p,g}(b)) : b \in \delta_2\} \Rightarrow \{a : a\in \delta_1\} = \{b : b\in \delta_2\}$

$\Rightarrow \delta_1 = \delta_2$

Therefore $F_{p,g}{}^*(\delta)$ is one to one.

For $\tau \in$ LFT(X,L), consider the collection $\delta = \{ H_{p,g}{}^{-I}(comb(a)) : a\in \tau\}$.

Then δ is a fuzzy topology on X and

$$\begin{aligned} F_{p,g}{}^*(\delta) &= \{ comp(H_{p,g}(H_{p,g}{}^{-I}(comp(a)))) : a\in\tau\} \\ &= \{comp(comp(a)) : a\in \tau\} \\ &= \{a : a \in \tau \} \\ &= \tau \end{aligned}$$

Therefore $F_{p,g}{}^*$ is onto. Also,

$$\begin{aligned} \delta_1 \subseteq \delta_2 &\Leftrightarrow \{a : a \in \delta_1\} \subseteq \{b : b \in \delta_2\} \\ &\Leftrightarrow \{H_{p,g}(a): a\in\delta_1\} \subseteq \{H_{p,g}(b): b\in\delta_2\} \\ &\Leftrightarrow \{comp(H_{p,g}(a)) : a\in\delta_1\} \subseteq \{comp(H_{p,g}(b)) : b\in\delta_2\} \\ &\Leftrightarrow F_{p,g}{}^*(\delta_1) \subseteq F_{p,g}{}^*(\delta_2) \end{aligned}$$

Hence $F_{p,g}{}^*(\delta)$ is an automorphism of LFT(X,L).

Example. 1

Let X = {a,b}, L = { 0,1/2,1}. Then the lattice $L^X = \{a^0b^0, a^1b^1, a^{1/2}b^{1/2}, a^0b^{1/2}, a^0b^1, a^{1/2}b^0, a^{1/2}b^1, a^1b^0, a^1b^{1/2}\}$ where $a^i b^j$; i,j = 0,1/2,1 is the map $a \to i$ and $b \to j$. Let S(X) denotes the group of bijections on X and A(L) denotes the group of automorphisms of L. Then $S(X) = \{p_1,p_2\}$ where p_1 is the identity map on X and p_2 is the map on X which sends $a \to b$ and $b \to a$. A(L) consists only one member g which is the identity map on L. Thus by Theorem 1 and Theorem 2, $H_{p1,g}{}^*$, $H_{p2,g}{}^*$, $F_{p1,g}{}^*$ and $F_{p2,g}{}^*$ are automorphisms of the lattice LFT(X,L).

Example. 2

Let X = {x,y}, L = {0,a,b,1} where a and b are not comparable. Then the lattice $L^X = \{ x^0y^0, x^ay^a, x^by^b, x^0y^1, x^0y^a, x^0y^b, x^1y^0, x^1y^a , x^1y^b, x^ay^b, x^by^a, x^ay^1, x^by^1, x^ay^0, x^by^0, x^1y^1\}$ where $x^i y^j$; i,j = 0,a,b,1 is the map on X which sends $x \to i$ and $y \to j$. Here, $S(X) = \{p_1, p_2\}$ where p_1 is the identity map on X and p_2 is the map which sends $x \to y$ and $y \to x$. $A(L) = \{g_1, g_2\}$ where g_1 is the identity map on L and g_2 is the map which sends $0 \to 0$, $a \to b$, $b \to a$, $1 \to 1$.

Thus by Theorem 1 and Theorem 2 , $H_{p1,g1}*$, $H_{p1,g2}*$, $H_{p2,g1}*$, $H_{p2,g2}*$, $F_{p1,g1}*$, $F_{p1,g2}*$, $F_{p2,g1}*$ and $F_{p2,g2}*$ are automorphisms of LFT(X,L).

Remark. 3

When $L = \{0,1\}$, LFT(X,L) coincides with LT(X), $H_{p,g}*$ coincides with A_p and $F_{p,g}*$ coincides with B_p where A_p and B_p are as defined in the beginning of this paper. Note that we are identifying the subsets of X as characteristic functions.

Acknowledgements

The author wishes to thanks Dr. P.T. Ramachandran, Department of Mathematics, University of Calicut for his valuable guidance during the preparation of this paper.

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