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# "Certain Case of Reducible Hypergeometric Functions of Hyperbolic Function as Argument"

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***Abstract:*** *In this paper,we specialized parameters and argument, Hypergeometric function* $F_E$ $(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3;$ *cosh x, cosh y, cosh z)* $F_G$, $F_k$ *and* $F_N$ *can be reduced to the hypergeometric function of Bailey's* $F_4(\alpha_1, \beta_2, \gamma_2, \gamma_3;$ *-coshy, - cosh z) and also discussed their reducible cases into Horn's function. In the journal we consider hypergeometric function of three variables and obtain its interesting reducible case into Bailey's* $F_4$ *& Horn's function.*
*In the section 2, hypergeometric function of four variables can be reduced to the hypergeometric function of one, two & three variables with some new and interesting particular cases.*

## I. On Hypergemoteric Integrals

### 1.1 INTRODUCTION

We will study Laplace's double integral for Saran's function $F_E$ $(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3;$ cosh x, cosh y, coshz) which has been reduced to Bailey's $F_4$ $(\alpha_1, \beta_2, \gamma_2, \gamma_3;$ - cosh y, - cosh z) and pochhamer type of Integrals for $F_E$, $F_G$, $F_K$ and $F_N$ and also discussed their reducible cases into Horn's functions. The purpose of studing only the function $F_E$, $F_G$, $F_K$ and $F_N$ is mainly due to the function in their integral representation contain Appell's function $F_1$ or $F_2$ or the product of Gauss's hypergeometric series which can be reduced by the following relations.

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) = \cosh^m y$ (1.1.1)

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) =$ (1.1.2)

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) =$ (1.1.3)

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) = (1- \cosh y)^{-\alpha}\ {}_2F_1 (\alpha, {}^3-\beta; {}^3; )$ (1.1.4)

Similarly

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) = (1- \cosh y)^{-\beta}\ {}_2F_1 (\gamma - \alpha, \beta; {}^3; )$ (1.1.5)

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) = (1- \cosh y)^{\gamma-\alpha-\beta}\ {}_2F_1 (\gamma-\alpha, {}^3-\beta; {}^3; \text{coshy } y)$ (1.1.6)

${}_2F_1 (\alpha, \beta; {}^3; \cosh y) = (1- \cosh y)^{-\alpha}$ (1.1.7)

$F_1(\alpha,\beta,\beta\_,\alpha: \cosh x, \cosh y) = (1 \_ \cosh y)^{-\alpha} (1 \_ \cosh y)^{-\beta}$ (1.1.8)

$F_1(\alpha,\beta,\beta\_,\beta,\beta\_; \cosh x, \cosh y) = (1 \_ \cosh x \_ \cosh y)^{-\alpha}$ (1.1.9)

$F_2(\alpha,\beta,\beta\_,\alpha,\beta\_; \cosh x, \cosh y) = (1–\cosh y)^{-\beta-\alpha} (1\_\cosh x\_\cosh y)^{-\beta}$ (1.10)

where

|cosh x| <1 | cosh y| <1|Cosh z|<1 for (1.1.1) to (1.1.10)

### 1.2 <u>Reduction of integrals of $F_E$(.) INTO BAIEY'S $F_4$</u> (.)

The hypergeometric function $F_E$ is defined by

$F_1$ $(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3;$ cosh x, cosh y, coshz)

= (1.2.1)

absolutely convergent if m+|coshx|<1, |coshy|<1,|coshz|<1 and |coshz|<1 and It's integral representation S. Saran (1957) is given by

$F_E$ =

$X\ {}_0F_1 (\gamma_2;- pt \cosh y)\ {}_0F_1(\gamma_3;- pt \cosh z)\ dpdt$ (1.2.2)

Where R $(\alpha_1)$> 0 and R $(\alpha_2)$ > 0

$F_4$ $(\alpha_1, \beta_2, \gamma_2, \gamma_3;$ - cosh y, - cosh z)

=

$X\ {}_0F_1 (\gamma_3;- pt \cosh y)$ dp dt (1.2.3)

Changing the variable t to T by the substitution $t = C^2T^2/4p$ and writing

$\alpha_1 = ½ (\lambda+\mu+\upsilon-\rho)$ $\beta_2 = ½ (\lambda+\mu+\upsilon+\rho)$

$\gamma_2 = \mu+1$ $\gamma_3 = \upsilon+1$, cosh y = , cosh z = (1.2.4)

F4

=

dpdt

We know that [E.T. Wittaker and G.N. Watson (1902)]

$\exp\{-(cx + a / x)\}dx = 2(c / a)K_v(2)$, (1.2.5)

so the p- integral

(1.2.6)

and changing $_0F_1$ in to Bessel function of the kind the relation

$J_k(z)={}_0F_1$ (1.2.7)

$F_4$ (1.2.8)

Where $F_4$ is fourth type of Appell's function

## 1.3 REDUCTION OF $F_E$, $F_G$, $F_K$ AND $F_N$ INTO HORN'S FUNCTION

Let us consider the integral S. Saran (1955) for $F_E$ viz.

$F_E=$

$F_2 (\alpha_1,\beta_1, \beta_2,\gamma_1, \gamma_2 + \gamma_3-1;$ cosh x, dt (1.3.1)

where

|cosh x|+< 1 along the contour.

Puttting $\beta_1 = \gamma_1$ and $\beta_2 = \gamma_2 + \gamma_3-1$ in equation (1.3.1) we get

$F_E=$

$\int (-t)^{-\gamma 2} (-t)^{-\gamma 3}(1\text{-cosh x -})^{-\alpha 1}$ dt

We can expand

$(1\text{-cosh x -})^{-\alpha 1} = (1\text{-cosh x – cosh y})^{-\alpha 1}$

….. (1.3.2)

Where

<1, < 1

along the contour

$= (1 – \text{cosh x – cosh y – cosh z})^{-\alpha 1}$

… (1.3.3)

where

< l and, < 1

along the contour using (1.3.2) and (1.3.3) and then evaluating the integral, after changing the order of the integration and summation, keeping

We will get,

$F_E(\alpha_1, \alpha_1, \alpha_1, \gamma_1, \gamma_2 + \gamma_3 -1, \gamma_2 + \gamma_3 -1, \gamma_1, \gamma_2, \gamma_3,$ : cosh x, cosh y, cosh z)

$= (1 – \text{cosh x – coxh y})^{-\alpha 1}$.

$H_1(1- \gamma_3, \alpha_1, \gamma_2 + \gamma_3-1, \gamma_2; )$ (1.3.4)

$= (1\text{-cosh x – coshy – cosh z})^{-\alpha 1}$.

$G_1(\alpha_1,1- \gamma_2, 1- \gamma_3, )$ (1.3.5)

When $H_1$ and $G_1$ are defined by Horn (1931)

$G_1(\alpha_1,\beta, \beta\_;$ cosh x, cosh y) = $\cosh^m x \cosh^n y$ (1.3.6)

$H_1(\alpha,\beta,^3,´;$ cosh x, cosh y) = $\cosh^m x \cosh^n y$ (1.3.7)

We also expand $(1\text{-cosh x-})^{-\alpha 1}$ by taking = $(1\text{-cosh x})^{-\alpha 1}$ as factor but that will only reduce $F_E$ to $F_4$.

Similarly, we have $F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3;$ cosh x, cosh y, cosh z)

$= \cosh^m x \cosh^n y \cosh^p z$ (1.3.8)

absolutely convergent if m + n = 1 and m + p = 1 while

|cosh x|<1, |cosh y|<1, and |cosh z|<1

Its Integral representation is given by S. Saran (1955)

$F_G=$

$_2F_1(\rho,\beta_1,:\gamma_1:)F_1 (\rho_1,; \beta_2;\beta_2;\gamma_2; )$dt (1.3.9)

$_2F_1(\alpha,\beta, 2\beta;\text{cosh x})=(1\text{-cosh x})^{-\alpha}\ {}_2F_1$ (1.3.10)

absolutely convergent if m + n = 1 and m + p = 1 while |cosh x| <1, |cosh y|< 1, and |cosh z|< 1

It's integral representation is given by

${}_2F_1$ (α,β,β_, β+β_; cosh x, cosh y)

(1-cosh y) $^{-\alpha}$ ${}_2F_1$(α,β,β+β_;) (1.3.11)

using (1.3.10) and then taking the new variable of integration u given by t = 1/2cosh x + (1-1/2 cosh x) u in $F_G$'s integral we will get

$F_G(\gamma_1 = 2\beta_1)$=(1-)$^{-\alpha 1}$

${}_2F_1$() (1.3.12)

$F_1(\rho,\beta_2, \beta_3, \gamma_3;)$du

Now using (1.3.11) after putting $\gamma_2 = \beta_2 + \beta_3$ and introducing a new variable of integration *v* given by:

*v* = (1- cosh z - (1-v)/ (1 - (1.3.13)

We will get, on Integration

$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3, 2\beta_1, \beta_2,+\beta_3, \beta_2+ \beta_3$; cosh x, cosh y, cosh z)

= (1 – cosh z - cosh x) $^{-\alpha 1}$

$H_4(\alpha_1, \beta_2,\beta_1+, \beta_2,+ \beta_3;)$ (1.3.14)

where

$H_4$(α,β,³,´; cosh x, cosh y) = $\cosh^m$ x $\cosh^n$ y (1.3.15)

We will now consider the integral S. Saran (1955) for $F_k$ to show that

$F_K(\alpha_1, \alpha_2, \alpha_3, \gamma_1+ \gamma_3 – 1, \gamma_2, \gamma_1+ \gamma_3-1; \gamma_1, \gamma_2, \gamma_3$; cosh x, cosh y, cosh z)

= (1 – cosh y – coshy z) $^{-\alpha 2}$ (1.3.16)

$H_2(1-\gamma_1, \alpha_2, \alpha_1, \gamma_1+ \gamma_3-1, \gamma_3;)$

= (1 – cosh x) $^{-\alpha 1}$ (1– coshy y) $^{-\alpha 2}$

$H_2(1-\gamma_3, \alpha_1, \alpha_2, \gamma_1+ \gamma_3-1, \gamma_1;)$ (1.3.17)

= (1-cosh x) $^{-\alpha 1}$ (1-cosh y – cosh z) $^{-\alpha 2}$

$G_2(\alpha_1, \alpha_2, 1-\gamma_1,1- \gamma_3;)$ (1.3.18)

where $H_2$ and $G_2$ are defined by

$H_2$(α,β,³,´; θ; cosh x, cosh y) = $\cosh^m$ x $\cosh^n$ y (1.3.19)

and

$G_2$(α,α_,β,β_ cosh x, cosh y)=$\cosh^m$ x $\cosh^n$ y (1.3.20)

$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1,2\alpha_1, \beta_2, \beta_3,$; cosh x, cosh y, cosh z)

= (1 – cosh x)$^{-\beta_1}$ (1- cosh y) $^{-\alpha 2}$

$H_4$ $(\beta_1, \alpha_2, \alpha_1+, \gamma_3;)$ (1.3.21)

This leads to Erdelyi's (1948) when y = 0

To prove these we know that

$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2,\beta_1; \gamma_1, \gamma_2, \gamma_3$; cosh x, cosh y, cosh z)

, $\cosh^m$ x $\cosh^n$ y $\cosh^p$ z (1.3.22)

absolutely convergent if p = (1-m) (1-n) where

|cosh x|< 1, |cosh y|<1, and |cosh z|<1

$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1;\gamma_1, \gamma_2, \gamma_3$; cosh x, cosh y, cosh z)

=

${}_2F_1(\rho,\alpha_1;\gamma_1+:).F_2(\alpha_2,\beta_2,\rho_1,\gamma_2, \gamma_3$; cosh y,)dt (1.3.23)

|1|>|cosh x| and |cosh z/1– t|< 1-|cosh y| along the contour and $\beta_1 = \rho+\rho_1-1$ putting $\gamma_2= \beta_2$ and $\gamma_3 = \rho_1$ we have

$F_2$ $(\alpha_2,\beta_2,\rho_1, \rho_2, \rho_1$; coshy, = (1-coshy -

= (1- coshy – cosh z) $^{-\alpha 2\ m}$ (1.3.24)

= (1- coshy) $^{-\alpha 2\ m}$ (1.3.25)

and

${}_2F_1$ $(\rho, \alpha_1, \rho$; coshy,

= (1- cosh x) $^{-\alpha 1}$ (1.3.26)

Thus if $\beta_1 = \gamma_1 + \gamma_3 -1$ and $\beta_2 = \gamma_2$ we will have

$F_K$=

dt (1.3.27)

Now writing down the expansion (1.3.24) and

(1-)$^{-\alpha 1}$ = $^m$ (1.3.28)

and the integrating term by term by term we will be table to prove (1.3.16) – (1.3.24) can also be proved by applying (1.3.24) and (1.3.26) and then integrating term by term.
Application of (1.3.25) and (1.3.26) will prove (1.3.17) the proof of (1.3.21) is however, similar to (1.3.14).
We can also prove the following results for $F_N$

$F_N(\alpha_1,\gamma_2, \gamma_2, \gamma_1+ k-1-\beta_2, \gamma_1+k-1; \gamma_1, \gamma_2,; \gamma_3,;$ cosh x, cosh y, cosh z)
= (1-cosh x) $^{-\alpha 1}$ (1-cosh y)$^{-\beta 2}$

$H_2(1-k, \alpha_1,k, \gamma_1+k-1, \gamma_2;)$ (1.3.29)

= (1-cosh x) $^{-\alpha 1}$ (1-cosh y)$^{-\beta 2}$

$G_2(\alpha_1,k,1- \gamma_1,1-k;)$ (1.3.30)

$F_N(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1,\gamma_1, \gamma_2, \gamma_3,;$ cosh x, cosh y, cosh z)
= $\cosh^m x \cosh^n y \cosh^p z$ (1.3.31)

absolutely convergent if m+p = 1 and n = 1
where |cosh x|<1, |cosh y|<1, and |cosh z|<1 and
it's integral representations

$F_N(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1,\gamma_1, \gamma_2, \gamma_3,;$ cosh x, cosh y, cosh z)
=

${}_2F_1(\rho, \alpha_1, \gamma_1;F_2(\alpha_2, \beta_2, \rho_1,\gamma_2;$cosh y, dt (1.3.32)

Where $\beta_1= \rho+\rho_1$-1 and |t|>|coshx| and |1-t|>|cosh z| along the contour.

## II. Reducible Cases For The Qardruple Hypergeometric Function

2.1 INTRODUCTION

The various Hypergeometric function of four variables are studied earlier by H. Exton, H. Srivastava and many others.

Here all 21 functions are given below in the terms of a table.

| | | |
|---|---|---|
| 1 | $K_1$(a,a,a,a;b,b,b,c;d,$e_1$,$e_2$,d; x,y,z,t) | = ∑ |
| 2. | $K_2$(a,a,a,a;b,b,b,c;$d_1$,$d_2$,$d_3$,$d_4$; x,y,z,t) | = ∑ |
| 3 | $K_3$(a,a,a,a;$b_1$,$b_1$,$b_2$,$b_2$,;$c_1$,$c_2$,$c_2$,$c_1$; x,y,z,t) | = ∑ |
| 4 | $K_4$(a,a,a,a;$b_1$,$b_1$,$b_2$,$b_2$;c,$d_1$,$d_2$,c; x,y,z,t) | = ∑ |
| 5 | $K_5$(a,a,a,a;$b_1$,$b_1$,$b_2$, $b_2$;$c_1$,$c_2$,$c_3$,$c_4$; x,y,z,t) | = ∑ |
| 6 | $K_6$(a,a,a,a;b,b,$c_1$,$c_2$;e,d,d,d; x,y,z,t) | = ∑ |
| 7. | $K_7$(a,a,a,a;b,b,$c_1$,$c_2$,$d_1$,$d_2$,$d_1$,$d_2$; x,y,z,t) | = ∑ |
| 8. | $K_8$(a,a,a,a;b,b,$c_1$,$c_2$;d,$e_1$,d,$e_2$; x,y,z,t) | = ∑ |
| 9. | $K_9$(a,a,a,a;b,b,$c_1$,$c_2$,$e_1$,$e_2$,d,d; x,y,z,t) | = ∑ |
| 10 | $K_{10}$(a,a,a,a;b,b,$c_1$,$c_2$;$d_1$,$d_2$,$d_3$,$d_4$; x,y,z,t) | = ∑ |
| 11 | $K_{11}$(a,a,a,a;$b_1$,$b_2$,$b_3$,$b_4$,c,c,c,d; x,y,z,t) | = ∑ |
| 12 | $K_{12}$(a,a,a,a;$b_1$,$b_2$,$b_3$,$b_4$,$c_1$,$c_1$,$c_2$,$c_2$; x,y,z,t) | = ∑ |
| 13 | $K_{13}$(a,a,a,a;$b_1$,$b_2$,$b_3$,$b_4$,c,c,$d_1$,$d_2$; x,y,z,t) | = ∑ |
| 14 | $K_{14}$(a,a,a,$c_3$;b,$c_1$,$c_1$,b;d,d,d,d; x,y,z,t) | = ∑ |
| 15 | $K_{15}$(a,a,a,$b_5$;$b_1$,$b_2$,$b_3$,$b_4$;c,c,c,c; x,y,z,t) | = ∑ |
| 16 | $K_{16}$($a_1$,$a_2$,$a_3$,$a_4$;b; x,y,z,t) | = ∑ |
| 17 | $K_{17}$($a_1$,$a_2$,$a_3$,$b_1$,$b_2$;c; x,y,z,t) | = ∑ |
| 18 | $K_{18}$($a_1$,$a_2$,$a_3$,$b_1$,$b_2$;c; x,y,z,t) | = ∑ |
| 19 | $K_{19}$($a_1$,$a_2$,$b_1$,$b_2$,$b_3$,$b_4$;c; x,y,z,t) | = ∑ |
| 20 | $K_{20}$($a_1$,$a_2$,$b_3$,$b_4$;$b_1$,$b_2$, $a_2$, $a_2$;c,c,c,c; x,y,z,t) | = ∑ |
| 21 | $K_{21}$(a,a,$b_6$,$b_5$;$b_1$,$b_2$, $b_3$, $b_4$;c,c,c,c; x,y,z,t) | = ∑ |

All results are having convergent conditions and restrictions on parameters including variables due to Exton (1972)

As far as known to me all the results investigation in this journal are new and interesting which have a wide range of application in the field mathematical series

**2.2 In this Section Quadruple hypergeometric function reduced to the hypergeometric function of one variable.**

**(1) <u>Theorem</u> :** By specializing the parameters of $K_{14}$, we obtain the following

$F(1,a,a,1;b_1, b_2, b_1, b_2;b_1, b_2 + b_1+ b_2, b_1+ b_2, b_1+ b_2; x, y,z,t)$
$= (x + y - xy)^{-1} (z + t - zt)^{-1}$.
$[x\, {}_2F_1(b_1+p,a_2+p;b_1+b_2+p+q;x)+y\, {}_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;y)]$

$.[z_2F_1(b_1,a_2;b_1+b_2;z)+y_2F_1(b_2,a_2,b_1+b_2;t)]$ (2.2.1)

condition and restrictions are given in Exton (1972)

proof:

$F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+ b_2, b_1+ b_2, b_1+ b_2; x, y,z,t)$

= (2.2.2)

=

= (2.2.3)

by virtue of the result the to carlitz formula in the term

$== (x + y – xy)^{-1}$ +

We have equation (2.2.3) in the term

$=(x + y - xy)^{-1}$

$(z + t - zt)^{-1}$ (2.2.4)

$= (x+y-xy)^{-1}(z+t-zt)^{-1}$

then theorem follows

$F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+ b_2, b_1+ b_2, b_1+ b_2; x, y,z,t)$

$=(x + y - xy)^{-1} (z + t -zt)^{-1}$

$[x_2F_1(b_1+p,a_2+p;b_1+b_2+p+q;x)+y_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;y)].$

$[z_2F_1(b_1,a_2;b_1+b_2;z)+y_2F_1(b_2,a_2,b_1+b_2;t)]$

The completes the derivation of (2.2.1).

(2) **Theorem** by specializing the parameters of $K_{14}$, we obtain the following.

(i) $F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+ b_2, b_1+ b_2, b_1+ b_2; 1, 1,1,1)$

=

(2.2.6)

(ii) $F(1,a_2,a_2,1;a_2,a_2,a_2,a_2;2a_2,2a_2,2a_2,2a_2;1,1,1,1)$

= (2.2.7)

(iii) $F(1,1,1,1;1,1,1,1;2,2,2,2;1,1,1,1)$

$= [\Gamma(q+1)+ \Gamma(p+1)]$ (2.2.8)

**(1) Proof :-**

now putting x = y = z = 1 in equation (2.2.1) than we get

$F(1,a_2,a_2,1;b_1,b_2,b_1,b_2;b_1+b_2,b_1+b_2,b_1+b_2,b_1+b_2;1,1,1,1)$

$=[{}_2F_1(b_1+p,a_2+p;b_1+b_2+p+q;1)+{}_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;1)].$

$[{}_2F_1(b_1,a_2;b_1+b_2;1)+ {}_2F_1(b_2,a_2,b_1+b_2;1)]$ (2.2.9)

now Apply Gauss's summation theorem (1812)

${}_2F_1(\alpha,\beta,{}^3;1)=$ (2.2.10)

using equation (2.2.10) in (2.2.9) we get

=

(2.2.11)

This completes the derivation of (2.2.6)

**(2) Proof:-** now putting $b_1 = b_2 = a_2$ in equation (2.2.11)

$F(1,a_2,a_2,a_21;a_2,a_2,a_2,a_2;2a_2,2a_2,2a_2,2a_2;1,1,1,1)$

=

= (2.2.12)

This competes the derivation of (2.2.7)

**(3) Proof:** now putting $b_1 = b_2 = a_2 = 1$ in equation (2.2.12)

$F(1,1,1,1;1,1,1,1;2,2,2,2;1,1,1,1)$

=

$= [\Gamma(q+1)+\Gamma(p+1)]$ (2.2.13)

This completes the derivation of (2.2.8)

### 2.3 In this Section Quadruple hypergeometric function reduced to the Appell hypergeometric function

**(1) Theorem** By specializing the parameters of $K_{12},K_{10},K_{15}$ we obtain the following

$K_{12}$ (a, a, a, a; $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_1$, $c_2$, $c_2$; x,y,z,t)

= F1 (a, $b_1$, $b_2$; $c_1$; x, y) $F_1$ (a+m+n, $b_3$, $b_4$ ; $c_2$ ; z,t) (2.3.1)

$K_{12}$ (a, a, a, a; $b_1$, $b_2$,$b_3$,$b_4$, $c_1$,$c_1$, $c_2$,$c_2$; 1,1,1,1)

= (2.3.2)

$K_{10}$ (a, a, a, a; b,b,$c_1$,$c_2$ ; $d_1$, $d_2$, $d_3$, $d_4$; x, y, z, t)

= $F_4$ (a, b; $d_1$, $d_2$; x, y) $F_2$ (a+m+n, $c_1$,$c_2$;$d_3$, $d_4$; z, t) (2.3.3)

$K_{15}$ (a, a, a, $b_5$; $b_1$, $b_2$, $b_3$, $b_4$; c, c, c, c; x, y, z, t)

= $F_1$ (a, $b_1$, $b_2$; c; x, y) $F_3$ (a+m+n, $b_5$, $b_3$, $b_4$; c+m+n; z, t) (2.3.4)

where ($F_1$, $F_2$, $F_3$, $F_4$,) are Appell hypergeometric function of two variables.

**(1) Proof:**

Now Quadruple hypergemotric function can be reduced to Appell function of two variable.

$K_{12}$ (a, a, a, a; $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_1$, $c_2$, $c_2$; x, y, z, t)

= (2.3.5)

=

=

= $F_1$ (a, $b_1$, $b_2$; c ; x, y) $F_1$ ( a+m+n,$b_3$,$b_4$; $c_2$; z, t) (2.3.6)

where $F_1$ is Appell function of two variable

This completes the derivation of (2.3.1)

**(2) Proof :-**

now putting x = y = z = t= 1 in equation (2.3.6)

$K_{12}$ ( a, a, a, a; $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_1$, $c_2$, $c_2$; 1,1,1,1 )

= $F_1$(a,$b_1$,$b_2$;$c_1$; 1,1) $F_1$ ( a+m+n,$b_3$,$b_4$; $c_2$; 1,1) (2.3.7)

Now Apply Gauss's summation theorem

$F_1(\alpha,\beta,\beta_1,\gamma;1,1)$= (2.3.8)

Using equation ( 2.3.7) and (2.3.8)

= (2.3.9)

This completes the derivation of (2.3.2)

**(3) Proof:**

$K_{10}$ (a, a, a, a; b, b, $c_1$, $c_2$; $d_1$, $d_2$, $d_3$, $d_4$; x, y, z, t)

= (2.3.10)

=

=

= $F_4$ ( a, b; $d_1$, $d_2$ ; x, y) $F_2$ ( a+m+n, $c_1$,$c_2$; $d_3$, $d_4$; z,t) (2.3.11)

where $F_4$ & $F_2$ are Appell function of two variable

This completes the derivation of (2.3.3)

**(4) Proof:**

$K_{15}$ (a, a, a, $b_5$; $b_1$, $b_2$, $b_3$, $b_4$; c, c, c, c; x, y, z, t)

= (2.3.12)

=

=

= $F_1$( a, $b_1$, $b_2$; c ; x, y) $F_3$(a+m+n, $b_5$,$b_3$,$b_4$; c+m+n; z, t) (2.3.13)

Where $F_1$ & $F_3$ are Appell function of two variable

This completes the derivation of (2.3.4)

## 2.4 In this Section Quadruple hypergeometric function reduced to the Appell hypergeometric function Lauricella's set and Saran

**(1) <u>Theorem</u>**

By specializing the parameters of $K_2$, $K_{12}$, $K_{15}$ , $K_6$ we obtain the following

$K_2$ ( a, a, a, a; b, b, b, c; $d_1$ $d_2$, $d_3$, $d_4$; x,y,z,t)

= $F_E$ ( a, a,a, b,b,c,; $d_1$, $d_2$,$d_3$ ; x,y,t) ${}_2F_1$( a+m+n+q,b+m+n, $d_4$; z ) (2.4.1)

$K_2$, (a, a, a, a; b, b, b, c; $d_1$, $d_2$, $d_3$, $d_4$; x,y,l,t)

$F_E$ (a,a,a,b,b,c,; $d_1$,$d_2$,$d_3$; x, y, t) (2.4.2)

$K_{12}$ ( a, a, a, a; $b_1$,$b_2$, $b_3$,$b_4$, $c_1$, $c_1$, $c_2$, $c_2$; x,y,z,t)

= $F_G$( a, a, a, $b_1$,$b_3$, $b_4$, $c_1$,$c_2$, $c_2$; x,z,t) ${}_2F_1$ (a+m+p+q,$b_2$;$c_1$+m;y) (2.4.3)

$K_{12}$ ( a, a, a, a; $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_1$,$c_2$, $c_2$ ; x,l,z,t)

$F_G$ ( a, a, a, $b_1$,$b_3$, $b_4$,; $c_1$, $c_2$, $c_2$; x, y, z, t ) (2.4.4)

$K_{15}$ ( a, a, a, $b_5$;$b_4$, $b_1$,$b_2$,$b_3$; c, c, c,c; x, y, z, t )

= $F_S$( $b_5$, a, a, $b_4$,$b_1$,$b_2$;c,c,c; x,y,t) ${}_2F_1$ (a+m+n,$b_3$ ;c+q+m+n; z) (2.4.5)

$K_{15}$ ( a, a, a, $b_5$; $b_4$, $b_1$, $b_2$, $b_3$;c, c, c, c; x, y, z, t)

$F_S$($b_5$, a, a, $b_4$, $b_1$, $b_2$,; c,c,c; x,y,t) (2.4.6)

$K_6$ (a,a,a,a; b,b, $c_1$,$c_2$;e, d,d,d;x,y,z,t)

= $F_F$( a, a, a,b, $c_1$, b, ;e,d,d; x,z,y) ${}_2F_1$ ( a+m+p+n,$c_2$ ;d+p+n; t) (2.4.7)

$K_6$ ( a,a,a,a; b,b, $c_1$,$c_2$;e, d,d,d;x,y,z,t)

= $F_F$( a, a, a,b, $c_1$, b, ;e,d,d; x,z,y) (2.4.8)

Where ($F_4$,$F_{14}$, $F_8$,$F_7$) & ($F_E$, $F_F$, $F_G$,$F_S$) are Lauriceila's set & Saran Triple hypergeometric Series.

**(1) Proof:-**

Now Quadruple hypergeometric function can be reduced to Lauricella's set and Saran Triple hypergometric Series.

$K_2$ ( a,a,a,a; b,b,b,c; $d_1 d_2, d_3, d_4$;x,y,z,t)

= (2.4. 9)

=

=

= $F_E$ ( a, a, a, b, b,c,; $d_1$ , $d_2$, $d_3$ , x,y, t) ${}_2F_1$ ( a+m+n+q, b+m+n, $d_4$; z) (2.4.10)

This completes the derivation of (2.4.1)

**(2) Proof:**

If z = 1, in euqation (2.4.10)

$K_2$ (a, a, a, a; b, b, b, c; $d_1$, $d_2$, $d_3$, $d_4$; x,y, l,t)

= $F_E$( a, a, a, b, b,c,; $d_1$, $d_2$, $d_3$, x,y, t) ${}_2F_1$ ( a+m+n+q, b+m+n, $d_4$; 1) (2.4.11)

Now Apply Gauss's summation theorem in equation ( 2.4.11)

$F_1(\alpha,\beta,{}^3;1)$=

$K_2$ ( a, a, a, a; b, b, b, c; $d_1$, $d_2$, $d_3$, $d_4$; x,y, l,t)

= $F_E$( a, a, a,b,b, c,; $d_1$,$d_2$,$d_3$; x,z,t) (2.4.12)

This completes the derivation of (2.4.2)

**(3) Proof:**

$K_{12}$ ( a, a, a, a; $b_1$, $b_2$, $b_3$,$b_4$,$c_1$,$c_2$, $c_2$ ; x,y, z,t)

= (2.4.13)

=

=

$F_G$ ( a, a, a, $b_1$, $b_3$, $b_4$; $c_1$, $c_2$,$c_2$; x, z, t) ${}_2F_1$ (a+m+p+q; $b_2$; $c_1$+m; y) (2.4.14)

This completes the derivation of (2.4.3)

**(4) Proof:**

When y = 1, in equation (2.4.14) Then

$K_{12}$ ( a, a, a, a; $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_1$, $c_2$, $c_2$; x,**l**, z,t)

=$F_G$ (a, a, a, $b_1$, $b_3$, $b_4$; $c_1$, $c_2$,$c_2$; x, z, t). ${}_2F_1$ (a+m+p+q; $b_2$; $c_1$+m; 1) (2.4.15)

Now Apply Gauss's summation theorem in equation ( 2.4.15)

$F_1(\alpha,\beta,{}^3;1)$=

= $F_G$( a, a, a,$b_1$,$b_3$, $b_4$,$c_1$, $c_2$, $c_2$,; x,z,t) (2.4.16)

This completes the derivation of (2.4.4)

**(5) Proof :-**

$K_{15}$ (a, a, a, $b_5$; $b_1$, $b_2$, $b_3$, $b_4$ ; c, c, c, c; x; y, z, t)

=

=

=

= $F_S$( $b_5$, a, a, $b_4$,$b_1$,$b_2$;c,c,c; x,y,t) ${}_2F_1$ ( a+m+n,$b_3$ ;c+q+m+n; z ) (2.4.17)

This completes the derivation of (2.4.5)

**(6) Proof :-**

When z = 1 in equation (2.4.17) Then

$K_{15}$ (a, a, a, $b_5$, $b_4$, $b_1$, $b_2$, $b_3$; c, c, c, c; x, y, 1, t )

= $F_s$( $b_5$, a, a, $b_4$,$b_1$,$b_2$;c,c,c; x,y,t) ${}_2F_1$ ( a+m+n,$b_3$ ;c+q+m+n;1) (2.4.18)

Now Apply Gauss's summation theorem in equation

$F_1(\alpha,\beta,{}^3;1)$=

= $F_S$($b_5$,a, a,$b_4$,$b_1$, $b_2$,c, c, c,; x,z,t) (2.4.19)

This completes the derivation of (2.4.6)

**(7) Proof:-**

$K_6$ ( a, a, a, a; b, b, $c_1$,$c_2$ ; e, d, d, d; x, y, z, t)

= (2.4.20)

=

=

= $F_F$( a, a, a,b, $c_1$, b, ;e,d,d; x,z,y) ${}_2F_1$ ( a+m+p+n,$c_2$ ;d+p+n; t) (2.4.21)

This complete the derivation of (2.4.7)

**(8) Proof :-**

When t = 1 in equation (2.4.21) Then

$K_6$ ( a, a, a, a; b, b, $c_1$,$c_2$; e, d, d, d; x, y, z, 1)
= $F_F$( a, a, a,b, $c_1$, b,; e,d,d; x,z,y) ${}_2F_1$ ( a+m+p+n,$c_2$ ;d+p+n; 1) (2.4.22)
Now Apply Gauss's summation theorem in equation
$F_1$($\alpha$,$\beta$,³;1)=
= $F_F$(a, a,a,b,$c_1$, b,;e,d,d; x,z,y) (2.4.23)
This completes the derivation of (2.4.8)

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