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Hypergeometric Functions and Lucas Numbers

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Abstract: *The main purpose of this paper is to give the relationship between hypergeometric series and the Lucas sequence. A variety of representations in terms of finite sums and infinite series involving coefficients are obtained. While many of them are well known and some identities appear to be now.*

I. Hypergeometric Functions:

The hypergeometric function ${}_2F_1(a,b;c;x)$ is defined by the series

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k \tag{1.1}$$

For |x|<1, and by continuous elsewhere (see[2] p.64) where the rising factorial $(a)_k$ is defined by $(a)_0=1$ and

$$(a)_k = a(a+1)(a+2)\ldots\ldots(a+k-1) \quad (k\geq 1) \tag{1.2}$$

II. Lucas Numbers

We define the n[th] Lucas number, denoted by L_n as

$$L_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{2.1}$$

Again we have

$${}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{a}{2}+\frac{1}{2}; \\ & z^2 \\ \frac{1}{2}; \end{matrix}\right]=\frac{1}{2}\left[(1-z)^{-a}+(1+z)^{-a}\right] \tag{2.2}$$

Setting a= – n,z= $\sqrt{5}$ we arrive at

$$L_n=\frac{1}{2^{n-1}}\;{}_2F_1\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}; \\ & 5 \\ \frac{1}{2}; \end{matrix}\right] \tag{2.3}$$

with (1.1)the representation of (2. 3)becomes

$$L_n=\frac{1}{2^{n-1}}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k}5^k \tag{2.4}$$

which is well known ([5].p.168)
Throughout this chapter the following results will be used

$$(\alpha)_{2n}=2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n \tag{2.5}$$

$$(1-n)_k=(-1)^k\,\frac{(n-1)!}{(n-1-k)!} \tag{2.6}$$

$$(1+n)_k = \frac{(n+k)!}{n!} \tag{2.7}$$

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{2.8}$$

$$\Gamma(-x) = -\frac{\pi}{\sin\pi\Gamma(x+1)} \tag{2.9}$$

$$\left(\frac{3}{2}\right)_k = \frac{(2k+1)!}{4^k k!} \tag{2.10}$$

III. Linear And Quadratic Transformation

In this section we will use the well-known linear and quadratic transformations for the hypergeometirc functions to derive some representation from (2.3). We begin by the following pair of linear transformations:

$${}_2F_1\left[\begin{matrix} a,b; & \\ & z \\ c; & \end{matrix}\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} a,\ c-b; & \\ & \frac{z}{z-1} \\ c; & \end{matrix}\right] \tag{3.1}$$

$${}_2F_1\left[\begin{matrix} a,b; & \\ & z \\ c; & \end{matrix}\right] = (1-z)^{-b}\,{}_2F_1\left[\begin{matrix} a,\ c-a; & \\ & \frac{z}{z-1} \\ c; & \end{matrix}\right] \tag{3.2}$$

That are linked together by the relation

$${}_2F_1\left[\begin{matrix} a,b; & \\ & z \\ c; & \end{matrix}\right] = (1-z)^{c-a-b}\,{}_2F_1\left[\begin{matrix} c-a,c-b; & \\ & z \\ c; & \end{matrix}\right] \tag{3.3}$$

([6].p.60.Theorem 21)

We also note the obvious relationship

$${}_2F_1\left[\begin{matrix} a,b; & \\ & z \\ c; & \end{matrix}\right] = {}_2F_1\left[\begin{matrix} b,\ a; & \\ & z \\ a; & \end{matrix}\right]$$

Applying (3.1) to (2.3) then RHS(3.3) of is a finite sum only when n is odd and we get the following identity

$$L_{2n+1} = (-1)^n\ {}_2F_1\left[\begin{matrix} -n,n+1; & \\ & \frac{5}{4} \\ \frac{1}{2}\ ; & \end{matrix}\right] \tag{3.4}$$

A companion relationship of (3.4) can be obtained by applying (3.2) to (2.3). In this case n has to be even, and

$$L_{2n} = (-1)^n\, 2\ {}_2F_1\left[\begin{matrix} -n,n; & \\ & \frac{5}{4} \\ \frac{1}{2}\ ; & \end{matrix}\right] \tag{3.5}$$

Our next transformation formula is

$$ {}_2F_1\left[\begin{matrix} a,b; \\ c; \end{matrix} z\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1\left[\begin{matrix} a,b; \\ a+b-c+1; \end{matrix} 1-z\right] + (1-z)^{c-a-b} $$

$$ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, {}_2F_1\left[\begin{matrix} c-a,c-b; \\ c-a-b+1; \end{matrix} 1-z\right] \tag{3.6} $$

However since a+b-c=-n in(2.3), one of the gamma terms in numerator is not defined. So we use the following transformation which is a special case where a or b is a negative integer and m is a non negative integer.

$$ {}_2F_1\left[\begin{matrix} a,b; \\ a+b+m; \end{matrix} z\right] = \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}\, {}_2F_1\left[\begin{matrix} a,b; \\ 1-m; \end{matrix} 1-z\right] \tag{3.7} $$

(see[8].p.559)

Also we use $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$

(see[6].p.24)

Then, we have

$$ {}_2F_1\left[\begin{matrix} -\frac{n}{2},\frac{n}{2}+\frac{1}{2}; \\ \frac{1}{2}; \end{matrix} 5\right] = 2^{n-1}\, {}_2F_1\left[\begin{matrix} -\frac{n}{2},-\frac{n}{2}+\frac{1}{2}; \\ 1-n; \end{matrix} -4\right] $$

$$ \Rightarrow L_n = {}_2F_1\left[\begin{matrix} -\frac{n}{2},-\frac{n}{2}+\frac{1}{2}; \\ 1-n; \end{matrix} -4\right] \tag{3.8} $$

Another transformation formula is

$$ {}_2F_1\left[\begin{matrix} a,b; \\ c; \end{matrix} z\right] = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\, {}_2F_1\left[\begin{matrix} a,1-c+a; \\ 1-b+a; \end{matrix} \tfrac{1}{z}\right] + $$

$$ \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\, {}_2F_1\left[\begin{matrix} b,1-c+b; \\ 1-a+b; \end{matrix} \tfrac{1}{z}\right] \tag{3.9} $$

We apply this to (2.3) and using the fact that Γ(z) has poles at non-positive integers, we see that one of two terms in (3.9) always disappears.
Evaluating the gamma terms in the remaining expression we arrive at

$$ L_{2n+1} = (2n+1)\left(\frac{5}{4}\right)^n\, {}_2F_1\left[\begin{matrix} -n,\frac{1}{2}-n; \\ \frac{3}{2}; \end{matrix} \frac{1}{5}\right] \tag{3.10} $$

And

$$L_{2n} = 2\left(\frac{5}{4}\right)^n {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-n; \\ \frac{1}{2}; \end{matrix} \; \frac{1}{5}\right] \tag{3.11}$$

From (3.3) we have

$$L_{2n+1} = (2n+1)\left(\frac{4}{5}\right)^{n+1} {}_2F_1\left[\begin{matrix} \frac{3}{2}+n, 1+n; \\ \frac{3}{2}; \end{matrix} \; \frac{1}{5}\right] \tag{3.12}$$

And

$$L_{2n} = 2\left(\frac{4}{5}\right)^n {}_2F_1\left[\begin{matrix} \frac{1}{2}+n, n; \\ \frac{1}{2}; \end{matrix} \; \frac{1}{5}\right] \tag{3.13}$$

These two formulas give us the first infinite series representation for the Lucas Numbers. Employing (3.10) and (3.11) to (3.7) we arrive at

$$L_{2n+1} = 5^n \; {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-n; \\ -2n; \end{matrix} \; \frac{4}{5}\right] \tag{3.14}$$

$$L_{2n} = 5^n \; {}_2F_1\left[\begin{matrix} -n, \frac{1}{2}-n; \\ 1-2n; \end{matrix} \; \frac{4}{5}\right] \tag{3.15}$$

The next transformation formula is

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} \; z\right] &= (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \; {}_2F_1\left[\begin{matrix} a, c-b; \\ a-b+1; \end{matrix} \; \frac{1}{1-z}\right] \\ &+ (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \; {}_2F_1\left[\begin{matrix} b, c-a; \\ b-a+1; \end{matrix} \; \frac{1}{1-z}\right] \end{aligned} \tag{3.16}$$

Applying (3.16) to (2.3) we get

$$L_{2n+1} = (2n+1) \; {}_2F_1\left[\begin{matrix} -n, 1+n; \\ \frac{3}{2}; \end{matrix} \; -\frac{1}{4}\right] \tag{3.17}$$

$$L_{2n} = 2 \; {}_2F_1\left[\begin{matrix} -n, n; \\ \frac{1}{2}; \end{matrix} \; -\frac{1}{4}\right] \tag{3.18}$$

Applying (3.3) to these we get

$$L_{2n+1}=(2n+1)\frac{\sqrt{5}}{2}\ {}_2F_1\left[\begin{matrix}\frac{3}{2}+n,\frac{1}{2}-n;\\ & -\frac{1}{4}\\ \frac{3}{2};\end{matrix}\right]$$

(3.19)

$$L_{2n}=\sqrt{5}\ {}_2F_1\left[\begin{matrix}\frac{1}{2}+n,\frac{1}{2}-n;\\ & -\frac{1}{4}\\ \frac{1}{2};\end{matrix}\right]$$

(3.20)

Now, we use quadratic transformations.
Our first quadratic transformation is

$$ {}_2F_1\left[\begin{matrix}a,b;\\ & z\\ a-b+1;\end{matrix}\right]=(1+z)^{-a}\ {}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{a}{2}+\frac{1}{2};\\ & \frac{4z}{(1+z)^2}\\ a-b+1;\end{matrix}\right] \tag{3.21}$$

(see[2].p.128.(3.19))
Employing this to (3.10) and (3.11) we get the following transformations respectively

$$L_{2n+1}=(2n+1)\left(\frac{3}{2}\right)^n\sqrt{\frac{5}{6}}\ {}_2F_1\left[\begin{matrix}\frac{1-2n}{4},\frac{3-2n}{4};\\ & \frac{5}{9}\\ \frac{3}{2};\end{matrix}\right] \tag{3.22}$$

And

$$L_{2n}=2\left(\frac{3}{2}\right)^n\ {}_2F_1\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2};\\ & \frac{5}{9}\\ \frac{1}{2};\end{matrix}\right] \tag{3.23}$$

Again applying (3.3) we get

$$L_{2n+1}=(2n+1)\left(\frac{2}{3}\right)^{n+1}\sqrt{\frac{5}{6}}\ {}_2F_1\left[\begin{matrix}\frac{5+2n}{4},\frac{3+2n}{4};\\ & \frac{5}{9}\\ \frac{3}{2};\end{matrix}\right]$$

(3.24)

And

$$L_{2n}=2\left(\frac{2}{3}\right)^n\ {}_2F_1\left[\begin{matrix}\frac{n+1}{2},\frac{n}{2};\\ & \frac{5}{9}\\ \frac{1}{2};\end{matrix}\right]$$

(3.25)

Employing (3.6) to (3.24) and (3.7) to (3.23) we get

$$L_{2n+1}=(2n+1)\frac{2^{-\frac{3}{2}}}{3^{n+1}}\sqrt{\frac{5}{6}}\ {}_2F_1\left[\begin{matrix}\frac{5+2n}{4},\frac{3+2n}{4};\\ & \frac{4}{9}\\ \frac{3}{2}+n\ ;\end{matrix}\right]$$

(3.26)

$$+\left(\frac{2}{3}\right)^{n+1}2^{n+\frac{1}{2}}\sqrt{\frac{5}{6}}\ {}_2F_1\left[\begin{matrix}\frac{1}{4}-\frac{n}{2},-\frac{1}{4}-\frac{n}{2};\\ & \frac{4}{9}\\ \frac{1}{2}+n;\end{matrix}\right]$$

And

$$L_{2n} = 3^{n}\ {}_2F_1\left[\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}; & \\ & \frac{4}{9} \\ 1-n; & \end{matrix}\right] \tag{3.27}$$

From (3.3) and (3.26) we get

$$L_{2n+1} = \frac{2n+1}{4.3^{n+\frac{1}{2}}}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}; & \\ & \frac{4}{9} \\ \frac{3}{2}+n; & \end{matrix}\right] \tag{3.28}$$

$$+\frac{2^{2n+\frac{3}{2}}}{3^{5n+2}}\cdot 5^{2n+1}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}+\frac{3n}{2}, \frac{3}{4}+\frac{3n}{2}; & \\ & \frac{4}{9} \\ \frac{1}{2}+n; & \end{matrix}\right]$$

Again applying(3.1) to (3.26) we get

$$L_{2n+1} = \frac{3(2n+1)}{4} 5^{-\frac{3}{4}-\frac{n}{2}}\ {}_2F_1\left[\begin{matrix} \frac{5+2n}{4}, +\frac{3+2n}{4}; & \\ & -\frac{4}{5} \\ \frac{3}{2}+n; & \end{matrix}\right]$$

(3.29)

$$+\left(\frac{2}{3}\right)^{2n+1}\cdot 5^{\frac{1}{4}+n}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}-\frac{n}{2}, \frac{3}{4}+\frac{3n}{2}; & \\ & -\frac{4}{5} \\ \frac{1}{2}+n; & \end{matrix}\right]$$

For odd n we have from (3.27)by using (3.1) that

$$L_{4n+2} = 2\left(\frac{\sqrt{5}}{2}\right)^{2n+1}(2n)!\ {}_2F_1\left[\begin{matrix} -n-\frac{1}{2}, -n; & \\ & -\frac{4}{5} \\ -2n; & \end{matrix}\right]$$

(3.30)

Again for even n, employing (3.2) to (3.27) we arrive at

$$L_{4n} = 2\frac{6}{\sqrt{5}}\left(\frac{5}{4}\right)^{n}(2n-1)!\ {}_2F_1\left[\begin{matrix} \frac{1}{2}-n, 1-n; & \\ & -\frac{4}{5} \\ 1-2n; & \end{matrix}\right]$$

(3.31)

Again the identity (3.29) can be further transformed by formula (3.3) as follows

$$L_{2n+1} = \frac{2n+1}{4}\cdot 5^{-\frac{1}{4}-\frac{n}{2}}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}; & \\ & -\frac{4}{5} \\ \frac{3}{2}+n; & \end{matrix}\right] +$$

$$\frac{1}{3}\cdot\left(\frac{2}{3}\right)^{2n+1} 5^{\frac{3}{4}+n}\ {}_2F_1\left[\begin{matrix} \frac{1}{4}+\frac{3n}{2}, -\frac{1}{4}-\frac{n}{2}; & \\ & -\frac{4}{5} \\ \frac{1}{2}+n; & \end{matrix}\right] \tag{3.32}$$

Applying (3.16) to identity (3.23) we get

$$L_{4n+2}=(-1)^n 3(2n+1)\ {}_2F_1\left[\begin{matrix}-n,n+1; \\ & \frac{9}{4} \\ \frac{3}{2}; \end{matrix}\right] \tag{3.33}$$

And

$$L_{4n}=2(-1)^n\ {}_2F_1\left[\begin{matrix}-n,\ n; \\ & \frac{9}{4} \\ \frac{1}{2}; \end{matrix}\right] \tag{3.34}$$

Again applying (3.1) to these two identities we arrive at

$$L_{4n+2}=3(2n+1)\left(\frac{5}{4}\right)^n\ {}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-n; \\ & \frac{9}{5} \\ \frac{3}{2}; \end{matrix}\right] \tag{3.35}$$

$$L_{4n}=2\left(\frac{5}{4}\right)^n\ {}_2F_1\left[\begin{matrix}-n,\frac{1}{2}-n; \\ & \frac{9}{5} \\ \frac{1}{2}; \end{matrix}\right] \tag{3.36}$$

Finally we use the following two quadratic transformation formulae

$${}_2F_1\left[\begin{matrix}a,b; \\ & z \\ a-b+1; \end{matrix}\right]=(1-z)^{-a}\ {}_2F_1\left[\begin{matrix}\frac{a}{2},\frac{a-2b+1}{2}; \\ & -\frac{4z}{(1-z)^2} \\ a-b+1; \end{matrix}\right] \tag{3.37}$$

$${}_2F_1\left[\begin{matrix}a,b; \\ & z \\ a-b+1; \end{matrix}\right]=\frac{1+z}{(1-z)^{a+1}}\ {}_2F_1\left[\begin{matrix}\frac{1+a}{2},\frac{a}{2}-b+1; \\ & -\frac{4z}{(1-z)^2} \\ a-b+1; \end{matrix}\right] \tag{3.38}$$

Applying these two to (2.3) we get

$$L_{4n+2}=3\ {}_2F_1\left[\begin{matrix}-n,n+1; \\ & -\frac{5}{4} \\ \frac{1}{2}; \end{matrix}\right] \tag{3.39}$$

$$L_{4n}=2\ {}_2F_1\left[\begin{matrix}-n,n; \\ & -\frac{5}{4} \\ \frac{1}{2}; \end{matrix}\right] \tag{3.40}$$

IV. Explicit Formulas

In this section we will simply rewrite the formulas obtained above in terms of combinatorial sums. Identities (3.8), (3.17) and (3.18) respectively lead to the sum

$$L_n=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{n}{n-k}\binom{n-k}{k} \tag{4.1}$$

$$L_{n+1} = \sum_{k=0}^{n} \frac{2n+1}{2k+1}\binom{n+k}{2k} \tag{4.2}$$

$$L_{2n} = 2\sum_{k=0}^{n} \frac{n}{n+k}\binom{n+k}{2k} \tag{4.3}$$

Again Identities (2.3), (3.40) and (3.5) give rise to

$$L_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} 5^k \tag{4.4}$$

$$L_{4n} = \sum_{k=0}^{n} \frac{2n}{n+k}\binom{n+k}{2k} 5^k \tag{4.5}$$

$$L_{2n} = \sum_{k=0}^{n} (-1)^{n+k} \frac{2n}{n+k}\binom{n+k}{2k} 5^k \tag{4.6}$$

Both (3.4) and (3.14) lead to the following identity

$$L_{2n+1} = \sum_{k=0}^{n} (-1)^{n+k} \binom{n+k}{2k} 5^k$$

(4.7)

Again both (3.30) and (3.39) lead to the following

$$L_{4n+2} = 3\sum_{k=0}^{n} \binom{n+k}{k} 5^k \tag{4.8}$$

Again from (3.23) and (3.27) we get respectively

$$L_{2n} = 2\left(\frac{3}{2}\right)^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} \left(\frac{5}{9}\right)^k \tag{4.9}$$

$$L_{2n} = 3^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k} 9^{\frac{1}{k}}$$

(4.10)

From the identities (3.19), (3.20),(3.12) and (3.13) we respectively arrive at

$$L_{2n+1} = \sqrt{5}\sum_{k=0}^{\infty} \binom{n+k+\frac{1}{2}}{2k+1}$$

(4.11)

$$L_{2n} = \sqrt{5}\sum_{k=0}^{\infty} \binom{n+k-\frac{1}{2}}{2k+1}$$

(4.12)

$$L_{2n+1} = (2n+1)\left(\frac{4}{5}\right)^{n+1} \sum_{k=0}^{\infty} \binom{2n+2k+1}{2k} \frac{1}{2k+1}\left(\frac{1}{5}\right)^k$$

(4.13)

$$L_{2n}=2\left(\frac{4}{5}\right)^{n}\sum_{k=0}^{\infty}\binom{2n+2k-1}{2k}\frac{1}{5^{k}}$$

(4.14)

Using (3.22),(3.24) and (3.25) respectively we arrive at

$$L_{2n+1}=(2n+1)\left(\frac{3}{2}\right)^{n}\sqrt{\frac{5}{6}}\sum_{k=0}^{\infty}\binom{n-\frac{1}{2}}{k}\frac{1}{2k+1}\left(\frac{5}{9}\right)^{k}$$

(4.15)

$$L_{2n+1}=(2n+1)\left(\frac{2}{3}\right)^{n+1}\sqrt{\frac{5}{6}}\sum_{k=0}^{\infty}\binom{n+2k+\frac{1}{2}}{2k}\frac{1}{2k+1}\left(\frac{5}{9}\right)^{k}$$

(4.16)

$$L_{2n}=2\left(\frac{2}{3}\right)^{n}\sum_{k=0}^{\infty}\binom{n+2k-1}{2k}\left(\frac{5}{9}\right)^{k}$$

(4.17)

V. Further Application

The Generalized Hypergeometirc Function is defined by

$${}_{p}F_{q}\left[\begin{matrix}\alpha_1,\alpha_2......,\alpha_p;\\ & ;z\\ \beta_1,\beta_2......,\beta_p;\end{matrix}\right]=1+\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_j)_n}\;\frac{z^n}{n!} \tag{5.1}$$

(see[6].p.73)

We use the following identity due to Clausent (see[8].p.180)

$$\left\{{}_{2}F_{1}\left[\begin{matrix}a,b;\\ & z\\ a+b+\frac{1}{2};\end{matrix}\right]\right\}^{2}={}_{3}F_{2}\left[\begin{matrix}2a,a+b,2b;\\ & z\\ a+b+\frac{1}{2},2a+2b;\end{matrix}\right] \tag{5.2}$$

Taking a= –n,b=1+n, $z=-\frac{1}{4}$ then (3.17) and (5.2) gives

$$L_{2n+1}^{2}=(2n+1)^{2}\;{}_{3}F_{2}\left[\begin{matrix}-2n,1,2+2n;\\ & -\frac{1}{4}\\ \frac{3}{2},2;\end{matrix}\right] \tag{5.3}$$

By way of (5.1) we get

$$L_{2n+1}^{2}=(2n+1)^{2}\sum_{k=0}^{2n}\binom{2n+k+1}{2k+1}\frac{1}{k+1} \tag{5.4}$$

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