Analysis of Modified Average Method for One Dimensional Non Linear Burgers Equation with Moving Mesh

Sachin S.Wani¹, Sarita Thakar²

¹ Department of Applied Mathematics, Pillai’s Institute Of Information Technology, Sector 16,Plot No.10,New Panvel ,Dist. Raigad,Navi Mumbai(MS) 410206, INDIA,
²Department of Mathematics, Shivaji University, Kolhapur (MS) 416004,INDIA

Abstract: A Moving Mesh method is proposed for numerical solution of one dimensional non linear Burgers Equation with homogeneous Dirichlets boundary conditions. Discretization of derivatives is obtained by forward Euler, Backward Euler and central difference formula. We proved that the method is stable in $L_2$ norm. Numerical solutions of one dimensional non linear Burgers equation obtained by Modified average method with Moving mesh are bounded. Numerical solutions are given in different domains for different values of $t$ and constant of diffusivity $k$.

AMS subject classification: 65M12

Keywords: Burgers equation, Moving mesh, Modified average Method, Discretization , Stability

I. Introduction

Moving mesh methods have important applications in a variety of physical and engineering areas such as solid and fluid dynamics, Combustion, heat transfer ,material science etc. Moving mesh method has become an important tool for computing singular or nearly singular problems such as interfaces, shock waves and boundary layers [1,2] and reaction diffusion systems in growing domains [3]. As the domain changes shape, it necessitates the use of a computational mesh that moves and deforms to the new spatial configuration. The numerical investigation of these problems may require extremely fine meshes over a small portion of the physical domain to resolve the large solution variations. Successful implementation of this strategy can increase the accuracy of the numerical approximation and decrease the computational cost.

Burgers equation is a natural first step towards developing methods for control of flows. The one dimensional non linear Burgers equation was first introduced by H. Bateman [4] who found its steady solutions descriptive of certain viscous flows. It was later proposed by J.M.Burger [5] as one of a class of equation describing mathematical model of turbulence. In the context of gas dynamics, it was discussed by E.Hopf [6] and J.D.Cole[7]. N Su (Australia) [8] demonstrates that the Nonlinear Burger’s equation can be mapped to the Burnoulli’s equation & Riccati equation respectively. For different convection terms for the analysis of water flows in soils undergoing, surface erosion on hill slopes, Burger’s equation is shown to be much easier to use without the need of including the unsaturated hyperbolic conductivity. In 2007 Idris Dag Ali Sahin [9] illustrate how the numerical solution of the Burger’s equation is obtained using the method of cubic B-spline collocation & quadratic B-spline Galerkin over the geometrically graded mesh. In 2005 Ronald E. Mckens[10] gives solutions of several different equation models of Burger’s equation for particular cases considered correspond to the diffusion free nonlinear steady –state & linear steady state situation. In 2008 [11] Alireza , Hashemian , Hossein M Shodja has proposed a new meshless method called gradient reproducing Kernel particle method for the numerical solution of 1-D Burger’s equation with various values of viscosity & different initial and boundary conditions . Discretization is first done in the space via GRKPM & subsequently the reduced system of nonlinear ordinary differential equation is discretized in time by the Gear’s Method. In 2009[12]Sachin wani and Sarita Thakar proved the stability and convergence of Mixed Euler method for one dimensional non linear Burgers equation.

The analysis of numerical methods using moving meshes has received limited attention.. Fezeria (1997) [13] analysed a moving mesh implicit Euler method that uses central finite differences to descretize the spatial derivatives of model linear convection reaction diffusion problems in 1-d using an energy type analysis . It was shown that the method was only conditionally stable depending on the temporal and spatial smoothness of the moving mesh. Ferrerra then went to show that the mesh would be sufficiently smooth if it was obtained by the equidistribution of a smooth monitor function . The result is rather surprising given that the method is fully implicit & is unconditionally stable when the mesh is stationary.

An important issue related to the use of moving mesh techniques to solve conservation laws is so called Geometric conservation Laws (GCL) which was first introduced by Thomas & Lambard (1979) [14]. In 1990 [15] R. M. Fuzzeland, J. G. Verwer gave their moving mesh techniques which is due to the moving finite
The governing equation (2.1) in the computational domain therefore takes the form

\[ u_t - ku_{xx} - xu_t + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2 \right) = 0 \]

\[ u = u_0, x \in \Omega_0, t > 0 \]
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

\[ u = 0, x \in \partial \Omega, t > 0 \] (2.2)

Now we will write the conservative form of (2.2)

\[ (x_u \cdot u) = \left( \frac{1}{2} u^2 + k \frac{u_x}{x} \right) = 0 \] (2.3)

III. Moving Mesh Discretizations

We consider the semi-discretization of (2.3) achieved using second order central finite difference approximations of the spatial derivatives of \( u \) and a discretization of the mesh velocity and meshmetric terms. We will assume that the domain \( \Omega = [x_j(t), x_{j+1}(t)] \) is covered by a nonuniform mesh of \( N \) cells with \( x_j(t) = x_0(t) < x_1(t) < \ldots < x_{N-1}(t) < x_N(t) = x_r(t) \)

The nonuniform moving mesh in physical space is assumed to be the image of a fixed uniform mesh covering the computational domain \( \Omega_0 = [0,1] \) via the mapping \( x(\xi, t) \), so that \( x_j(t) = x(\xi_j, t) = \left( \frac{j}{N}, t \right), j=0,1,\ldots,N \).

The measure of each physical cell will be denoted by \( h_j(t) = x_j(t) - x_{j-1}(t) \), \( j=1,2,\ldots,N \) and the midpoints of the cell are defined as \( x_{j-1/2}(t) = \frac{1}{2} (x_j(t) + x_{j-1}(t)) \).

The location of the physical mesh points at time level \( t = t_n \) and \( t = t_{n+1} \) is well defined given the mapping \( x(\xi, t) \). To obtain the numerical approximation of (2.3) we require an approximation \( x^h(\xi, t) \) of \( x(\xi, t) \). We will assume that \( x^h(\xi, t) \) is piecewise linear in space and time and hence

\[ x^h(\xi, t) = x_j^{1/2}(t) + (\xi - x_j^{1/2}) (x_{j+1/2}(t) - x_{j-1/2}(t)) \]

for \( \xi_j^{1/2} \leq \xi \leq \xi_{j+1/2} \), and

\[ x_j^{1/2}(t) = x_j^{1/2}(t_n) + (t - t_n) \left( \frac{x_{j+1/2}(t_n) - x_{j-1/2}(t_n)}{t_n} \right), t_n \leq t \leq t_{n+1} \]

Assuming this form for the mapping we have

\[ x^h_\xi(t_n) = \left( \frac{x_{j+1/2}(t_n) - x_{j-1/2}(t_n)}{h^\xi} \right) = \frac{1}{h^\xi} \left( \frac{h_j^n + h_j^{n+1}}{2} \right) \]

\[ \xi_j^{1/2} \leq \xi \leq \xi_{j+1/2} \] (3.1)

And

\[ x^h_{\xi_{j-1/2}}(t) = \left( \frac{x_{j+1/2}(t_n) - x_{j-1/2}(t_n)}{t_{n+1} - t_n} \right), t_n \leq t \leq t_{n+1} \] (3.2)

With these definitions it is easy to see that \( x^h(\xi, t) \) satisfies the DGCL, which states that

\[ \left( \dot{x}^h_\xi \right)_{j+1} = \left( \dot{x}^h_\xi \right)_j + \frac{t_{n+1} - t_n}{h^\xi} \left( \dot{x}^h_{\xi_{j+1/2}} - \dot{x}^h_{\xi_{j-1/2}} \right) \]

(3.3)

To define the semidiscetization of (2.3) we will use the notation \( u^n(j) \) to denote the approximation of \( u(x^h_\xi, t_n) \) and \( u^n = \left( u^n_0, u^n_1, \ldots, u^n_{N-1}, u^n_N \right) \). We will use the forward and backward divided differences

\[ (D^+ u)_j = \frac{u_{j+1} - u_j}{h_{j+1}}, (D^- u)_j = \frac{u_j - u_{j-1}}{h_j} \]

average operator \( (\ddot{u})_j = \frac{1}{2} \left( u_j + u_{j+1} \right) \) and the central divided difference \( \ddot{u}^n_j = \frac{u^n_{j+1} - u^n_{j-1}}{h_{j+1} + h_j} \)

Using the notations (3.1) and (3.2), the semidiscretization of (2.3) takes the form

\[ \text{www.iosrjournals.org} \]

26 | P a g e
(\chi^h_x u)_j = \frac{1}{\Delta x^2} \left[ (k(D_+ - D_-)u)_j + \chi^h_{j+1/2}(\partial u)_j - \frac{1}{2} \partial (\partial u^2)_j - \chi^h_{j-1/2}(\partial u)_j + \frac{1}{2} \partial (\partial u^2)_j \right] 

(3.4)

In section 4, we consider various temporal discretization of (3.3) and the stability of the resulting fully discrete schemes. The analysis will be carried out using the following mesh dependent norms. For the numerical solution, we use the \( L_2 \) norm (noting the Homogeneous boundary conditions)

\[
\|u\|_{n+\theta} = \left( \sum_{j=1}^{N-1} \left( \frac{h_j^\theta + h_j^\theta}{2} \right) u_j^n \right)^{1/2}
\]

(3.5)

\[
<u, v>_{n+\theta} = \sum_{j=1}^{N-1} \left( \frac{h_j^\theta + h_j^\theta}{2} \right) u_j^n v_j^n
\]

(3.6)

Approximations of the derivatives will be measured in the cell-based norm.

\[
<u, v>_{(n+\theta)} = \sum_{j=1}^{N-1} h_j^\theta v_j^n w_j^n
\]

(3.7)

\[
\|v\|_{n+\theta} = \left[ \sum_{j=1}^{N-1} (h_j^\theta)^2 v_j^n \right]^{1/2}
\]

(3.8)

\[
(\mu f)_j = \frac{f_{j+1} - f_{j-1}}{h_{j+1} + h_j}
\]

(3.9)

IV. Stability analysis of Modified Average Method using Moving Mesh

For one dimensional non linear Burgers equation forward Euler as well as Backward Euler method are unstable. These methods are unstable due to the nonlinear term \( (u^2)_x \). To avoid the nonlinearity, in Backward Euler method, we approximate \( (u^2)_x \), by central difference at \( t = t_n \). This gives Mixed Euler Method [23] which is superior than the existing methods. Therefore, it is natural to extend the finite difference scheme for moving mesh.

4.1 Modified Average method

Discretizing (3.4) using Mixed Euler temporal discretization and assuming that \( t_{n+1} - t_n = \Delta t \) yields the fully discrete scheme.

\[
(\chi^h_x u)^{n+1}_j = (\chi^h_x u)^n_j + \frac{\Delta t}{\Delta x^2} \left[ (k(D_+ - D_-)u)^n_j + \chi^h_{j+1/2}(\partial u)^n_j - \frac{1}{2} \partial (\partial u^2)^n_j - \chi^h_{j-1/2}(\partial u)^n_j + \frac{1}{2} \partial (\partial u^2)^n_j \right]
\]

(4.1)
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with and Modified Forward Euler scheme is

\[
(x_{t}^{\text{ME}})_{j}^{n+1} = (x_{t}^{\text{ME}})_{j}^{n} + \frac{\Delta t}{\Delta x} \left[ k \left( \frac{u_{j+1}^{n+1} - u_{j}^{n+1}}{h_{j+1}^{\text{ME}}} - \frac{u_{j}^{n+1} - u_{j-1}^{n+1}}{h_{j}^{\text{ME}}} \right) \right] + \chi_{j}^{h+1/2} (\delta u_{j}^{n+1})_{j+1/2} - \chi_{j-1/2}^{h} (\delta u_{j}^{n+1})_{j-1/2} \]

(4.2)

Let us consider a weighted combination of the modified ME and FE schemes of the form

\[
\theta (4.1) + (1 - \theta) (4.2)
\]

which gives

\[
(x_{t}^{\text{ME}})_{j}^{n+1} = (x_{t}^{\text{ME}})_{j}^{n} + \frac{\Delta t}{\Delta x} \left[ k \left( \theta \left( \frac{u_{j+1}^{n+1} - u_{j}^{n+1}}{h_{j+1}^{\text{ME}}} - \frac{u_{j}^{n+1} - u_{j-1}^{n+1}}{h_{j}^{\text{ME}}} \right) \right) + (1 - \theta) \left( \frac{u_{j+1}^{n} - u_{j}^{n}}{h_{j+1}^{\text{ME}}} - \frac{u_{j}^{n} - u_{j-1}^{n}}{h_{j}^{\text{ME}}} \right) \right] + \chi_{j}^{h+1/2} (\delta u_{j}^{n+1})_{j+1/2} - \chi_{j-1/2}^{h} (\delta u_{j}^{n+1})_{j-1/2} \]

(4.3)

We call scheme (4.3) as Modified Average scheme with Moving mesh.

For this scheme we have the following stability result.

**Theorem 4.1** The numerical solution of equation (2.1) obtained by equation (4.3) with

\[
\theta = \max_{j} \frac{(x_{t}^{\text{ME}})_{j}^{n+1}}{(x_{t}^{\text{ME}})_{j}^{n}}
\]

is bounded and the following a priori bound holds.

\[
\| u^{n+1} \|_{n+1}^2 = \| u^n \|_n^2 - 2k \Delta t (\theta D_{+} u^{n+1} + (1 - \theta) D_{-} u^n + (1 - \theta) D_{+} u^{n+1} + (1 - \theta) D_{-} u^n) \|_n^2

- \Delta t (\mu(u^n)^2, \theta u^{n+1} + (1 - \theta) u^n)_{n+\theta}
\]

(4.4)

**Proof:**- Multiplying through the Modified average Method (4.3) by \( \theta u_{j}^{n+1} + (1 - \theta) u_{j}^{n} \) we obtain

\[
\theta^2 (\overline{\text{ME}}) u_{j}^{n+1} + \theta (1 - \theta) (\overline{\text{ME}}) u_{j}^{n} + \theta (1 - \theta) (\overline{\text{FE}}) u_{j}^{n+1} + (\theta^2) u_{j}^{n} (\overline{\text{FE}})
\]

(4.5)

Where \( \overline{\text{ME}} \) and \( \overline{\text{FE}} \) denote the Modified Mixed Euler and Modified Forward Euler schemes.

To evaluate

\[
\sum_{j=1}^{N-1} (4.5)
\]

(4.6)

We first evaluate

\[
\sum_{j=1}^{N-1} (\overline{\text{ME}}) u_{j}^{n}
\]

Multiply (4.1) by \( u_{j}^{n} \) and sum over all interior nodes we obtain

\[
\sum_{j=1}^{N-1} (x_{t}^{\text{ME}})_{j}^{n+1} u_{j}^{n} = \sum_{j=1}^{N-1} (x_{t}^{\text{ME}})_{j}^{n} u_{j}^{n} + I + II
\]

Using the identity \( ab = \frac{1}{2} a^2 + \frac{1}{2} b^2 - \frac{1}{2} (a - b)^2 \)

(4.7)

to \( u_{j}^{n+1} u_{j}^{n} \) in l.h.s gives
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

\[
\frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1})^2 + \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^n)^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1} - u_j^n)^2 = \sum_{j=1}^{N-1} (x_j^h)^n (u_j^n)^2 + I + II
\]

Using (3.3) we get

\[
\frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1})^2 + \frac{\Delta t}{2\Delta x} \left( \dot{x}_{j+1/2}^h - \dot{x}_{j-1/2}^h \right) (u_j^n)^2
\]

\[= \sum_{j=1}^{N-1} (x_j^h)^n (u_j^n)^2 + \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1} - u_j^n)^2 + I + II
\]

Where

\[
I = k \frac{\Delta t}{\Delta x} \sum_{j=1}^{N-1} \left( \frac{u_{j+1}^{n+1} - u_j^{n+1} - u_j^{n+1} - u_{j-1}^{n+1}}{h_{j+1}^n} \right) u_j^n
\]

And

\[
II = \frac{\Delta t}{2\Delta x} \sum_{j=1}^{N-1} \left[ \dot{x}_{j+1/2}^h (u_j^{n+1} + u_{j+1}^{n+1}) - \dot{x}_{j-1/2}^h (u_j^{n+1} + u_{j-1}^{n+1}) - \frac{1}{2} (u_{j+1}^{n+1} - (u^n_{j-1} - u_{j-1}^{n+1})) \right] u_j^n
\]

to evaluate

\[
\sum_{j=1}^{N-1} (FE) u_j^{n+1}
\]

Multiply (4.2) by \(u_j^{n+1}\) and sum over all interior nodes we obtain

\[
\sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1})^2 = \sum_{j=1}^{N-1} (x_j^h)^n (u_j^n)^2 + III + IV
\]

\[\therefore \text{ using (3.3) we get}
\]

\[
\frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (u_j^{n+1})^2 = \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^n (u_j^n)^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^n (u_j^n - u_j^{n+1})^2
\]

\[= \frac{\Delta t}{2\Delta x} \sum_{j=1}^{N-1} \left\{ \dot{x}_{j+1/2}^h - \dot{x}_{j-1/2}^h \right\} (u_j^{n+1})^2 + III + IV
\]

Where

\[
III = k \frac{\Delta t}{\Delta x} \sum_{j=1}^{N-1} \left( \frac{u_{j+1}^{n+1} - u_j^n - u_j^n - u_{j-1}^{n+1}}{h_{j+1}^n} \right) u_j^{n+1}
\]

\[= -k \frac{\Delta t}{\Delta x} (D_u u^n, D_u u^{n+1})_{n+\theta}
\]

And

\[
IV = \frac{\Delta t}{2\Delta x} \sum_{j=1}^{N-1} \left[ \dot{x}_{j+1/2}^h (u_j^{n+1} + u_{j+1}^{n+1}) - \dot{x}_{j-1/2}^h (u_j^n + u_{j-1}^{n+1}) - \frac{1}{2} (u_{j+1}^{n+1} - (u^n_{j-1} - u_{j-1}^{n+1})) \right] u_j^{n+1}
\]

www.iosrjournals.org
To evaluate
\[ \sum_{j=1}^{N-1} (ME) u_j^{n+1} \]

Multiply (4.1) by \( u_j^{n+1} \) and sum over all interior nodes we obtain
\[ \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} = \sum_{j=1}^{N-1} (x_j h u_j)^n + I + II \]

Using 4.6
\[ \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} (u_j^{n+1})^2 = \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)^n \left[ (u_j^{n+1})^2 + (u_j^n)^2 - (u_j^{n+1} - u_j^n)^2 \right] + V + VI \]

Using 3.3
\[ \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} (u_j^{n+1})^2 = \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)^n \left[ \left( u_j^{n+1} - \frac{\Delta t}{\Delta x} \left( x_j^{h/2} - x_{j-1/2}^{h/2} \right) \right) (u_j^{n+1})^2 + \right. \]
\[ \left. + \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)^n (u_j^{n+1} - u_j^n)^2 \right] + V + VI \]

Where
\[ V = k \frac{\Delta t}{\Delta x} \left( \sum_{j=1}^{N-1} \frac{u_j^{n+1} - u_j^n}{h_j^{n+1}} - \sum_{j=1}^{N-1} \frac{u_j^{n+1} - u_j^{n-1}}{h_j^n} \right) \]
\[ = -k \frac{\Delta t}{\Delta x} (D_u u^{n+1}, D_u u^{n+1})_{n+\theta} \]

And
\[ VI = \frac{\Delta t}{2 \Delta x} \sum_{j=1}^{N-1} \left[ x_j^{h/2} (u_j^{n+1} + u_j^{n-1}) - x_{j-1/2}^{h/2} (u_j^{n+1} + u_j^{n-1}) - \frac{1}{2} (u_j^{n+1} - u_j^{n-1}) \right] u_j^{n+1} \]
\[ - \frac{\Delta t}{2 \Delta x} \sum_{j=1}^{N-1} \left( x_j^{h/2} - x_{j-1/2}^{h/2} \right) (u_j^{n+1})^2 - \frac{\Delta t}{2 \Delta x} (\mu u^n, u^{n+1})_{n+\theta} \]

To evaluate
\[ \sum_{j=1}^{N-1} (FE) u_j^n \]

Multiply (4.2) by \( u_j^n \) and sum over all interior nodes we obtain
\[ \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} u_j^n = \sum_{j=1}^{N-1} (x_j h u_j)^n + VII + VIII \]

Using 4.6 and 3.3
\[ \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} (u_j^{n+1})^2 + \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)^n \left( \frac{\Delta t}{\Delta x} \left( x_j^{h/2} - x_{j-1/2}^{h/2} \right) \right) (u_j^n)^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j h u_j)_{n+1} (u_j^{n+1} - u_j^n)^2 \]
\[ = \sum_{j=1}^{N-1} (x_j h u_j)^n u_j^n + VII + VIII \]

(4.11)

Where
\[ VII = k \frac{\Delta t}{\Delta x} \left\{ \sum_{j=1}^{N-1} \frac{u_j^{n+1} - u_j^n}{h_j^{n+1}} - \sum_{j=0}^{N-2} \frac{u_j^{n+1} - u_j^n}{h_j^{n+1}} \right\} \]
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

\[ Vlll = \frac{\Delta t}{2\Delta x} \sum_{j=1}^{N-1} \left[ x_j^{h+1/2} (u_j^n + u_{j+1}^n) - x_j^{h-1/2} (u_j^n + u_{j-1}^n) - \frac{1}{2} (u^2)_{j+1}^n - (u^2)_{j-1}^n \right] u_j^n \]

And

\[ \frac{\Delta t}{2\Delta x} \sum_{j=1}^{N-1} \left( x_j^{h+1/2} - x_j^{h-1/2} \right) (u_j^n)^2 - \frac{\Delta t}{2\Delta x} (\mu(u^2)_n, u^n)_n + \theta \left( \frac{\partial u}{\partial x} \right)_n \]

Noting that

\[ \theta^2 + 2\theta(1 - \theta) + (1 - \theta)^2 = 1 \]

and substituting (4.8),(4.9),(4.10) and (4.11) in (4.6) we get

\[ \frac{1}{2} \| u^{n+1} \|^2_{n+1} - \frac{1}{2} \| u^n \|^2_n + \frac{1}{2} (1 - \theta) \| u^{n+1} - u^n \|^2_{n+1} - \frac{\theta}{2} \| u^{n+1} - u^n \|^2_n - (1 - \theta)^2 k \Delta t \| D_x u^n \|^2_{n+\theta} \]

\[ \frac{\Delta t}{2} \frac{\partial u}{\partial x} \| D_x u^{n+1} \|^2_{n+\theta} - \frac{\Delta t \theta}{4} \left[ \| \mu(u^2)_n \|^2_{n+\theta} + \| u^{n+1} \|^2_{n+\theta} \right] \]

\[ \frac{\Delta t}{2} (1 - \theta)^2 (\mu(u^2), u^n)_{n+\theta} + IX + XI \]

(4.12)

Where

\[ IX = \theta (1 - \theta) \Delta t \sum_{j=1}^{N-1} \left( u_j^{n+1} - u_j^n \right)^2 \left[ \left( x_j^h \right)^{n+1} - \left( x_j^h \right)^n \right] \]

\[ X = \theta (1 - \theta) \Delta t \sum_{j=0}^{N-1} \left( \frac{u_j^{n+1} - u_j^n}{h_j^{n+1}} - \frac{u_j^{n+1} - u_j^{n+1}}{h_j^n} \right) u_j^{n+1} \]

\[ \frac{\theta (1 - \theta)}{4} \sum_{j=1}^{N-1} \left( (u^2)_{j+1}^n - (u^2)_{j-1}^n \right) u_j^n + \left( (u^2)_{j+1}^n - (u^2)_{j-1}^n \right) u_j^n \]

\[ \frac{\theta (1 - \theta) \Delta t}{4} \left( \mu(u^2)_n, u^n + u^{n+1} \right)_{n+\theta} \]

We can see from (4.12) that the method will be stable if we can ensure that the term IX is a nonpositive. As this term arises from the mesh movement and has no physical meaning it would be ideal if we could choose \( \theta \) so that it is zero.

We choose \( \theta = \max \left( \frac{\left( \left( x_j^h \right)^{n+1} \right)}{\left( \left( x_j^h \right)^n + \left( x_j^h \right)^{n+1} \right)} \right) \) as given in [22] so that IX becomes zero.

With this choice of \( \theta \) and \( \epsilon \) on simplifying X and XI and substituting in (4.12) we get (4.4)

**Theorem 4.2**

\[ \Lambda = [\langle \mu(x^2)^n, \theta \epsilon^{n+1} + (1 - \theta) \epsilon^n \rangle_{n+\theta} + \langle \mu(\epsilon x)^n, (1 - \theta) \epsilon^2 + \theta(2 - \theta) \epsilon^{n+1} \rangle_{n+\theta}] \geq 0 \]

then the scheme 4.3 is stable and

\[ \| \epsilon^{n+1} \|^2_{n+1} = \| \epsilon^n \|^2_n \]

\[ -2 k \Delta t \langle \theta D_x \epsilon^{n+1} + (1 - \theta) D_x \epsilon^n, \theta D_x \epsilon^{n+1} + (1 - \theta) D_x \epsilon^n \rangle_{n+\theta} - A \Delta t \]

(4.13)

**Proof** Let \( U \) be numerical solution and \( u \) be the exact solution of the difference scheme(4.3) then the error \( \epsilon = U - u \).Since U is the numerical solution of the scheme (4.3)
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

\[(x^u_j)^{n+1} = (x^u_j)^n + \frac{\Delta t}{\Delta x} \left\{ k \left[ \theta \left( \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{h_j^{n+1}} - \frac{U_j^{n+1} - U_j^{n+1}}{h_j^n} \right) + (1 - \theta) \left( \frac{U_{j+1}^n - U_j^n - U_j^{n-1}}{h_j^{n+1}} + \frac{U_j^n - U_j^{n-1}}{h_j^n} \right) \right] \right\}

+ \frac{\Delta t}{\Delta x} \left\{ k \left[ \frac{U_j^n}{h_j^{n+1}} \left( \theta (\delta U_j^{n+1})_{j+1/2} + (1 - \theta)(\delta U_j^{n+1})_{j+1/2} \right) - \frac{U_j^n}{h_j^n} \left( \theta (\delta U_j^{n+1})_{j-1/2} - (1 - \theta)(\delta U_j^{n+1})_{j-1/2} \right) \right] \right\} \right\}

\[(x^u_j^{n+1} = (x^u_j^{n+1})^n + \theta \left( \frac{x_j^{n+1} - x_j^n}{\Delta x} \right)^n + (1 - \theta) \left( \frac{x_j^n - x_j^{n-1}}{\Delta x} \right)^n \]

\[= \left\{ \frac{\theta}{\Delta x} \left[ x_j^{n+1} + (1 - \theta) x_j^n \right] + \frac{\Delta t}{\Delta x} \left\{ k \left[ \theta \left( \frac{x_j^{n+1} - x_j^n}{\Delta x} \right)^n + (1 - \theta) \left( \frac{x_j^n - x_j^{n-1}}{\Delta x} \right)^n \right] \right\} \right\}

Since \( U = u + \varepsilon \) we get

\[(x^u_j^{n+1} = (x^u_j^{n+1})^n + \theta \left( \frac{x_j^{n+1} - x_j^n}{\Delta x} \right)^n + (1 - \theta) \left( \frac{x_j^n - x_j^{n-1}}{\Delta x} \right)^n \]

The exact solution must satisfy difference equation 4.3 on subtracting equation (4.3) from (4.14) we get an error equation

\[(x^u_j^{n+1} = (x^u_j^{n+1})^n + \theta \left( \frac{x_j^{n+1} - x_j^n}{\Delta x} \right)^n + (1 - \theta) \left( \frac{x_j^n - x_j^{n-1}}{\Delta x} \right)^n \]

\[= \left\{ \frac{\theta}{\Delta x} \left[ x_j^{n+1} + (1 - \theta) x_j^n \right] + \frac{\Delta t}{\Delta x} \left\{ k \left[ \theta \left( \frac{x_j^{n+1} - x_j^n}{\Delta x} \right)^n + (1 - \theta) \left( \frac{x_j^n - x_j^{n-1}}{\Delta x} \right)^n \right] \right\} \right\}

Multiplying 4.15 by \( \theta x_j^{n+1} + (1 - \theta)x_j^n \) we obtain

\[\theta^2 (EME) x_j^{n+1} + \theta (1 - \theta)(EME) x_j^n + \theta (1 - \theta)(EFE) x_j^{n+1} + (1 - \theta)^2 x_j^n (EFE) \]

Where \( EME \) and \( EFE \) denote the error equation of Modified Mixed Euler and Modified forward Euler respectively. To evaluate

\[\sum_{j=1}^{N-1} (4.16) \]

We first evaluate

\[\sum_{j=1}^{N-1} (EME) x_j^n \]
Multiply error equation of Modified Mixed Euler by \( e_j^{n+1} \) and sum over all interior nodes we obtain

\[
\sum_{j=1}^{N-1} (x_j^h e_j^{n+1}) e_j^n = \sum_{j=1}^{N-1} (x_j^h e_j^n) e_j^n + I + II + III
\]

Using the identity 4.6 to \( E_j^{n+1} E_j^n \) in l.h.s gives

\[
\frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^{n+1})^2 + \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^n)^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^{n+1} - e_j^n)^2 = \sum_{j=1}^{N-1} (x_j^h)^n (e_j^n)^2 + I + II + III
\]

Using 3.3

\[
\begin{align*}
\frac{1}{2} & \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^{n+1})^2 + \frac{\Delta t}{2\Delta \xi} (\delta x_j^h)^{n+1} (\delta e_j^{n+1})^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^{n+1} - e_j^n)^2 = \sum_{j=1}^{N-1} (x_j^h)^n (e_j^n)^2 + I + II + III
\end{align*}
\]

(4.18)

\[
I = k \frac{\Delta t}{\Delta \xi} \sum_{j=1}^{N-1} \left( \frac{e_j^{n+1} - e_j^n}{h_j^{n+1}} - \frac{e_j^{n+1} - e_j^{n-1}}{h_j^n} \right) e_j^n
\]

\[
II = \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left[ \delta (u e_j)^{n+1} - \delta (u e_j)^{n-1} \right] e_j^n
\]

\[
III = -\frac{\Delta t}{\Delta \xi} (\mu e_j^n) e_j^n + e_j^n
\]

To evaluate

\[
\sum_{j=1}^{N-1} (\Delta F E) e_j^{n+1}
\]

Multiply error equation of Modified Forward Euler by \( e_j^{n+1} \) and sum over all interior nodes we obtain

\[
\sum_{j=1}^{N-1} (x_j^h e_j^{n+1}) e_j^{n+1} = \sum_{j=1}^{N-1} (x_j^h e_j^n) e_j^{n+1} + IV + VI
\]

\[
\begin{align*}
\frac{1}{2} & \sum_{j=1}^{N-1} (x_j^h)^{n+1} (e_j^{n+1})^2 = \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^n (e_j^n)^2 - \frac{1}{2} \sum_{j=1}^{N-1} (x_j^h)^n (e_j^n - e_j^{n+1})^2
\end{align*}
\]

\[
- \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} (x_j^h)^{n+1} (x_j^h - x_j^{n+1/2}) (e_j^{n+1})^2 + IV + VI
\]

(4.19)

Where

\[
IV = k \frac{\Delta t}{\Delta \xi} \sum_{j=1}^{N-1} \left( \frac{e_j^{n+1} - e_j^n}{h_j^{n+1}} - \frac{e_j^n - e_j^{n-1}}{h_j^n} \right) e_j^{n+1}
\]

\[
=-k \frac{\Delta t}{\Delta \xi} (D e^n, D e^{n+1})_{n+\theta}
\]
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

And

\[ V = \frac{\Delta t}{\Delta \xi} \sum_{j=1}^{N-1} \left( \frac{\chi_j^{n+1/2}(\delta e)_j^{n+1/2} - \chi_j^{n-1/2}(\delta e)_j^{n-1/2}}{2} - \frac{\delta(e^2)_j^{n+1/2} - \delta(e^2)_j^{n-1/2}}{2} \right) e_j^{n+1} \]

\[ = \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left( \chi_j^{n+1/2}(e_j^{n+1} + e_j^n) - \chi_j^{n-1/2}(e_j^{n+1} + e_j^{n-1}) - \frac{1}{2} \left( (e^2)_j^{n+1} - (e^2)_j^{n-1} \right) \right) e_j^{n+1} \]

\[ VI = \frac{\Delta t}{\Delta \xi} \sum_{j=1}^{N-1} \left( \delta(ue)_j^{n+1/2} - \delta(ue)_j^{n-1/2} \right) e_j^{n+1} \]

\[ = \frac{\Delta t}{\Delta \xi} \langle \mu(ue)^n, e^{n+1} \rangle_{n+\theta} \]

To evaluate

\[ \sum_{j=1}^{N-1} (EME) e_j^{n+1} \]

Multiply error equation by \( e_j^{n+1} \) and sum over all interior nodes we obtain

\[ \sum_{j=1}^{N-1} (x_j^n e_j^n) e_j^{n+1} = \sum_{j=1}^{N-1} (x_j^n e_j^n) e_j^{n+1} + VII + VIII + IX \]

Where

\[ VII = k \frac{\Delta t}{\Delta \xi} \left( \sum_{j=1}^{N-1} \frac{e_j^{n+1} - e_j^n}{h_{j+1}^n} e_j^{n+1} - \sum_{j=1}^{N-1} \frac{e_j^{n+1} - e_j^{n-1}}{h_{j}^n} e_j^{n+1} \right) \]

\[ = -k \frac{\Delta t}{\Delta \xi} \langle D^+, e^{n+1} \rangle_{n+\theta} \]

And

\[ VIII = \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left( \chi_j^{n+1/2}(e_j^{n+1} + e_j^n) - \chi_j^{n-1/2}(e_j^{n+1} + e_j^{n-1}) - \frac{1}{2} \left( (e^2)_j^{n+1} - (e^2)_j^{n-1} \right) \right) e_j^{n+1} \]

\[ = \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left( (x_j^{n+1/2} - x_j^{n-1/2}) (e_j^{n+1})^2 - \Delta t \langle \mu(e^2)^n, e^{n+1} \rangle_{n+\theta} \right) \]

\[ IX = \frac{\Delta t}{\Delta \xi} \sum_{j=1}^{N-1} \left( \delta(ue)_j^{n+1/2} - \delta(ue)_j^{n-1/2} \right) e_j^{n+1} \]

\[ = -\frac{\Delta t}{\Delta \xi} \langle \mu(ue)^n, e^{n+1} \rangle_{n+\theta} \]

To evaluate

\[ \sum_{j=1}^{N-1} (EFE) e_j^n \]

Multiply error equation by \( e_j^n \) and sum over all interior nodes we obtain

\[ \sum_{j=1}^{N-1} (x_j^n e_j^n) e_j^n = \sum_{j=1}^{N-1} (x_j^n e_j^n) e_j^n + X + XI + XII \]

where

\[ X = k \frac{\Delta t}{\Delta \xi} \left( \sum_{j=1}^{N-1} \frac{e_j^{n+1} - e_j^n}{h_{j+1}^n} e_j^n - \sum_{j=0}^{N-2} \frac{e_j^{n+1} - e_j^{n+1}}{h_{j+1}^n} e_j^{n+1} \right) \]
Analysis of Modified Average Method For One Dimensional Non Linear Burgers Equation with

\[-k \frac{\Delta t}{\Delta \xi} (D_{\xi} e^n, D_{\xi} e^n)_{n+\theta}\]

And

\[X_I = \frac{\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left( \frac{x_j^{h+1/2} (e^n + e^n_{j+1}) - x_j^{h-1/2} (e^n_{j-1} + e^n_{j})}{\Delta \xi} - \frac{1}{2} \left( (e^2)^{n+1} - (e^2)^{n-1} \right) \right) e_j^n\]

\[= \frac{-\Delta t}{2\Delta \xi} \sum_{j=1}^{N-1} \left( \frac{\Delta(x_j)^{n+1/2} - \Delta(x_j)^{n-1/2}}{\Delta \xi} \right) e_j^n\]

Proceeding same way as in theorem 4,1 and substituting (4.18),(4.19),(4.20) and (4.21) in (4.17) we get 4.13

V. Numerical Experiments and analysis

Numerical solution of 1-dimensional nonlinear Burgers equation are obtained by Modified Average Method with moving mesh method by the difference formula (4.3). The solutions are obtained for k=1 and k=0.5 for t=0.1,0.15,0.2 with \(\Delta t = 0.001\) \& \(\Delta \xi = 0.01\) for different domains. The graphs of solution in these domains are given for the analysis.

Example 1:- To test the analysis, we consider the solution of (2.1) in the domain \(x_l(t) = 1 - e^{t/2}\) and \(x_r(t) = e^{t/2}\) with Homogeneous Dirichlet boundary conditions. We use a set of mesh points which are uniformly spaced between \(x_l(t)\) & \(x_r(t)\). The initial condition is taken to be \(u(x,0) = u_0(x) = \sin \pi x\) and we choose k=1 and k=0.5 with \(\Delta t = 0.001\) and \(\Delta \xi = 0.01\). Figure 1 and 2 shows the solution of (2.1) for different values of t.

Example 2 Consider the one dimensional non linear Burgers equation (2.1) in the domain \(x_l(t) = 0\) and \(x_r(t) = e^{2t}\) with Homogeneous Dirichlet boundary conditions and the initial condition \(u_0(x) = \frac{x_r - x}{x_r - x_l}\). We choose \(\Delta t = 0.001\) and \(\Delta \xi = 0.01\). The numerical solution of above equation with given initial and boundary conditions are shown in figure 1 and figure 2 for k=1 and k=0.5 respectively for different values of t.
VI. Conclusion

A Modified Average method with Moving Mesh is constructed for 1-D nonlinear Burgers equation with Homogeneous Dirichlets boundary conditions. For this method discretization of the terms containing linear differentials are approximated by backward difference where as the nonlinear terms by central difference in state variable. The solutions are proved to be bounded and the method is made stable in $L_2$ norm. The boundedness of error is achieved. Numerical solutions are obtained for different domains.

References: