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Free Transverse Vibrational Analysis of Shearing Effects on Rectangular Prismatic Beams

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Abstract*: Beams are structural elements that primarily resist loads applied laterally by bending. They exhibit dynamic behaviours and hence the need to predict their dynamic characteristics for proper structural design. In the cause of some disturbances, beams start to vibrate which may result to some shear deformation on the fundamental frequency. Analysis of shearing effects on beams resulting from vibration is necessary in order to know how the member behaves. With this knowledge one can predict and avoid structural element from failure. Hence, this paper investigates shearing effects on free transverse vibration of beams. Classical method was adopted to determine the natural frequencies of beams with uniformly distributed mass along its length. Differential equations were formulated to describe the dynamic behaviour of Euler-Bernoulli beam and for when shear effects were considered for elastic beams. These governing equations were solved by applying boundary conditions of the beam. The beams considered are; simply supported beams, fixed-ended beams and propped cantilever beams. The shear effects on the fundamental natural frequencies of these beams were evaluated and result showed that in long beams, shear can be ignored with negligible effects in design whereas in short deep beams, the shearing effects are significant and have great influence on the frequency of such beam.*

Keywords: *Prismatic Beams, Shear Deformation, Vibration, Natural Frequency, Slenderness, Resonance.*

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I. Introduction

Beams are basic structural members that are widely used in the fields of mechanical, aeronautical, and civil engineering. Because the dynamic behaviour of a beam is of great importance in engineering, it is necessary to be able to accurately predict the dynamic characteristics of beams. The free vibration characteristics of isotropic beams have been investigated by many researchers. The fundamental vibration behaviour of the slender beams can be studied using the classical Bernoulli–Euler beam theory. However, this theory can lead to a significant over-prediction of the natural frequencies when it is used to study either thick beams or the higher-order vibrational modes. This is because the effects of shear deformation and rotary inertia are ignored in this theory. An improved beam theory is the Timoshenko beam theory in which both the transverse shear deformation and rotary inertia effects are included [1]. He demonstrated the free vibration of laminated beams by using the analytical solutions of the governing equations of beams formulated for a generally layered composite beam on the basis of third-order shear deformation theory.

Structural Engineering has been evolving continuously in some areas of emerging significance. This area of emerging significance is the shearing effects in free vibration of prismatic beams. Most often, the behaviour of structures or structural elements in static mode has been considered. However, there comes the possibility when the dynamic response of a structure need be considered as result of vibration effects. These effects can be from free or forced vibration [2]. Beams are structural elements that primarily resist loads applied laterally by bending and exhibit dynamic behaviours that make it very important in engineering, hence the need to predict their dynamic characteristics for proper structural design [3]. In a vibrating structure, the dynamic force arises from vibration of the element mass of the structure in which the mass and deformation properties are uniformly distributed along its length [4]. If the structure is sufficiently flexible, very large forces can be developed as a result of apparently small vibratory forces.

Timoshenko was the first to develop and consider the vibrational theory of shear effects in beam analysis. This theory requires the determination of a well-known shear correction factor, which is the ratio of the shear strain within the cross-section to the shear strain at the centre of the section [5] and [6] calculated the shear correction factor for a variety of cross-sections of the beams.

In Timoshenko beams of rectangular cross-section, the investigation showed that the shear correction factor is a factor of the aspect ratio of the beam [7]. Euler-Bernoulli's beam theory is the most commonly used because it is simple and provides reasonable engineering approximations for many problems but it slightly over-estimates the natural frequency of the system. Though the Euler-Bernoulli formulation for a thin beam vibration at relatively low frequency is sufficient, the effects of shear as well as rotary inertia are not negligible for thick beams or even thin beams that are vibrating at high frequencies [8] [9]. The proneness of a structure to vibratory forces is assessed by comparing natural frequencies of the structure with frequencies of the vibratory force, the fundamental frequency of a structure being the most important of other frequencies. In analysis of structures subjected to static forces only, the forces acting on the structure are in general independent of deformations of the structure provided that displacements are small [2]. The fundamental frequencies of the beam types were calculated and used to check the shear effect on free vibration of the element and the occurrence of resonance.

The shearing effects on large amplitude vibration of beams were investigated and showed that shear has significant effects on thick and short beams [10] [11]. Zohoor and Kakavand [12], in their work recommended Timoshenko model that shows the significance of shear effects on structures. In practice, it is common that these shearing effects can be safely neglected with little or no error for beams of normal proportion or long beams and shear included for short deep beams. But in principle, it is better to check these shear effects on beams (either long or short) to eliminate any form of failure. This paper shows the effects of shear on long and short prismatic beams as it is freely vibrating.

II. System Description and Formulation of Governing Equation

Considering a beam subjected to action of time dependent loads which will result in dynamic response. The cross-section of the beam was small in comparison with its length. Considering the transverse vibration of the prismatic beam in x-y plane as in Fig.1, the beam is subjected to time-dependent loads which will result in dynamic response. If the load is removed, the beam will commence to vibrate on its own. Since the beam is un-excited by an external force, the vibration is free.

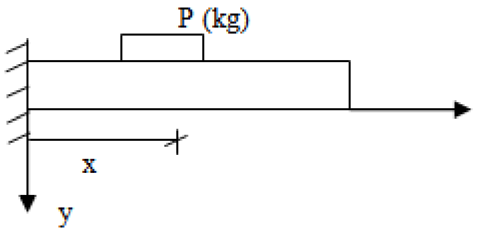

Fig.1

Cutting an element of this beam and showing the free body diagram under free vibration, of length dx, with internal and inertial actions upon it as in Fig. 2;

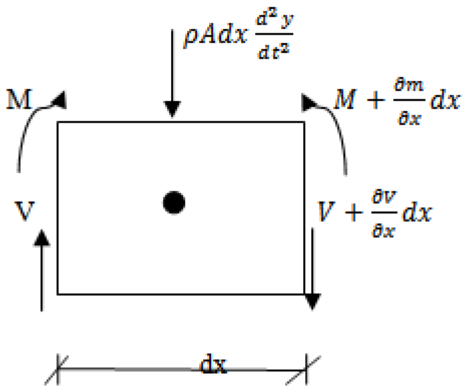

Fig.2

From Fig.2, applying Newton's second law of motion, the equilibrium condition of forces in y-direction when vibrating transversely is;

$$\rho A \frac{\partial^2 y}{\partial t^2} = \frac{\partial V}{\partial x} \qquad 1$$

Where: ρ is mass density of the beam
A is cross-sectional area of the beam.
M is moment acting on the beam.
V is shear acting on the beam.

And considering the moment equilibrium condition from centre of gravity of the beam element in Fig.2 gives;

$$M - (M + \partial M) + V\frac{dx}{2} + (V + \partial V)\frac{dx}{2} = 0 \qquad 2$$

Expanding and solving Eq.2 gives;

$V = \frac{\partial m}{\partial x}$ 3

From Euler-Bernoulli equation for bending of beam;

$M = -EI\frac{\partial^2 y}{\partial x^2}$ 4

Referencing Eq. 4, Eq.3 and then substituting into Eq.1 gives the basic partial differential equation governing flexural transverse free vibration of a prismatic beam when shear is ignored as;

$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = 0$ 5

When shear stress develops on beam as shown in Fig.3, this shear is assumed to be uniformly distributed over the cross-section.

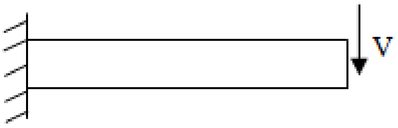

Fig.3

$from\ this, the\ total\ slope\ '\theta'\ = \frac{dy}{dx}$ 6

This slope comprises of bending slope 'θ_b' and shear slope 'θ_s' acting on the beam, i.e;

$\theta = \frac{dy}{dx} = \theta_b + \theta_s$ 7

$Shear\ stress\ '\tau'\ = Shear\ strain\ '\gamma' = \frac{V}{GA}$ 8

But most times, shear stress is not usually uniformly distributed over the cross-section then, Fig.4;

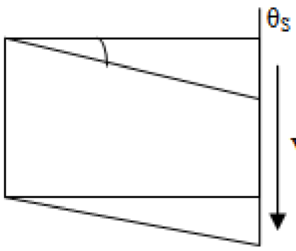

Fig.4

$Shear\ stress, \theta s = \frac{V}{KGA}$ 9

Where G = modulus of elasticity in shear

K = Timoshenko shear coefficient and a factor that depends on the shape of the cross-section. For a rectangular cross-section used, k = 0.833.

Using Eq.7 and Eq.9 and the elementary flexure theory, the modified bending moment expression due to shear gives;

$M = EI\left[-\frac{\partial^2 y}{\partial x^2} + \frac{1}{KGA}\frac{\partial v}{\partial x}\right]$ 10

Consequently, substituting Eq.3 into Eq.10, gives;

$EI\frac{\partial^2 y}{\partial x^2} - \frac{EI}{KGA}\frac{\partial^2 m}{\partial x^2} + M = 0$ 11

Referring to Eq.1;

$\rho A\frac{\partial^2 y}{\partial t^2} = \frac{\partial v}{\partial x} = \frac{\partial^2 m}{\partial x^2}$ 12

Differentiating Eq.11 twice and substituting Eq.12 and then dividing with 'EI' gives the equations;

$\frac{\partial^4 y}{\partial x^4} - \frac{\rho}{KG}\frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\rho A}{EI}\frac{\partial^2 y}{\partial t^2} = 0$ 13

Equation 13 is the partial differential equation that governs the transverse free vibration of prismatic beams with shear effects.

III. Solution of the Equation Of Motion

To solve Eq. 13 which is the partial differential equation that governs the transverse free vibration of prismatic beams with shear effects, assume that the motion is simple harmonic;

$y = y(x,t) = Y(x)cos\omega_n t$ 15

Where; ω_n = Angular or circular frequency. Differentiating Eq. 15 and substituting into Eq.13 gives the ordinary differential equation governing the vibration of a beam when shear is considered as;

$$\frac{d^4y}{dx^4}+\frac{\rho\omega_n{}^2}{KG}\frac{d^2y}{dx^2}-\frac{\rho A\omega^2{}_n}{EI}y=0 \qquad 16$$

Assume that

$$\alpha^4=\frac{\rho A\omega^2{}_n}{EI} \qquad 17$$

Referencing Eq.17,

$$\omega_n={}^{\alpha^2}\sqrt{\frac{EI}{\rho A}} \qquad 18$$

Also form Eq.17,

$$\rho\omega^2{}_n=\frac{EI\alpha^4}{A} \qquad 19$$

Calling from Eq.16,

$$\frac{\rho\omega^2{}_n}{KG}=\frac{EI\alpha^4}{KGA} \qquad 20$$

From Eq.20, let;

$$\frac{EI}{KGA}=2f^2 \qquad 21$$

Where f is a factor accounting for shear in the beam;

$$f=\sqrt{\frac{EI}{2KGA}} \quad \text{And} \quad \frac{\rho\omega^2{}_n}{KG}=2f^2\alpha^4 \qquad 22$$

Using these expressions and substituting into Eq.16 gives;

$$y^{iv}+2f^2\alpha^4y''-\alpha^4y=0 \qquad 23$$

To solve Eq. 23, assume that $y=e^{\lambda x}$, the ordinary differential equation (ODE) becomes;

$$\lambda^4+2f^2\alpha^4\lambda^2-\alpha^4=0 \qquad 24$$

This quadratic equation in 'λ' gives the solution that $\lambda_{1,2}=\pm\beta_1$ and $\lambda_{3,4}=\pm\beta_2 i$, where $i=\sqrt{-1}$.
Finally,

$$\beta_1=\alpha\left(\sqrt{1+f^4\alpha^4}-\alpha^2f^2\right)^{1/2} \qquad 25$$

$$\beta_2=\alpha\left(\sqrt{1+f^4\alpha^4}+\alpha^2f^2\right)^{1/2} \qquad 26$$

Then the differential equation will have the general solution as;

$$y(x)=D_1{}^1e^{\beta_1x}+D_2{}^1e^{-\beta_1x}+D_3{}^1e^{\beta_2ix}+D_4{}^1e^{-\beta_2ix} \qquad 27$$

Since;

$$sinh\beta x=\frac{e^{\beta x}-e^{-\beta x}}{2} \quad \text{And} \quad cosh\beta x=\frac{e^{\beta x}+e^{-\beta x}}{2} \qquad 29$$

From Eq. 29, it can be shown that;

$$e^{\beta x}=cosh\,\beta x+sinh\,\beta x \quad \text{And} \quad e^{-\beta x}=cosh\,\beta x-sinh\,\beta x \qquad 30$$

The Euler's formulae for complex analysis gives;

$$e^{\beta ix}=cos\,\beta x+isin\,\beta x \quad \text{And} \quad e^{-\beta ix}=cos\,\beta x-isin\,\beta x \qquad 31$$

Putting Eq.30 and 31 into Eq.27 gives the expression to determine or define the mode shape of a vibrating beam when shear is considered as;

$$Y(x)=D_1\cosh\beta_1x+D_2\sinh\beta_1x+D_3\cos\beta_2x+D_4\sin\beta_2x \qquad 32$$

From Eq.11 and Eq.12,

$$M=-EI\frac{\partial^2y}{\partial x^2}-\frac{\rho EI}{KG}\frac{\partial^2y}{\partial t^2} \qquad 33$$

Eq.33 is a partial differential equation. In solving Eq.33, to ordinary differential equation, let's say;

$$M(x,t)=M(x)cosw_nt \qquad 34$$

$$Y(x,t)=Y(x)cosw_nt \qquad 35$$

Substitute Eq.34 and 35 into Eq.33 gives that;

$$M(x)=-EI\frac{d^2y}{dx^2}-\frac{\rho EI\omega^2{}_n}{KG}Y \qquad 36$$

From Eq.22, let; $\Phi=\frac{\rho\omega^2{}_n}{KG}=2f^2\alpha^4$ 37

Then with Eq. 37 and referencing Eq.36;

$$M(x)=-EI(Y''+\Phi Y) \qquad 38$$

$$V(x)=-EI(Y'''+\Phi Y') \qquad 39$$

Eq. 38 and 39 is applied to determine the effects of shear on vibration of any given beam using boundary condition of the beam. Generally calling up Eq.32 which is,

$Y(x) = D_1 \cosh\beta_1 x + D_2 \sinh\beta_1 x + D_3 \cos\beta_2 x + D_4 \sin\beta_2 x$

$Y'(x) = \beta_1 D_1 \sinh\beta_1 x + \beta_1 D_2 \cosh\beta_1 x - \beta_2 D_3 \sin\beta_2 x + \beta_2 D_4 \cos\beta_2 x$ 40

$Y''(x) = \beta_1^2 (D_1 \cosh\beta_1 x + D_2 sinh\beta_1 x) - \beta_2^2 (D_3 \cos\beta_2 x + D_4 \sin\beta_2 x)$ 41

$Y'''^{(x)} = \beta_1^3 (D_1 \sinh\beta_1 x + D_2 cosh\beta_1 x) + \beta_2^3 (D_3 \sin\beta_2 x - D_4 \cos\beta_2 x)$ 42

Referencing Eq.38 and 39, Eq.41 and 42 which represent the governing equations when shearing effect is accounted for in a beam can be written as.

$Y''^{(x)} = -EI[(\beta_1^2 + \Phi)(D_1 \cosh\beta_1 x + D_2 sinh\beta_1 x) + (\Phi - \beta_2^2)(D_3 \cos\beta_2 x + D_4 \sin\beta_2 x)]$ 43

$Y'''^{(x)} = -EI[\beta_1(\beta_1^2 + \Phi)(D_1 \sinh\beta_1 x + D_2 cosh\beta_1 x) + \beta_2(\Phi - \beta_2^2)(-D_3 \sin\beta_2 x + D_4 \cos\beta_2 x]$ 44

IV. Solution of Beam Problems

In solving any given beam problem, essential boundary conditions must be applied to the formulated equations. For this analysis, the beam cases to consider include; simply supported beam, fixed ended beam, propped cantilever beam.

Simply Supported Beam (SSB)

The boundary conditions are

At x = 0; y(0) = 0 m(0) = 0

At x = L y(L) = 0 m(L) = 0

Fixed Ended Beam (FEB)

The boundary conditions are

At x = 0; y(0) = 0 θ(0) = 0

At x = L; y(L) = 0 θ(L) = 0

Propped Cantilever Beam (PCB)

The boundary conditions are

At x = 0; y(0) = 0 θ(0) = 0

At x = L; y(L) = 0 m(L) = 0

Using the boundary conditions and with Eq. 32, 40, 43 and 44, gives the characteristic equation for the vibration of the beam. Consequently, a factor 'S' that indicates the influence of shear on the beam is obtained as;

$$S = \frac{1}{\sqrt{1+\frac{2F^2n^2\pi^2}{L^2}}} \quad 45$$

Solving for the numerical values of shear on a beam, let $\frac{F}{L}$ be a parameter for evaluating shearing influence on the frequency, using a rectangular beam for illustration;

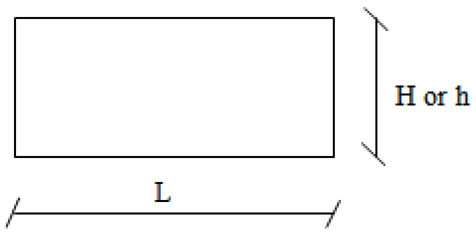

Fig.5: A rectangular beam section

$I = \frac{bh^3}{12}$, $A = b$h , $\frac{E}{G} = 2(1 + \nu)$

Where ν = 0.25 is the Poisson's ratio, k = 0.833 for a rectangular cross-section

Dividing Eq.22 with 'L' and substituting the above derivatives results to:

$$\frac{F}{L} = 0.354\frac{H}{L} \quad 46$$

Using trial and error method to solve for the value of $\frac{F}{L}$ and also using Eq.25 and 26 and then substituting in the characteristic equation to find out if it is satisfied. The numerical effects of shear on rectangular beams are presented in tables below assuming; L = 10m, H = 1m.

V. Results

Table 1: Aspect Ratio (H/L) and Shear Factor (F/L) effects on Beam length

Length (m)	10	8	6	4	2	1
H/L ratio	0.100	0.125	0.167	0.250	0.500	1.000
F/L factor	0.020	0.040	0.060	0.100	0.200	0.400

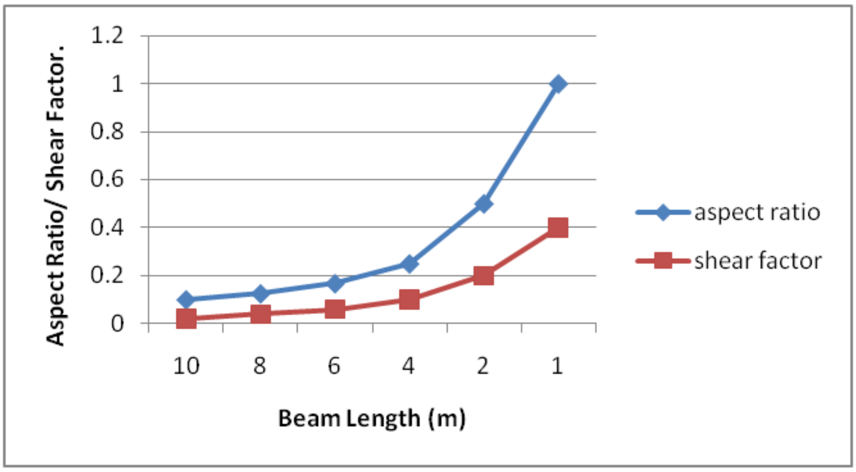

Fig.6: graph of Aspect ratio and Shear Factor on the beam length.

Table 2: The Factor 'S' Indicating Shear Influence on the Beams

	BEAM TYPE		
	SSB	PCB	FEB
SHEAR FACTOR 'S' AT ASPECT RATIO:			
0.100	0.996	0.995	0.995
0.125	0.985	0.982	0.981
0.167	0.966	0.961	0.958
0.250	0.914	0.901	0.895
0.500	0.748	0.720	0.708
1.000	0.491	0.459	0.446

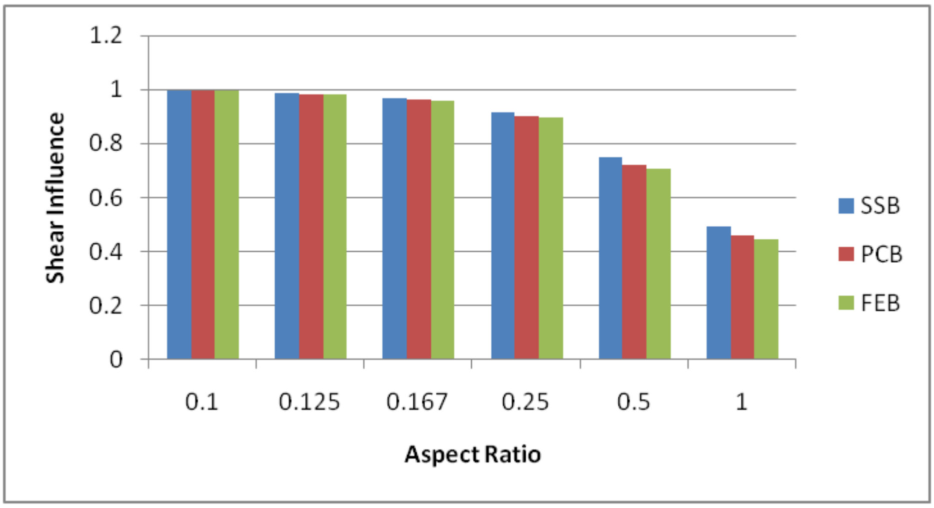

Fig.7: Chart showing shear influence on the beam with respect to the Aspect ratio.

Table 3: beams fundamental frequencies when shear is ignored.

	BEAM TYPE FUNDAMENTAL FREQUENCY		
	SSB	PCB	FEB
Shear ignored	9.869	15.421	22.373

Table 4: The Beams Fundamental Frequencies with Shear Effects.

	BEAM TYPE FUNDAMENTAL FREQUENCIES WITH SHEAR EFFECTS		
	SSB	PCB	FEB
Aspect Ratio:			
0.100	9.830	15.343	22.264
0.125	9.722	15.141	21.945
0.167	9.534	14.816	21.442
0.250	9.021	13.896	20.033
0.500	7.382	11.099	15.829
1.000	4.846	7.077	9.976

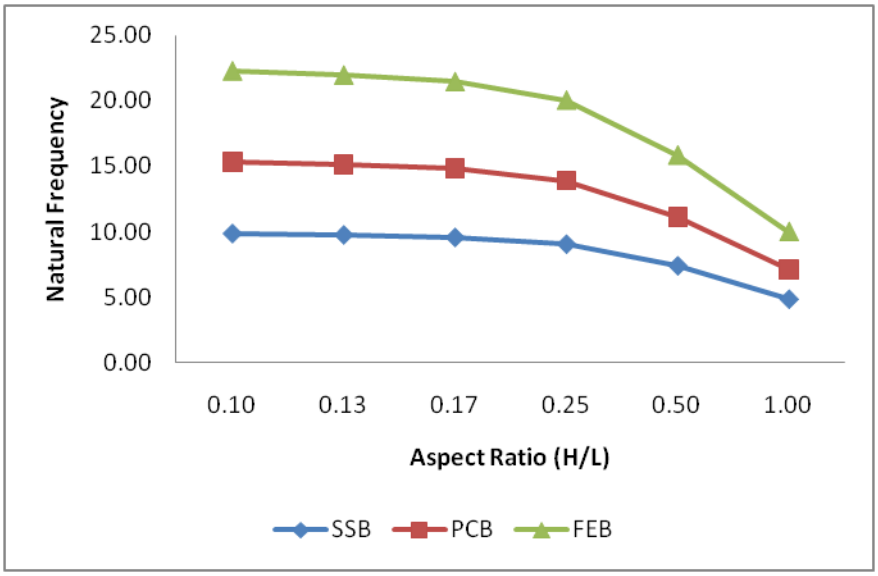

Fig.8: graph of shear effects on the Natural Frequency of the beams.

VI. Discussion of Results and Conclusion

The effect of shear on these beams; simply supported beams, fixed ended beams, propped cantilever beams using their fundamental natural frequencies were investigated and the results presented in tables above. From Fig.6, it can be observed that as the beam increases in length, shear factor and aspect ratio have significant effects on the beam which must be accounted for in the analysis and design of beams. The shear influence on the beam becomes stronger as the aspect ratio increases as shown in Fig.7.

From the results gotten, by applying the shear effects on the natural frequencies as in Fig.8, the natural frequencies were significantly affected as aspect ratio increases. For instance, at aspect ratio from 0.25 and 1.0, the natural frequencies with shear considered of a fixed ended beam are 20.033 and 9.976 as compared to the frequency when shear effects was ignored in table 3. This is applicable in short deep beams or at higher frequency modes when a continuous beam vibrates with many nodes in a span implying that shear effects in flexure design has great influence on the frequency of such beam.

In practical structures, higher frequencies are of very little interest and the effects of shearing deformation are most times ignored. This is because long beams are mostly used in construction. But for stability of structures, majority of design checks for structures subjected to vibration involves calculating its natural frequency and comparing with the frequency of the vibratory force acting on the structure to avoid resonance as the structure starts vibrating freely. This analysis shows that natural frequencies of vibrating beams with shear considered depends on the aspect ratio and slenderness of that beam and should be properly checked to prevent resonsnsce.

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