Strong Result for Level Crossings of Random Polynomials

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Abstract: Let \( N_n \) be the number of real roots of the algebraic equation

\[ f_n(x) = \sum_{k=0}^{n} \xi_k x^k = 0 \]

where \( \xi_k x^k \) are independent random variables assuming real values only.

Then there exists an integer \( n_0 \) such that for each \( n > n_0 \) the number of real roots of most of the equations \( f(x) = 0 \) is at least \( en \log n \) except for a set of measure at most \( \frac{\mu}{(e_{n_0} \log n_0)} \).


Keywords and Phrases: Independent identically distributed random variables, random algebraic polynomial, random algebraic equation, real roots

I. Theorem

Let \( f_n(x, w) \) be a polynomial of degree \( n \) whose coefficients are independent random variables with a common characteristics function \( \exp \left( -C|t|^\alpha \right) \), where \( \alpha = 1 \) and \( C \) is a positive constant. Take, \( \{e_n\} \) to be any sequence tending to zero such that \( e_n \log n \) tends to infinity as \( n \) tends to infinity. Then there exists an integer \( n_0 \) such that for each \( n > n_0 \) the number of real roots of most of the equations \( f(x) = 0 \) is at least \( en \log n \) except for a set of measure at most \( \frac{\mu}{(e_{n_0} \log n_0)} \).

II. Introduction

Let \( N_n \) be the number of real roots of the algebraic equation

\[ f_n(x) = \sum_{k=0}^{n} \xi_k x^k = 0 \]

where \( \xi_k x^k \) are independent random variables assuming real values only. Several authors have estimated bounds for \( N_n \) when the random variables satisfy different distribution laws. Littlewood and Offord [2] made the first attempt in this direction. They considered the cases when the \( \xi_k x^k \) are normally distributed or uniformly distributed in (-1, +1) or assume only the values +1 and -1 with equal probability. They obtained in each case that

\[ P \left( N_n > \frac{\mu_{\log n}}{\log \log n} \right) > 1 - \frac{A}{\log n} \]

Samal [3] has considered the general case when the \( \xi_k x^k \) have identical distribution, with exception zero, variance and third absolute moment finite and non-zero. He has shown that \( N_n > s_n \log n \) outside an exceptional set whose measure tends to zero as \( n \) tends to infinity, where \( s_n \) tends to zero, but \( s_n \log n \) tends to infinity.

Samal and Mishra [4] have considered the case the \( \xi_k x^k \) have a common characteristics function \( \exp \left( -C|t|^\alpha \right) \) where \( C \) is a positive constant and \( \alpha \geq 1 \). They have shown that

\[ N_n > \frac{\mu_{\log n}}{\log \log n} \]
outside an exceptional set measure at most

\[
\begin{cases}
\frac{\mu}{(\log \log n)(\log n)^{\alpha-1}}, & \text{if } 1 \leq \alpha < 2, \text{if } \alpha = 2 \\
\frac{\mu \log \log n}{\log n} & \text{if } \alpha = 1
\end{cases}
\]

In all the above cases the exceptional set depends upon \( n \). Evans [1], was the first to obtain ‘strong result’ for these bounds. In such case the exceptional set is independent of the degree \( n \) of the polynomial. We use the term ‘strong result’ in the following sense:

All the above results are of the form

\[
P\left(P^\left(\frac{N_n}{\Delta_n} > \mu\right) \rightarrow 1 \text{ as } n \text{ tends to infinity}
\]

whereas the theorem of Evans is of the form

\[
P\left(\sup_{n > n_0} \frac{N_n}{\Delta_n} > \mu\right) \rightarrow 1 \text{ as } n_0 \text{ tends to infinity.}
\]

Evans [1] has shown, in case of normally distributed coefficients, that there exists an integer \( n_0 \) such that for \( n > n_0 \),

\[
N_n > \frac{\mu \log n}{\log \log n}
\]

e xcept for a set of measure at most \( \frac{\mu' \log \log n_0}{\log n_0} \).

Samal and Mishra [5] have shown in the case of characteristic function \( \exp\left(-C|t|^\alpha\right) \) that for \( n > n_0 \),

\[
N_n > \frac{\mu \log n}{\log \log n}
\]

outside an exceptional set of measure at most

\[
\frac{\mu'}{\left(\log \left(\frac{\log n_0}{\log \log n_0}\right)\right)^{\alpha-1}}
\]

where \( \alpha > 1 \).

In [7], they have considered the ‘strong result’ in the general case. Assuming that the random variables (not necessarily identically distributed) have exception zero, variance and third absolute moment non-zero finite, they have shown that for \( n \geq n_0 \),

\[
N_n > (\mu \log n) / \log\left\{(K_n/t_n) \log n\right\}
\]

outside a set of measure at most

\[
\frac{\mu'}{\log \left(\frac{\log n_0}{
\log \left(\frac{K_{n_0}}{t_{n_0}} \log n_0\right)\right)}\right}\]

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provided \( \lim_{n \to \infty} \frac{p_n}{t_n} \) is finite and \( \log \left( \frac{K_{n\theta}}{t_{n\theta}} \log n_0 \right) = 0 \) (logn) where \( K_n = \max_{\theta \leq a} \sigma_{\theta} t_n = \max_{\theta \leq a} \sigma_{\theta} \text{ and } \max_{\theta \leq a} \sigma_{\theta}^2, \sigma_{\theta}^3 \) being the variance and third absolute moment respectively of \( \xi_{\theta} \).

Our object is to improve the ‘strong result’ for lower bound in case of characteristic function exp \( -C|t|^\alpha \). We have shown that for \( n>n_0 \).

\( N_n > \in_n \log n \)

Outside an exceptional set of measure at most \( \left( \frac{\mu'}{\in_{n_0} \log n_0} \right) \) where \( \in_n \to 0, \text{ but } \in_{n_0} \log n \to \infty \).

The result of Evans [1] is a special case of ours and is obtained by taking \( \alpha=2 \) and \( \in_n = (\log \log n)^{-1} \) in our theorem 1. The result of Samal and Mishra [5] is also a special case of our theorem 1. On the other hand our exceptional set is smaller.

All authors who have estimated bounds for \( N_n \) have used one kind of basic technique originally used by Littlewood and Offord [2].

We shall denote \( \mu \) for positive constants which may have different values in different occurrences.

We suppose always that \( n \) is large so that any inequalities true when \( n \) is large may be taken as satisfied.

Throughout the paper, \([x]\) will denote the greatest integer not exceeding \( x \).

It may be noted that although Evans [1] is a special case of ours, a much better estimate for the lower bound with smaller exceptional set can be derived from our theorem 1. For example, if we take \( \alpha=2, \)

\( \in_n = (\log \log n)^{-p} \) where \( 0<p<1 \), then for \( n>n_0 \).

\( N_n > \frac{\log n}{(\log \log n)^p} \)

outside an exceptional set of measure at most \( \mu(\log \log n_0)^p \log n_0 \)

Lemma 1.2.

If a random variable \( \zeta \) has characteristic function exp \( -C|t|^\alpha \), then for every \( \epsilon>0 \)

\( P \{ |\zeta| > \epsilon \} \leq \frac{2^{1+\alpha} C}{1+\alpha} \frac{1}{\epsilon^2}. \)

This lemma is due to Samal and Mishra [4].

Proof of the Theorem

Take constant \( A \) and \( B \) such that \( 0<B<1 \) and \( A>1 \). Choose \( \beta_m \) such that \( \beta_m \) and \( \frac{\log m}{\log \beta_m} \) both tend to infinity as \( m \) tends to infinity. Let

\( \lambda_m = m^{2/\alpha} \beta_n, M_n = \left[ 2^\alpha \beta_n^{\alpha} \left( \frac{Ae}{B} \right) \right] + 1. \) (1.1)

So

\( \mu_1 \beta_n^{\alpha} \leq M_n \leq \mu_2 \beta_n^{\alpha}. \)

We define

\( \Phi(X) = x^{[\log x]+x} \)
Let $k$ be the integer determined by
\[
\varphi(8k + 7)M^{8k+7}_n \leq n < \varphi(8k + 11)M^{8k+11}_n \tag{1.2}
\]
The first inequality gives $k \leq \frac{\log n}{\log \beta_m}$. The second inequality gives
\[
\log \leq \left(\log(8k + 11)\right)^2 + (8k + 11)\log(8k + 11) + (8k + 11)\log M_n
\]
\[
< 2(8k + 11) + (8k + 11)^2 + (8k + 11)\log M_n
\]
\[
< \mu k^2 \log M_n
\]
So
\[
k > \mu \frac{\log n}{\log M_n} > \mu \frac{\log n}{\log \beta_m}
\]
Thus
\[
\mu \frac{\log n}{\log \beta_m} < k \leq \mu \frac{\log n}{\log \beta_m} \tag{1.3}
\]
By the condition imposed on $\beta_n$ it follows that $k$ tends to infinity as $n$ tends to infinity. We have
\[
f(x) = U_m + R_m \text{ at the points}
\]
\[
X_m = \left\{1 - \frac{1}{\varphi(4m + 1)M^{4m}_n}\right\}^{1/u}
\]
for $m = \lceil k/2 \rceil + 1, \lceil k/2 \rceil + 2, \ldots, k$ \textit{where}
\[
U_m = \sum \xi v_X, R_m = \left(\sum + \sum \right) \xi v_X
\]
the index $v$ ranging from $\varphi(4m + 1)M^{4m-1}_n + 1$ to $\varphi(4m + 3)M^{4m+3}_n$ in $\sum$ from 0 to $\varphi(4m + 1)M^{4m-1}_n$ and from $\varphi(4m + 3)M^{4m+3}_n + 1$ to $n$ in $\sum$. We also have
\[
f(x_{2m}) = U_{2m} + R_{2m}, f(x_{2m+1}) = U_{2m+1} + R_{2m+1} \tag{1.5}
\]
Obviously $U_{2m}$ and $U_{2m+1}$ are independent random variables. Again it follows from (1.3) that $2k+1<n$ for larger $n$. Also the maximum index in $U_{2m+1}$ for $m=k$ is $\varphi(8k + 7)M^{8k+7}_n$, which, by (1.2) is consistent with (1.5).

Let $V_m = \left(\sum x^{\alpha v}_m\right)^{1/2}$. Then
\[
V^\alpha_m = \sum x^{\alpha v}_m > \sum \varphi(4m - 1)M^{4m-1}_n + 1 x^{\alpha v}_m
\]
\[
> \varphi(4m + 1)M^{4m}_n - \varphi(4m - 1)M^{4m-1}_n x^{2\varphi(4m+1)M\alpha v}_m
\]
\[
> \varphi(4m + 1)M^{4m}_n \left\{1 - \varphi(4m - 1) \frac{1}{\varphi(4m + 1) M_n}\right\} (e^{-1}/A)
\]
\[
> \varphi(4m + 1)M^{4m}_n (B/A)e^{-1}
\]
when $n$ is large
Now we estimate
\[
P = P \left( (U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}) \cup (U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1}) \right)
\]
\[
P_1 \left( U_{2m} > V_{2m}, \Pr(U_{2m} < -V_{2m}) + \Pr(U_{2m} < -V_{2m}) \Pr(U_{2m+1} > V_{2m+1}) \right)
\]

Since the characteristic function of \( \xi \) is \( \exp \left\{-C|t|^\alpha \right\} \), the characteristic function of \( U_{2m} \) is therefore
\[
\exp \left\{-C|t|^{\alpha} \sum_{m} x_{mv} \right\} = \exp \left\{-C|t|^{\alpha} V_{2m} \right\}
\]

where the index \( V \) ranges from \( \phi(8m - 1)M_{m-1} + 1 \) to \( \phi(8m + 3)M_{n+3} \). Therefore the characteristic function of \( U_{2m}/V_{2m+1} \) is \( \exp \left\{-C|t|^{\alpha} \right\} \), which is similarly also the characteristic function of \( U_{2m}/V_{2m+1} \).

1.2. We shall need the following lemmas.

**Lemma 1.2.**

\[
\sum_{m} x_{mv} < \frac{\phi(4m + 1)M_{m+3}}{1 - x_{m}} \text{ for sufficiently large } n.
\]

**Proof.**

The characteristics function of
\[
\sum_{m} x_{mv} = \exp \left\{-C|t|^{\alpha} \sum_{m} x_{mv} \right\}
\]

\[
\leq \frac{2^{1+2\alpha} C A e}{B(1 + \alpha)} \exp \left\{(4m + 1)^2 M_{m}^2 \right\}
\]

But
\[
\sum_{m} x_{mv} \leq \sum_{m} \phi(4m + 3)M_{m+3+1} \left\{ \frac{1}{\phi(4m + 1)M_{m+3+1}} \right\}
\]

Since
\[
\phi(4m + 3)M_{m}^2 < \phi(4m + 1)(4m + 1)M_{m}^2
\]

Hence using (1.6), we obtain
\[
P_1 < \frac{2^{1+2\alpha} C A e}{B(1 + \alpha)} \exp \left\{- (4m + 1)^2 M_{m}^2 \right\}
\]

as required

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Lemma 1.3.

\[ \sum_{m=1}^{\infty} \xi v x^m \leq \lambda \left( \sum_{m=1}^{\infty} \xi v x^m \right)^{1/a} \]

This follows directly from lemma 1.1.

1.3. Now

\[ \lambda \left( \sum_{m=1}^{\infty} \xi v x^m \right)^{1/a} < \lambda \left\{ \varphi(4m-1)M_n^{4m-1} + 1 \right\}^{1/a} \]

\[ \leq \lambda \left\{ \varphi(4m-1)M_n^{4m-1} \left( 1 + \frac{1}{\varphi(4m-1)M_n^{4m-1}} \right) \right\}^{1/a} \]

\[ < 2^{1/a} \lambda \left\{ \varphi(4m-1)M_n^{4m-1} \right\}^{1/a} \]

\[ = 2^{1/a} \lambda \left\{ \varphi(4m-1)(4m-1)^{\left\lfloor \log(4m-1)+(4m-1) \right\rfloor} M_n^{4m-1} \right\}^{1/a} \]

\[ \leq \frac{2^{1/a} \lambda M_n \varphi(4m+1)M_n^{4m}}{16m^2 M_n^2} \]

\[ < \frac{2^{1/a} \lambda \frac{Ae}{B} V_n^{4m}}{16m^2 M_n^2} \]

\[ < \frac{2^{1/a} \lambda \frac{Ae}{B} V_n^{4m}}{16m^2 M_n^2} \]

\[ < \frac{1}{2} V_m \]

The last two steps above follow from (1.1) and (1.6). Hence by using lemmas 1.2 and 1.3, we have \( R_m \) < \( V_m \) for every sufficiently large \( n \) except for a set of measure at most

\[ \mu \exp\left\{- (4m+1)^2 M_n^2 \right\} + \frac{\mu'}{\lambda_m^a} \leq \mu \exp\left\{- (m^2 M_n^2) \right\} + \frac{\mu'}{\lambda_m^a} \]

Thus we have

\[ |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \]

for \( m = m_0, m_0 + 1, \ldots, k \), where \( m_0 = \lceil k/2 \rceil + 1 \)

The measure of the exceptional set is at most
\[
\mu \exp \left\{ \left(4m^2M_2^n\right)^2 \right\} + \frac{\mu'}{\lambda^2} \leq \mu \exp \left\{ \left(2m+1\right)M_2^n \right\} + \frac{\mu'}{\lambda^2} \\
< \mu \exp \left\{ \left(-m^2M_2^n\right)^2 \right\} + \frac{\mu_2'}{\lambda^2} 
\] (1.7)

1.4. We define the events \(E_m\) and \(F_m\) as follows:
\[
E_m = \{U_{2m} > V_{2m}, \quad U_{2m+1} < V_{2m+1}\} \\
F_m = \{U_{2m} < V_{2m}, \quad U_{2m+1} > V_{2m+1}\}
\]
We have shown earlier that
\[
P_r(E_m \cup \overline{F_m}) = \delta > 0
\]
Let \(\eta_m\) be a random variable such that it takes value 1 on \(E_m \cup F_m\) and zero elsewhere. In other words
\[
\eta_m = \begin{cases} 1, & \text{with probability } \delta \\ 0, & \text{with probability } 1 - \delta \end{cases}
\]
Let \(\eta_m\) are thus independent random variables with \(E(\eta_m) = \delta\) and \(V(\eta_m) = \delta^2 < 1\).
We write
\[
S_m = \begin{cases} 0, & \text{if } |R_{2m}| < V_{2m}, \quad |R_{2m+1}| < V_{2m+1} \\ 1, & \text{otherwise} \end{cases}
\]

III. Conclusion

By considering the polynomial
\[
f_n(x) = \sum_{k=0}^{n} \xi_k x^k = 0
\]
where \(\xi_k x^k\) are independent random variables assuming real values only we found that the number of zeros of the above polynomial of the equations \(f(x) = 0\) is at least \(\left( en \log n \right) \) except for a set of measure at most \(\frac{\mu}{\varepsilon_0 \log n_0}\)

References