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Rhotrix Sub-Element Subgroups of The LoubéRé Magic Square Infinite Additive Abelian Group

Adamu H.[1], Babayo A.M.[2]
[1]*School of Preliminary and General Studies, Ibrahim Badamasi Babangida University, P.M.B. 0011, Lapai, Niger State, Nigeria.*
[2]*Department of Mathematics and Computer Science, Faculty of Science, Federal University, P.M.B. 0182, Kashere, Gombe State, Nigeria.*

Abstract : *By the requisite infinite additive abelian group over the Loube΄re΄ Magic Square, it is needless to be clarified that the underlining multiset of entries of the square is of the integer number. The Rhotrix Sub-elements of the squares are the rhomboid array of number that are sitting in the magic squares. The matrix sub-elements have analogous meaning and are considered babyish tautology, but we eclectic the two sub-elements to get the Rhotrix-Matrix Method of Construction of the Loube΄re΄ Magic Squares. It is also showcased via concrete examples that the rhotrix sub-element of the Loube΄re΄ Magic Squares form infinite additive abelian group. And, the rhotrix sub-element subgroup of the Loube΄re΄ Magic Square of roots of unity is explicated.*
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I. Introduction

The concept of rhotrix was first introduced by [1] in trying to extend the ideas of matrix-tertion and matrix-noitret. The construction of magic squares via Rhotrix-Matrix Method is one of the finest achievement of the newest concept, Rhotrix Theory because Rhotrix play an irresistible role in this method.

The set of magic squares constructed with De La Loube΄re΄ Procedure is what is aptly termed Loube΄re΄ Magic Squares. Before delving the Rhotrix Sub-element Subgroups of the Loube΄re΄ Magic Squares of roots of unity, we will examine two important notions. Firstly so, we will examine Rhotrix-Matrix Construction of 3×3 and 5×5 Magic Squares, where the 3×3 is Loube΄re΄. Secondly, we will examine Rhotrix Sub-element of the Loube΄re΄ Magic Square Infinite Additive Abelian Group.

Be that as it may, $x^n - a$ is solvable by radicals. Definitely the quartic polynomial $x^4 - 1$ is solvable whence S_4, the symmetry group of length 4, is solvable since A_4 is the maximal normal subgroup of S_4 and there are series of maximal normal subgroups in its composition down to 1.
Therefore,

$$x^4 - 1 = (x^2+1)(x^2-1) = (x-1)(x+1)(x^2+1) = 0 \qquad (1.1)$$

Because every field is an integral domain,

$$x - 1 = x + 1 = 0 \text{ or } x^2 + 1 = 0 \qquad (1.2)$$

That is, $x = 1, \ -1, \ i \text{ or } -i$.

One of the significance of group theory first introduced by Galois in 1830 is checking the nature of the polynomials having roots forming group. For example, the roots of $x^4 - 1$ form the group of roots of unity. The Cayley Hamilton Theorem followed the original definition by Galois so that group will be viewed as an extension of the Galois' Symmetric Group. Then finally comes the present general refined definition of group by Heinrich Martin Weber and Walter Von Dick in 1882 [2]; the roots 1, -1, i, and $-i$ (not far-fetched) of the aforementioned quartic equation if equipped with the complex multiplication of binary operation and the unary operations of inverse and identity form a group. There are two primes (inclusive) between 2 and 4, the oddest (even) prime (2) and an odd prime (3). Thus, the group of units of set of integer number is the maximal normal P-Sylow Subgroup of the group of roots of the quartic equation. That is,

$$(\{1,-1\},*) \trianglelefteq (\{1,-1,i,-i\},*) \qquad (1.3)$$

II. Preliminaries

We present preliminary definitions, notions and results here that are mostly original.

2.1. Loubéré Procedure (NE-W-S or NW-E-S, the cardinal points). Consider an empty $n \times n$ square of grids (or cells) . Start − from the central position $\left[\frac{n}{2}\right]$, where $[x]$ is the greater integer number less than or equal to x −with the number 1. The fundamental movement for filling the square is diagonally up, right (clockwise or NE or SE) or up left (anticlockwise or NW or SW) and one step at a time. If a filled cell(grid) is encountered, then the next consecutive number moves vertically down ward one square instead. Continue in this fashion until when a move would leave the square, it moves due N or E or W or S (depending on the position of the first term of the sequence) to the last row or first row or first column or last column, where n is an odd integer greater than or equal to 2. See also [3] for such a procedure.

The square grid of cells $[a_{ij}]_{n\times n}$ is said to be Loubéré Magic Square if the following conditions are satisfied .

1. $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = k;$
2. trace$[a_{ij}]_{n\times n}$= trace$[a_{ij}]^{T}{}_{n\times n} = k;$ and
3. $a_{1,\left[\frac{n}{2}\right]}, a_{\left[\frac{n}{2}\right],\left[\frac{n}{2}\right]}, a_{n,\left[\frac{n}{2}\right]}$ are on the same main column or row and $a_{\left[\frac{n}{2}\right],n}, a_{\left[\frac{n}{2}\right],\left[\frac{n}{2}\right]}, a_{\left[\frac{n}{2}\right],1}$ are on the same main column or row.

$[x]$ is the greatest integer less or equal to x, T is the transpose(of the square), k is the magic sum (magic product is defined analogously) usually expressed as $k = \frac{n}{2}[2a + (n-1)j]$ − derived from the sum of arithmetic sequence, where j is the common difference along the main column or row and a is the first term of the sequence − and $a_{\left[\frac{n}{2}\right],\left[\frac{n}{2}\right]} = \frac{k}{n}$.

Definition 2. 2. A square element $n-1 \times n-1$ removed from an $n \times n$ Loubéré Magic Square is called the sub-element square.

A set of 3 dimensional rhotrices (the plural of rhotrix) is cited as an example by [1] as follows:

$$\hat{R}(3) = \left\{\left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix}\right\rangle : a, b, c \in \mathbb{C}\right\} \qquad (2.1)$$

Where $\mathrm{C} = \mathrm{h(R)}$ is called the heart of any rhotrix R in $\hat{R}$.

Let $\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix}\right\rangle$ and $\left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & h & \end{matrix}\right\rangle$ be two 3 dimensional rhotrices. Then the rhotrix addition of binary operation is defined by

$$\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix}\right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & h & \end{matrix}\right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix}\right\rangle$$

(2.2)

The heart of rhotrix corresponds to the centre piece of Loubéré Magic Square . Thus, the set of the hearts equipped with an addition operation forms an abelian group . As like the Loubéré Magic Squares , rhotrices are of odd dimensions and $n = 1$ is considered trivial for it is isomorphic to the underlining set of entries of the rhotrix. The underlining set of entries of the rhotrix and of the aforementioned square is the set (multiset) of integer number. The $\ddot{R}(5)$, $\ddot{R}(7)$ and so forth are defined analogously.

Let C be the midterm of the arithmetic sequence arranged in an $n \times n$ magic square with the Loubéré Procedure, let d be the common difference of the sequence, j be the common difference of the sequence along the main column and $\mathrm{b = j - d}$ be the common difference of the sequence along the main row. Then the generalized 3×3 Loubéré Magic Square is available in [5] and we made a little adjustment to have:

$$\begin{bmatrix} c+b & c-j & c+d \\ c-k & c & c+k \\ c-d & c+j & c-b \end{bmatrix} \qquad (2.3)$$

Where $\mathrm{k = b - d}$.

The corresponding rhotrix sub-element is:

$$\left\langle \begin{array}{ccc} & c-j & \\ c-k & c & c+k \\ & c+j & \end{array} \right\rangle \qquad (2.4)$$

Definition 2.3. A non-empty set G equipped (or enclosed) with a binary operation * by which it means $*: G \times G \longrightarrow G$ i.e. a map inputting a pair and outputting a single element of G is said to be a group if it satisfies the following axioms:

i. Closure. $a, b \in G \Longrightarrow a * b \in G$

ii. Associativity. $a, b, c \in G \Longrightarrow a * (b * c) = (a * b) * c$

iii. Existence of Identity Element. $\exists e \in G \ni a * e = e * a = a \quad \forall a \in G$

iv. Existence of Inverse Element. $\forall a \in G \ \exists a^{-1} \in G \ \ni a * a^{-1} = a^{-1} * a = e$

A group is said to be abelian if it has an additional property $a * b = b * a \quad \forall a, b \in G.$

See [4] for more group theoretic concepts.

Remark 2.4. Every group has a unique identity and each element in a group has a unique inverse. The operation in axiom iii is referred to as the unary operation, i.e. an operation inputting one element of G and outputting one element of G.

The Rhotrix-Matrix Method Of Construction Of Magic Squares

A rhotrix is a rhomboid array of number and a matrix is a rectangular array of number. The rhotrix less the matrix of equal dimension with a multiple of 3 number of elements, and all rhotrices are of odd dimensions. So far, the application of rhotrices are usefully verbatim virtuoso of that of matrices.

We set to firstly so provide the construction of 3×3 for 1×1 is isomorphic to any chosen underlined set. Then it is followed by the construction of 5×5. We stop to economize space and to sidetrack the fact that we do not go to nth degree and conclude that we prove mathematics theorem.

The Construction of the 3×3 Square: An Example

Consider an arbitrary arithmetic sequence: 1, 2, … , 9. Write the sequence into a square matrix as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad (3.1)$$

Transform the matrix to a rhotrix as follows:

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 0 & 2 & \\ 7 & 0 & 5 & 0 & 3 \\ & 8 & 0 & 6 & \\ & & 9 & & \end{array} \right\rangle \qquad (3.2)$$

Swap the row r_2 with row r_4 and column c_1 with column c_5 (elementary row and/or column operation alike).

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 8 & 0 & 6 & \\ 3 & 0 & 5 & 6 & 7 \\ & 4 & 0 & 2 & \\ & & 9 & & \end{array} \right\rangle \qquad (3.3)$$

Overlap the row r_2 into the r_1 and the row r_5 into the row r_4 and overlap the column c_1 into the column c_2 and the column c_5 into the column c_4 to transform the rhotrix back to matrix again as follows:

$$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} \qquad (3.4)$$

Apply the square grid of cells as follows:

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \qquad (3.5)$$

This is a 3×3 Loube´re´ Magic Square.

The Rhotrix-Matrix Construction of 5×5 Magic Square

The following is one of the rhotrix-matrix method of construction of 5×5 non-Loube´re´ Magic Square . This paper recommends provision of the rhotrix-matrix construction of the 5×5 Loube´re´ Magic Square. Consider an arbitrary arithmetic sequence: 1, 2 , … , 25. Write the sequence into a square matrix as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix} \qquad (3.6)$$

Transform the matrix to a rhotrix as follows:

$$\left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 6 & 0 & 2 & & & \\ & & 11 & 0 & 7 & 0 & 3 & & \\ & 16 & 0 & 12 & 0 & 8 & 0 & 4 & \\ 21 & 0 & 17 & 0 & 13 & 0 & 9 & 0 & 5 \\ & 22 & 0 & 18 & 0 & 14 & 0 & 10 & \\ & & 23 & 0 & 19 & 0 & 15 & & \\ & & & 24 & 0 & 20 & & & \\ & & & & 25 & & & & \end{array} \right\rangle \qquad (3.7)$$

Draw a 5×5 Sub-element Draught on the rhotrix where the heart of the rhotrix is the centre piece of the Draught and fill the eastern empty cells with the triangular array of number outside the Draught on the west of the heart (centre piece), the southern empty cells with number on the north, the western with the number on the east and the northern with the number on the south.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	9	2	5

Magic Rhotrix Sub-Element Subgroups Of The Loube´re´ Magic Squares Abelian Group

We know that the set of Loube´re´ Magic Squares over the set of natural number equipped with the matrix addition of binary operation forms a semigroup, and over the set of integer number forms a group-enclosed with the same operation.

Consider the following 3 general Magic Rhotrix Sub-elements.

$$\left\langle \begin{array}{ccc} & c-j & \\ c-k & c & c+k \\ & c+j & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & e-i & \\ e-h & e & e+h \\ & e+i & \end{array} \right\rangle \text{ and } \left\langle \begin{array}{ccc} & f-m & \\ f-l & f & f+l \\ & f+m & \end{array} \right\rangle \qquad (4.1)$$

We show the **axioms** of the group as follows:

i. **Closure**. Let: $f = c + e, q = b + p, m = i + j$ and $l = k + h$. Then

$$\left\langle \begin{array}{ccc} & c-j & \\ c-k & c & c+k \\ & c+j & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & e-i & \\ e-h & e & e+h \\ & e+i & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & f-m & \\ f-l & f & f+l \\ & f+m & \end{array} \right\rangle \qquad (4.2)$$

ii. **Associativity**. Since we are dealing with integer number as entries of the square, then

$$\left\langle \begin{array}{ccc} & c-j & \\ c-k & c & c+k \\ & c+j & \end{array} \right\rangle + \left(\left\langle \begin{array}{ccc} & e-i & \\ e-h & e & e+h \\ & e+i & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f-m & \\ f-l & f & f+l \\ & f+m & \end{array} \right\rangle \right) = \left(\left\langle \begin{array}{ccc} & c-j & \\ c-k & c & c+k \\ & c+j & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & e-i & \\ e-h & e & e+h \\ & e+i & \end{array} \right\rangle \right) + \left\langle \begin{array}{ccc} & f-m & \\ f-l & f & f+l \\ & f+m & \end{array} \right\rangle \qquad (4.3)$$

We show the two miscellany axioms as follows:

iii. **Identity.** Since we are dealing with the integer number, then $\{0\}$, a singleton set with the arithmetic sequence 0, 0+0, 0+0+0,…,0+0+0+0+0+0+0+0+0 can be arranged in the square to form an identity element as follows:

$$\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle \qquad (4.4)$$

It is close to triviality that $a + e = e + a = a$ for all a in the rhotrix sub-elements set.

iv. **Inverse.** Let arbitrarily the following be a rhotrix sub-element:

$$\left\langle \begin{matrix} & u-v & \\ u-h & u & u+h \\ & u+v & \end{matrix} \right\rangle \quad (4.5)$$

Then its inverse is

$$\left\langle \begin{matrix} & -u+v & \\ u+h & u & u-h \\ & u-v & \end{matrix} \right\rangle$$ whence

$$\left\langle \begin{matrix} & u-v & \\ u-h & u & u+h \\ & u+v & \end{matrix} \right\rangle + \left\langle \begin{matrix} & -u+v & \\ u+h & u & u-h \\ & u-v & \end{matrix} \right\rangle = \left\langle \begin{matrix} & -u+v & \\ u+h & u & u-h \\ & u-v & \end{matrix} \right\rangle + \left\langle \begin{matrix} & u-v & \\ u-h & u & u+h \\ & u+v & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle \quad (4.6)$$

v. **Commutativity**.

$$\left\langle \begin{matrix} & c-j & \\ c-k & c & c+k \\ & c+j & \end{matrix} \right\rangle + \left\langle \begin{matrix} & e-i & \\ e-h & e & e+h \\ & e+i & \end{matrix} \right\rangle = \left\langle \begin{matrix} & f-m & \\ f-l & f & f+l \\ & f+m & \end{matrix} \right\rangle = \left\langle \begin{matrix} & e-i & \\ e-h & e & e+h \\ & e+i & \end{matrix} \right\rangle + \left\langle \begin{matrix} & c-j & \\ c-k & c & c+k \\ & c+j & \end{matrix} \right\rangle \quad (4.7)$$

This completes the proof that the rhotrix sub-element subgroups of the Loubere Magic Square form Infinite Additive Abelian Group.

Rhotrix Sub-Element Subgroup Of The Loubere Magic Squares Of Roots Of Unity

Here, we present the rhotrix sub -element subgroups of the Loubere Magic Squares of t he roots of unity as follows:

Let $\hat{G}$ is the rhotrix sub-element subgroup of the aforementioned squares. Then

$$\hat{G} := \left(\left\{ \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & -1 & \\ -1 & -1 & -1 \\ & -1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & i & \\ i & i & i \\ & i & \end{matrix} \right\rangle, \left\langle \begin{matrix} & -i & \\ -i & -i & -i \\ & -i & \end{matrix} \right\rangle \right\}, * \right) \quad (4.8)$$

Where $i = \sqrt{-1}, i^4 = 1, -i^4 = 1$ and $(-1)^2 = 1$.

The proof that $\hat{G}$ forms a group with respect to the complex multiplication of binary operation * is by the following Cayley Table.

*	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

Where $I = \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, A = \left\langle \begin{matrix} & -1 & \\ -1 & -1 & -1 \\ & -1 & \end{matrix} \right\rangle, B = \left\langle \begin{matrix} & i & \\ i & i & i \\ & i & \end{matrix} \right\rangle$ and $C = \left\langle \begin{matrix} & -i & \\ -i & -i & -i \\ & -i & \end{matrix} \right\rangle$ (4.9)

Let

$$\hat{H} := \left(\left\{ \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & -1 & \\ -1 & -1 & -1 \\ & -1 & \end{matrix} \right\rangle \right\}, * \right) \quad (4.10)$$

Then $\hat{H}$ is a subgroup of $\hat{G}$ because $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is a Magic Square since it is a Cayley Table of the group of units of set of integer number.

Evidently, $\hat{H}$ is a normal subgroup of $\hat{G}$ because of the following theorem.

Theorem 5.1. If H is a subgroup of index 2 in G, then H is a normal subgroup of G and $G/_H$ is a cyclic group of order 2.

Proof. Since the index $[G:H] = 2$, there are only two right cosets of H in G. One must be H and the other can be written as Hg, where g is an element of G that is not in H. To show that H is a normal subgroup of G, we need to show that $g^{-1}hg \in H \quad \forall g \in G$ and $h \in H$. If $g \in H$, it is clear that $g^{-1}hg \in H \quad \forall h \in H$.

If g is not an element of H, suppose that $g^{-1}hg \notin H$, $g^{-1}hg \in Hg$ and we can write $g^{-1}hg = h_1 g \in Hg$ for some $h_1 \in H$. It follows that $g = hh_1^{-1}$ which contradicts the fact that $g \notin H$.

Hence, $g^{-1}hg \in H \quad \forall g \in G$ and $h \in H$. H is normal in G.

Theorem 5.2. $(\ddot{N}, *)$ is a normal subgroup of $(\ddot{G}, *)$.

III. Conclusion

The order of $(\ddot{N},*)$ is the oddest prime and 3 does not divide 4. So, $(\ddot{N},*)$ is a P-Sylow Subgroup of $(\ddot{G},*)$. This implies that one of the newest realms of mathematics, the Rhotrix Theory, has gotten another genuine application . We build some algebraic theorems and some algebraic theories on these new algebraic realms: Loubéré Magic Square Realm and Rhotrix Realm. Banach told Ulam that Good Mathematicians see similarities between theories and theorems and the best ones see similarity between similarities.

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