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Mutually Unbiased Bases in Multipartite Systems

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Abstract: *We generalize the notion of mutually unbiased bases from bipartite system to multipartite system. First, we study mutually unbiased bases in tripartite systems. A general method of constructing a pair of mutually unbiased bases in tripartite systems is presented, and two pair different mutually unbiased bases in* $C^3 \otimes C^3 \otimes C^4$ *is given. Furthermore, we study mutually unbiased bases in multipartite systems, and we present the two pairs of mutually unbiased bases in* $C^3 \otimes C^3 \otimes C^4 \otimes C^2$.

Keywords: *Mutually unbiased bases, tripartite systems, multipartite systems*

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I. Introduction

Quantum entanglement lies in the heart of the quantum information processing. It plays important roles in many fields such as quantum teleportation[1], quantum coding[2], quantum key distribution protocol[3], quantum non-locality[4]. As one of the intrinsic feature of quantum computation and information, quantum is closely related to some of the fundamental problems in quantum mechanics such reality and locality. A important issue concerns with notion of unextendible product basis (UPBs), which is a set of imcomplete orthonormal product basis whose complementary space has no product states. It is proven that the UPBs reveal some nolocality without entanglement. UPBs can also demonstrate Bell inequalities without a quantum violation[5].

Another related interesting notion is that of mutually unbiased bases (MUBs), which also plays an important role in quantum information. The maximum number of MUBs in C^d is known to be no more that $d+1$ for any given d. It has been confirmed that there are indeed $d+1$ MUBs when d is a prime power[6]. However, the maximum number of MUBs is still open for general d.

MUBs are recently combined with other bases, such as product basis (PB)[6], unextendibleproduct basis (UPB) [7], unextendible maximally entangled basis (UMEB) [8], and maximally entangled basis (MEB) [9]. In [9], by systematically constructing MEBs, the concrete construction of pairs of MUMEBs in $C^d \otimes C^{kd} \left(k \in Z^+\right)$ is studied. In [10], the author first time considered the mutually unbiased bases in which all the bases are UMEB. And a systematic way of constructing a set of d^2 orthonormal maximally entangled states in $C^d \otimes C^{d'} \left(\frac{d'}{2} < d < d'\right)$ is provide. Necessary conditions of conditions of constructing a pair of MUUMEBs in $C^2 \otimes C^3$ are derived in [11].

In this paper, we study MUBs in multipartite systems. The paper is organized as follows: In Sect. 2. we first generalize MUBs in bipartite systems to tripartite systems. In Sect. 3, we construct different pairs MUBs in multipartite systems, and show two pairs MUBs in $C^3 \otimes C^3 \otimes C^4 \otimes C^2$ as example. Conclusion and discussion are given in Sect. 4.

II. MUBs in tripartite systems

MUBs has significant application in quantum information processing. In this section, we first recall the definition of MUBs .Then,we construct MUBs in tripartite systems.

Defintion 1.[9]A state $|\psi\rangle$ is said to be a $C^d \otimes C^{d'} \left(d < d'\right)$ maximally entangled if for a arbitrary given orthonrmal complete basis $\left\{|i_A\rangle\right\}$ of system A, there exists an orthonrmal basis $\left\{|i'_B\rangle\right\}$ of system B such

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that $|\psi\rangle$ can be written as $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i_A\rangle\otimes|i'_B\rangle$. A basis constituted by maximally entangled states is called maximally entangled basis(MEB).

Defintion 2.[11]A set of $\{|\phi_i\rangle\in C^d\otimes C^{d'}:i=1,2,...,n,n<dd'\}$ is said to be an n-member unextendible maximally entangled bases (UMEBs) if (1) $|\phi_i\rangle,i=1,2,...,n$,are maximally entangled; (2) $\langle\phi_i|\phi_j\rangle=\delta_{ij}$; (3) if $\langle\phi_i|\psi\rangle=0$ for all $i=1,2,...,n$,then $|\psi\rangle$ cannot be maximally entangled.

Defintion 3.[9]Two orthonormal bases $B_1=\{|\phi_i\rangle\}_{i=1}^{d}$ and $B_1=\{|\psi_i\rangle\}_{i=1}^{d}$ of a complex vector space C^d are said to be mutually unbiased (MU) if and only if

$$\left|\langle\phi_i|\psi_j\rangle\right|=\frac{1}{\sqrt{d}},i,j=1,2,...,d\,. \tag{1.1}$$

Theorem 1. If $\{|\phi_i\rangle\}_{i=0}^{d_1d_2-1}$ and $\{|\phi'_i\rangle\}_{i=0}^{d_1d_2-1}$ are two MUBs in $C^{d_1}\otimes C^{d_2}$, $\{|\psi_i\rangle\}_{i=0}^{d_3-1}$ and $\{|\psi'_i\rangle\}_{i=0}^{d_3-1}$ are two MUBs in C^{d_3} , then $\{|\phi_i\rangle\otimes|\psi_j\rangle\}_{i=0,j=0}^{d_1d_2-1,d_3-1}$ and $\{|\phi'_i\rangle\otimes|\psi'_j\rangle\}_{i=0,j=0}^{d_1d_2-1,d_3-1}$ are two MUBs in $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$.

Proof. Because $\{|\phi_i\rangle\}_{i=0}^{d_1d_2-1}$ and $\{|\phi'_i\rangle\}_{i=0}^{d_1d_2-1}$ are two MUBs in $C^{d_1}\otimes C^{d_2}$, we can get

$$.\left|\langle\phi_i|\phi'_j\rangle\right|=\frac{1}{\sqrt{d_1d_2}},i,j=0,1,...,d_1d_2-1 \tag{1.2}$$

And $\{|\psi_i\rangle\}_{i=0}^{d_3-1}$ and $\{|\psi'_i\rangle\}_{i=0}^{d_3-1}$ are two MUBs in C^{d_3} ,we have

$$\left|\langle\psi_i|\psi'_j\rangle\right|=\frac{1}{\sqrt{d_3}},i,j=0,1,...,d_3-1\,, \tag{1.3}$$

then

$$\left|\left(\langle\phi_i|\otimes\langle\psi_j|\right)\left(|\phi'_{i'}\rangle\otimes|\psi'_{j'}\rangle\right)\right|=\left|\langle\phi_i||\phi'_{i'}\rangle\otimes\langle\psi_j||\psi'_{j'}\rangle\right|=\frac{1}{\sqrt{d_1d_2}}\cdot\frac{1}{\sqrt{d_3}}=\frac{1}{\sqrt{d_1d_2d_3}}\,,$$

where $i,j=0,1,...,d_1d_2-1,i',j'=0,1,...,d_3-1$,

therefore, $\{|\phi_i\rangle\otimes|\psi_j\rangle\}_{i=0,j=0}^{d_1d_2-1,d_3-1}$ and $\{|\phi'_i\rangle\otimes|\psi'_j\rangle\}_{i=0,j=0}^{d_1d_2-1,d_3-1}$ are two MUBs in $C^{d_1}\otimes C^{d_2}\otimes C^{d_3}$.

Next, we establish two kinds of construction of MUBs in $C^3\otimes C^3\otimes C^4$. The first one is MUBs in $C^3\otimes C^3\otimes C^4$ from MUMEBs in $C^3\otimes C^3$ and MUBs in C^4. The second one is MUBs in $C^3\otimes C^3\otimes C^4$ from MUBs in C^3 and MUUMEBs in $C^3\otimes C^4$.

Example 1. MUBs in $C^3\otimes C^3\otimes C^4$ from MUMEBs in $C^3\otimes C^3$ and MUBs in C^4.

According to Ref. [12], we have two MUMEBs $\{|\phi_i\rangle\}_{i=0}^{8}$ and $\{|\phi'_j\rangle\}_{j=0}^{8}$ in $C^3\otimes C^3$ as follows,

$$\begin{cases}|\phi_{0-2}\rangle=\frac{1}{\sqrt{3}}\left(|00\rangle+\alpha|11\rangle+\alpha^2|22\rangle\right),\\ |\phi_{3-5}\rangle=\frac{1}{\sqrt{3}}\left(|01\rangle+\alpha|12\rangle+\alpha^2|20\rangle\right),\\ |\phi_{6-8}\rangle=\frac{1}{\sqrt{3}}\left(|02\rangle+\alpha|10\rangle+\alpha^2|21\rangle\right),\end{cases}\qquad \begin{cases}|\phi'_{0-2}\rangle=\frac{1}{\sqrt{3}}\left(|02\rangle+\alpha|11\rangle+\alpha^2|20\rangle\right),\\ |\phi'_{3-5}\rangle=\frac{1}{\sqrt{3}}\left(|01\rangle+\alpha|10\rangle+\alpha^2|22\rangle\right),\\ |\phi'_{6-8}\rangle=\frac{1}{\sqrt{3}}\left(|00\rangle+\alpha|12\rangle+\alpha^2|21\rangle\right),\end{cases}$$

where $\alpha=1,\omega_3,\omega_3^2$.

According to Ref. [11], we have two MUBs $\{|\psi_i\rangle\}_{i=0}^{3}$ and $\{|\psi'_i\rangle\}_{i=0}^{3}$ in C^4 as follows,

$$\begin{cases} |\psi_0\rangle = |0\rangle, \\ |\psi_1\rangle = |1\rangle, \\ |\psi_2\rangle = |2\rangle, \\ |\psi_3\rangle = |3\rangle, \end{cases} \qquad \begin{cases} |\psi'_0\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle + |2\rangle + |3\rangle\right), \\ |\psi'_1\rangle = \frac{1}{2}\left(|0\rangle + |1\rangle - |2\rangle - |3\rangle\right), \\ |\psi'_2\rangle = \frac{1}{2}\left(|0\rangle - |1\rangle - |2\rangle + |3\rangle\right), \\ |\psi'_3\rangle = \frac{1}{2}\left(|0\rangle - |1\rangle + |2\rangle - |3\rangle\right), \end{cases},$$

therefore, according to theorem 1, we can get the following MUBs $\left\{|\phi_i\rangle_{i=0}^{8} \otimes |\psi_j\rangle_{j=0}^{3}\right\}$ and $\left\{|\phi'_i\rangle_{i=0}^{8} \otimes |\psi'_j\rangle_{j=0}^{3}\right\}$ in $C^3 \otimes C^3 \otimes C^4$,

$$\begin{cases} |\phi''_{0-11}\rangle = \frac{1}{\sqrt{3}}\left(|00m\rangle + \alpha|11m\rangle + \alpha^2|22m\rangle\right), \\ |\phi''_{12-23}\rangle = \frac{1}{\sqrt{3}}\left(|01m\rangle + \alpha|12m\rangle + \alpha^2|20m\rangle\right), \\ |\phi''_{24-35}\rangle = \frac{1}{\sqrt{3}}\left(|02m\rangle + \alpha|10m\rangle + \alpha^2|21m\rangle\right), \end{cases}, \qquad \begin{cases} |\psi''_{0-11}\rangle = \frac{1}{\sqrt{3}}\left(|02\psi'_m\rangle + \alpha|11\psi'_m\rangle + \alpha^2|20\psi'_m\rangle\right), \\ |\psi''_{12-23}\rangle = \frac{1}{\sqrt{3}}\left(|01\psi'_m\rangle + \alpha|10\psi'_m\rangle + \alpha^2|22\psi'_m\rangle\right), \\ |\psi''_{24-35}\rangle = \frac{1}{\sqrt{3}}\left(|00\psi'_m\rangle + \alpha|12\psi'_m\rangle + \alpha^2|21\psi'_m\rangle\right), \end{cases},$$

where $m = 0,1,2,3$.

Example 2. MUBs in $C^3 \otimes C^3 \otimes C^4$ from MUBs in C^3 and MUUMEBs in $C^3 \otimes C^4$.

According to Ref. [13], we have two MUBs in C^3 as follows,

$$\begin{cases} |\phi_0\rangle = |0\rangle, \\ |\phi_1\rangle = |1\rangle, \\ |\phi_2\rangle = |2\rangle, \end{cases} \qquad \begin{cases} |\phi_0\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right), \\ |\phi_1\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \alpha|1\rangle + \alpha^2|2\rangle\right), \\ |\phi_2\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \alpha^2|1\rangle + \alpha|2\rangle\right), \end{cases}$$

where $\alpha = 1, \omega_3, \omega_3^2$.

According to Ref. [14], we have two MUUMEBs in $C^3 \otimes C^4$ as follows,

$$\begin{cases} |\psi_{0-2}\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + \alpha|11\rangle + \alpha^2|22\rangle\right), \\ |\psi_{3-5}\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + \alpha|12\rangle + \alpha^2|20\rangle\right), \\ |\psi_{6-8}\rangle = \frac{1}{\sqrt{3}}\left(|02\rangle + \alpha|10\rangle + \alpha^2|21\rangle\right), \\ |\psi_9\rangle = |03\rangle, \\ |\psi_{10}\rangle = |13\rangle, \\ |\psi_{11}\rangle = |23\rangle, \end{cases} \qquad \begin{cases} |\psi_{0-2}\rangle = \frac{1}{\sqrt{3}}\left(|0a\rangle + \alpha|1b\rangle + \alpha^2|2c\rangle\right), \\ |\psi_{3-5}\rangle = \frac{1}{\sqrt{3}}\left(|0b\rangle + \alpha|1c\rangle + \alpha^2|2a\rangle\right), \\ |\psi_{6-8}\rangle = \frac{1}{\sqrt{3}}\left(|0c\rangle + \alpha|1a\rangle + \alpha^2|2b\rangle\right), \\ |\psi_{9-11}\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \alpha|1\rangle + \alpha^2|2\rangle\right)|d\rangle, \end{cases}$$

where $\begin{cases} |a\rangle = \frac{1}{2}\left(|0\rangle - \alpha|1\rangle + |2\rangle + |3\rangle\right), \\ |b\rangle = \frac{1}{2}\left(\alpha|0\rangle + \alpha^2|1\rangle - \alpha|2\rangle + \alpha|3\rangle\right), \\ |c\rangle = \frac{1}{2}\left(-|0\rangle + |1\rangle + \alpha^2|2\rangle + \alpha|3\rangle\right), \\ |d\rangle = \frac{1}{2}\left(+\alpha^2|0\rangle + \alpha^2|1\rangle + \alpha|2\rangle - \alpha^2|3\rangle\right), \end{cases}$, $\alpha = 1, \omega_3, \omega_3^2$.

therefore, according to theorem 1, we can get the following we can get MUBs $\left\{|\phi_i\rangle_{i=0}^{3} \otimes |\psi_j\rangle_{j=0}^{11}\right\}$ and $\left\{|\phi'_i\rangle_{i=0}^{2} \otimes |\psi'_j\rangle_{j=0}^{8}\right\}$ in $C^3 \otimes C^3 \otimes C^4$,

$$\begin{cases} |\psi''_{0-8}\rangle = \frac{1}{\sqrt{3}}\left(|00m\rangle + \alpha|11m\rangle + \alpha^2|22m\rangle\right), \\ |\psi''_{9-17}\rangle = \frac{1}{\sqrt{3}}\left(|01m\rangle + \alpha|12m\rangle + \alpha^2|20m\rangle\right), \\ |\psi''_{18-26}\rangle = \frac{1}{\sqrt{3}}\left(|02m\rangle + \alpha|10m\rangle + \alpha^2|21m\rangle\right), \\ |\psi''_{27-29}\rangle = |03m\rangle, \\ |\psi''_{30-32}\rangle = |13m\rangle, \\ |\psi''_{33-35}\rangle = |23m\rangle, \end{cases} \begin{cases} |\psi_{0-8}\rangle = \frac{1}{\sqrt{3}}\left(|0a\phi_m\rangle + \alpha|1b\phi_m\rangle + \alpha^2|2c\phi_m\rangle\right), \\ |\psi_{9-17}\rangle = \frac{1}{\sqrt{3}}\left(|0b\phi_m\rangle + \alpha|1c\phi_m\rangle + \alpha^2|2a\phi_m\rangle\right), \\ |\psi_{18-26}\rangle = \frac{1}{\sqrt{3}}\left(|0c\phi_m\rangle + \alpha|1a\phi_m\rangle + \alpha^2|2b\phi_m\rangle\right), \\ |\psi_{27-35}\rangle = \frac{1}{\sqrt{3}}\left(|0d\phi_m\rangle + \alpha|1d\phi_m\rangle + \alpha^2|2d\phi_m\rangle\right), \end{cases}$$

where $\begin{cases} |a\rangle = \frac{1}{2}\left(|0\rangle - \alpha|1\rangle + |2\rangle + |3\rangle\right), \\ |b\rangle = \frac{1}{2}\left(\alpha|0\rangle + \alpha^2|1\rangle - \alpha|2\rangle + \alpha|3\rangle\right), \\ |c\rangle = \frac{1}{2}\left(-|0\rangle + |1\rangle + \alpha^2|2\rangle + \alpha|3\rangle\right), \\ |d\rangle = \frac{1}{2}\left(+\alpha^2|0\rangle + \alpha^2|1\rangle + \alpha|2\rangle - \alpha^2|3\rangle\right), \end{cases}$, $\alpha = 1, \omega_3, \omega_3^2$, $m = 0,1,2,3.$

III. MUBs in multipartite systems

In this section, we construct MUBs in multipartite systems. The following theorem provides a systematic way of constructing MUBs in multipartite systems.

Theorem 2. If $\left\{|\phi_i\rangle\right\}_{i=0}^{d_1 d_2 \cdots d_m - 1}$ and $\left\{|\phi'_i\rangle\right\}_{i=0}^{d_1 d_2 \cdots d_m - 1}$ are two MUBs in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_m}$, $\left\{|\psi_i\rangle\right\}_{i=0}^{d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ and $\left\{|\psi'_i\rangle\right\}_{i=0}^{d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ are two MUBs in $C^{d_{m+1}} \otimes C^{d_{m+2}} \otimes \cdots \otimes C^{d_{m+k}}$, then $\left\{|\phi_i\rangle \otimes |\psi_j\rangle\right\}_{i=0, j=0}^{d_1 d_2 \cdots d_m - 1, d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ and $\left\{|\phi'_i\rangle \otimes |\psi'_j\rangle\right\}_{i=0, j=0}^{d_1 d_2 \cdots d_m - 1, d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ are two MUBs in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_{m+k}}$.

Proof. Because $\left\{|\phi_i\rangle\right\}_{i=0}^{d_1 d_2 \cdots d_m - 1}$ and $\left\{|\phi'_i\rangle\right\}_{i=0}^{d_1 d_2 \cdots d_m - 1}$ are two MUBs in $C^{d_1} \otimes C^{d_2} \otimes \cdots \otimes C^{d_m}$, we can get

$$\left|\langle \phi_i | \phi'_j \rangle\right| = \frac{1}{\sqrt{d_1 d_2 \cdots d_m}}, i, j = 0,1,..., d_1 d_2 d_m - 1. \qquad (2.1)$$

And $\left\{|\psi_i\rangle\right\}_{i=0}^{d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ and $\left\{|\psi'_i\rangle\right\}_{i=0}^{d_{m+1} d_{m+2} \cdots d_{m+k} - 1}$ are two MUBs in $C^{d_{m+1}} \otimes C^{d_{m+2}} \otimes \cdots \otimes C^{d_{m+k}}$, we have

$$\left|\langle \psi_i | \psi'_j \rangle\right| = \frac{1}{\sqrt{d_{m+1} \cdots d_{m+k}}}, i, j = 0,1,..., d_{m+k} - 1, \qquad (2.2)$$

then

$$\left|\left(\langle\phi_i|\otimes\langle\psi_j|\right)\left(|\phi'_{i'}\rangle\otimes|\psi'_{j'}\rangle\right)\right|=\left|\langle\phi_i||\phi'_{i'}\rangle\otimes\langle\psi_j||\psi'_{j'}\rangle\right|=\frac{1}{\sqrt{d_1d_2\cdots d_m}}\cdot\frac{1}{\sqrt{d_{m+1}\cdots d_{m+k}}}=\frac{1}{\sqrt{d_1d_2\cdots d_{m+k}}},$$

where $i,j=0,1,...,d_1d_2d_m-1$, $i,j=0,1,...,d_{m+k}-1$.

Therefore, $\left\{|\phi_i\rangle\otimes|\psi_j\rangle\right\}_{i=0,j=0}^{d_1d_2\cdots d_m-1,d_{m+1}d_{m+2}\cdots d_{m+k}-1}$ and $\left\{|\phi'_i\rangle\otimes|\psi'_j\rangle\right\}_{i=0,j=0}^{d_1d_2\cdots d_m-1,d_{m+1}d_{m+2}\cdots d_{m+k}-1}$ are two MUBs in $C^{d_1}\otimes C^{d_2}\otimes\cdots\otimes C^{d_{m+k}}$.

Next, we establish two kinds of construction of MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$. The first one is MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$ from MUBs in $C^3\otimes C^3\otimes C^4$ and MUBs in C^2. The second one is MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$ from MUMEBs in $C^3\otimes C^3$ and MUMEBs in $C^4\otimes C^2$.

Example 3. MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$ from MUBs in $C^3\otimes C^3\otimes C^4$ and MUBs in $C^3\otimes C^3\otimes C^4$.

According to example 1, we can get two MUBs in $C^3\otimes C^3\otimes C^4$. And we can find two MUBs in C^2 from Ref. [13] as follows,

$$\begin{cases}|\varphi_0\rangle=|0\rangle,\\|\varphi_1\rangle=|1\rangle,\end{cases}\qquad\begin{cases}|\varphi'_0\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right),\\|\varphi'_1\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right),\end{cases}$$

according to theorem 2, we can get two MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$ as follows,

$$\begin{cases}|\varphi''_{0-23}\rangle=\frac{1}{\sqrt{3}}\left(|00mn\rangle+\alpha|11mn\rangle+\alpha^2|22mn\rangle\right),\\|\varphi''_{24-47}\rangle=\frac{1}{\sqrt{3}}\left(|01mn\rangle+\alpha|12mn\rangle+\alpha^2|20mn\rangle\right),\\|\varphi''_{48-71}\rangle=\frac{1}{\sqrt{3}}\left(|02mn\rangle+\alpha|10mn\rangle+\alpha^2|21mn\rangle\right),\end{cases}$$

$$\begin{cases}|\varphi'''_{0-23}\rangle=\frac{1}{\sqrt{3}}\left(|02\psi'_m\varphi'_n\rangle+\alpha|11\psi'_m\varphi'_n\rangle+\alpha^2|20\psi'_m\varphi'_n\rangle\right),\\|\varphi'''_{24-47}\rangle=\frac{1}{\sqrt{3}}\left(|01\psi'_m\varphi'_n\rangle+\alpha|10\psi'_m\varphi'_n\rangle+\alpha^2|22\psi'_m\varphi'_n\rangle\right),,\\|\varphi'''_{48-71}\rangle=\frac{1}{\sqrt{3}}\left(|00\psi'_m\varphi'_n\rangle+\alpha|12\psi'_m\varphi'_n\rangle+\alpha^2|21\psi'_m\varphi'_n\rangle\right),\end{cases}$$

where $m=0,1,2,3,n=0,1,$ $\begin{cases}|\psi'_0\rangle=\frac{1}{2}\left(|0\rangle+|1\rangle+|2\rangle+|3\rangle\right),\\|\psi'_1\rangle=\frac{1}{2}\left(|0\rangle+|1\rangle-|2\rangle-|3\rangle\right),\\|\psi'_2\rangle=\frac{1}{2}\left(|0\rangle-|1\rangle-|2\rangle+|3\rangle\right),\\|\psi'_3\rangle=\frac{1}{2}\left(|0\rangle-|1\rangle+|2\rangle-|3\rangle\right).\end{cases}$.

Example 4. MUBs in $C^3\otimes C^3\otimes C^4\otimes C^2$ from MUMEBs in $C^3\otimes C^3$ and MUMEBs in $C^4\otimes C^2$.

According to Ref. [15] , we recall two MUMEBs in $C^4\otimes C^2$ as follows,

$$\begin{cases} |\xi_{0-1}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\ |\xi_{2-3}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |12\rangle), \\ |\xi_{4-5}\rangle = \frac{1}{\sqrt{2}}(|02\rangle \pm |13\rangle), \\ |\xi_{6-7}\rangle = \frac{1}{\sqrt{2}}(|03\rangle \pm |10\rangle), \end{cases} \qquad \begin{cases} |\xi'_{0-1}\rangle = \frac{1}{\sqrt{2}}(|0\alpha_0\rangle \pm |1\alpha_1\rangle), \\ |\xi'_{2-3}\rangle = \frac{1}{\sqrt{2}}(|0\alpha_1\rangle \pm |1\alpha_2\rangle), \\ |\xi'_{4-5}\rangle = \frac{1}{\sqrt{2}}(|0\alpha_2\rangle \pm |1\alpha_3\rangle), \\ |\xi'_{6-7}\rangle = \frac{1}{\sqrt{2}}(|0\alpha_3\rangle \pm |1\alpha_0\rangle), \end{cases}.$$

where
$$\begin{cases} |\alpha_0\rangle = \frac{1}{2}(-i|0\rangle - i|1\rangle - i|2\rangle - i|3\rangle), \\ |\alpha_1\rangle = \frac{1}{2}(-|0\rangle - |1\rangle + |2\rangle + |3\rangle), \\ |\alpha_2\rangle = \frac{1}{2}(|0\rangle - |1\rangle - |2\rangle + |3\rangle), \\ |\alpha_3\rangle = \frac{1}{2}(i|0\rangle + i|1\rangle - i|2\rangle i|3\rangle). \end{cases}$$

according to **theorem 2**, the $\left\{|\phi_i\rangle_{i=0}^{8} \otimes |\xi_j\rangle_{j=0}^{7}\right\}$ and $\left\{|\phi_i'\rangle_{i=0}^{8} \otimes |\xi_j'\rangle_{j=0}^{7}\right\}$ are MUBs in $C^3 \otimes C^3 \otimes C^4 \otimes C^2$.

IV. Conclusion

We have generalized the notion of MUBs from bipartite to multipartite quantum systems. Based on this, we first generalize MUBs in bipartite systems to tripartite systems. Moreover, we have constructed two types of MUBs in $C^3 \otimes C^3 \otimes C^4$. Then,we construct MUBs in multipartite quantum systems. and show two pairs MUBs in $C^3 \otimes C^3 \otimes C^4 \otimes C^2$ as example. Our results may shed light on further investigation on MUBs for multipartite quantum states.

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