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A New Mathematical Approach based on the Friedmann Equation

Saddam Hossain[1, *], Mir Md. Moheuddin[2], Muhammad Abdus Sattar Titu[3]
[1]*Dept. of Basic Science (Mathematics), World University of Bangladesh (WUB), Dhaka-1205, Bangladesh*
[2]*Dept. of CSE (Mathematics), Atish Dipankar University of Science and Technology (ADUST), Dhaka-1230, Bangladesh*
[3]*Dept. of Mathematics (General Science), Mymensingh Engineering College (MEC), Mymensingh-2200, Bangladesh*

Abstract
Cosmological models can describe the physical universe, which shows that the universe expanding from the big bang. The expansion of space in homogeneous and isotropic models of the universe in the context of general relativity is Friedmann equations. Friedmann equation is the standard model in cosmology. The solutions of the Friedmann equations explain the flat, open and close universe. In this paper, we have attempted to present the Friedmann equation mathematically. We also demonstrated the process of how to solve our proposed equation easily.
Keywords: *Cosmological principle, Hubble rate, Flat universe, Close universe, Open universe.*

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I. Introduction

The increase of the distance between two distant parts of the universe with time is the expansion of the universe. Which is a continuous process from the beginning of the universe. The universe suddenly expanded after the Big Bang(about 10^{-32} of a second).According to Newton's law of gravity, every object of the universe attracts every other object. Now if every object attracts each other than a fundamental question rose that what has prevented them to fall on each other during the evolution of the universe? These questions regarding the evolution of the universe as a whole were first discussed and analyzed by Friedmann in 1922, from Einstein's General Relativity. He used the theory to obtain a set of equations that described the universe's shape, properties, and evolution, revealing that the universe is not stationary. This model is so successful that it is now being considered as the Standard Model of Cosmology. Friedmann derived the equations from Einstein general relativity for a homogeneous and isotropic universe. The equations also derived by Milne form Newtonian mechanics.

By the literature view, we can see that many researchers have tried to analyze and discuss the importance of mathematical cosmology and universe using the Friedmann equation. Moreover, in 2013 Ch'ng Han Siong et al. [1] derived the spatially flat Friedmann equation from Wheeler-DeWitt equation. The general solution of the Friedmann equation had been discussed by Dr. Balsa Terzic in [2] on his lecture. Hoque M. A. et al. [3] studied the general Friedmann-Robertson–Walker (FRW) metric (k = +1, 0, - 1) into explicit "Schwarzschild" or "Curvature" [1] form, which is important from the view point of cosmology. In their paper, they also presented cosmological model in which the gravitational and cosmological constants G and Λ respectively are time–dependent. In [4], Naser Mostaghel has proposed a new equation, describing the evolution of the scale factor through assuming a flat universe expanding under a constant pressure and combining the first and the second Friedmann equations. Besides, Numerous discussion about the mathematical cosmology and the Universe is described in the books of J.N. Islam [5, 6], Steven Weinberg [7], A.R. Liddle [8], S. Dodelson [9], S. W. Hawking et al. [10], E. Kolb et al. [11], J. D. Barrow [12] .J. P. Uzan et al. [13] recalled the Newtonian derivation of the Friedmann equations which govern the dynamics of our universe and discuss the validity of such a derivation. In addition, In [14], Ahmad Sheykhi was able to derive the corresponding Friedmann equations describing the dynamics of the universe with any spatial curvature by taking the entropy associated with the apparent horizon of the Friedmann–Robertson–Walker (FRW) Universe in the form of Tsallis entropy, and assuming the first law of thermodynamics, holds on the apparent horizon. R. G. Cai et al. [15] had derived the Friedmann equations of a Friedmann-Robertson-Walker universe by use of the holographic principle together with the equipartition law of energy and the Unruh temperature.

What's more, in [16], Ž. Mijajlović et al. applied the theory of regularly varying functions to the asymptotical analysis at infinity of solutions of Friedmann cosmological equations. A. Sheykhi et al. [17]

considered the Debye entropic gravity and following the strategy of Verlinde and then they derived the modified Newton's law of gravitation and the corresponding Friedmann equations which are valid in all range of temperature. In [18], B. Majumder expressed the proposed GUP in a more general form and the effect was studied for the modification of the Friedmann equations of the FRW universe. In [19], Ahmad Sheykhi derived corrections to Newton's law of gravitation as well as modified Friedmann equations by adopting the viewpoint that gravity can emerge as an entropic force. Also in their article, their study further supports the universality of the log correction and provides a strong consistency check on Verlinde's model. Further, S. Ghassemi et al. [20] presented the generalized Friedmann equations describing the cosmological evolution of a finite thick brane immersed in a five-dimensional Schwarzschild Anti-de Sitter space time. In [21], M. Trodden et al. provided a pedagogical introduction to cosmology aimed at advanced graduate students in particle physics and string theory.

Apart from these tasks, in our article, we have discussed the equations derived by Friedmann from Einstein's general relativity. The simplification form of the Balsa Terzic's solution presented here for the matter and radiation dominates the universe as well.

The paper is constructed as follows: in section 2, we analysis the mathematical development of Friedmann equations that are used in the next section. Section 3 shows the solutions to the Friedmann Equation. In section 4, we discussed the findings of this paper. Finally, we highlight a conclusion of our works in section 5.

II. Mathematical development of Friedmann Equation

From the cosmological principle, we know the metric of the universe is

$$d\delta^2 = a(t)^2 d\delta_3^2 - c^2 dt^2 \dots\dots\dots\dots\dots\dots (1)$$

Where $d\delta_3^2$ is one of flat space, which is constantly positive and negative curvature in the sphere and hyperbolic space.

The first Friedmann equations for the homogeneous and isotropic universe is

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G\rho + \Lambda c^2}{3}$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 + k\left(\frac{c}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3}$$

$$\Rightarrow H^2 + k\left(\frac{c}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} \dots\dots\dots\dots\dots\dots (2)$$

Where $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

And the second one is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3} \dots\dots\dots\dots\dots\dots (3)$$

Where, a is the scale factor, Λ is the cosmological constant, and c is the speed of light in vacuum.

Where$k$is the curvature. The physics between a and k is when $k = +1, -1$ then a and is the radius of curvature of the universe and when $k = 0$, then a may be fixed to any arbitrary positive number at one particular time. For the positive value of k the universe is hyperspherical, for negative the universe is hyperbolicand for zero, then the universe is flat.

From (1) and (2) we get

$$\dot{p} = -3H\left(\rho + \frac{p}{c^2}\right) \dots\dots\dots\dots\dots\dots (4)$$

which eliminates Λ. Now using $\rho \to \rho - \frac{\Lambda c^2}{8\pi G}$ and $p \to p + \frac{\Lambda c^4}{8\pi G}$

We get the simplification from of (2) and (3) as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \dots\dots\dots\dots\dots\dots (5)$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \dots\dots\dots\dots\dots\dots (6)$$

If other parts of the equation are time-dependent then the Hubble parameter can change over time.
Some cosmologists call the second of these two equations the Friedmann acceleration equation.

III. Solutions of Friedmann Equation

From the definition of the Hubble rate H we have

$$H = \frac{\dot{a}}{a} \dots\dots\dots\dots\dots (7)$$

Now from (6), we get

$$\dot{H} = -H^2 + \frac{\ddot{a}}{a}$$

$$= -H^2\left(1 - \frac{\ddot{a}}{H^2 a}\right)$$

$$= -H^2(1+q) \dots\dots\dots\dots\dots (8)$$

Where

$$q = -\frac{\ddot{a}}{H^2 a} \dots\dots\dots\dots\dots (9)$$

For non-relativistic matter-dominated Universe consider $P = 0$.
Then, from eq. (3), we have

$$\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\rho = 0$$

Using (9) in terms of H

$$-H^2 q + \frac{4\pi G}{3}\rho = 0$$

Therefore

$$\rho = \frac{3H^2}{4\pi G} q \dots\dots\dots\dots\dots (10)$$

Then the first Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho = -\frac{k}{a^2} \dots\dots\dots\dots\dots (11)$$

$$H^2 - 2H^2 q = -\frac{k}{a^2} \dots\dots\dots\dots\dots\dots (12)$$

So

$$-k = a^2 H^2(1-2q); a \neq 0, H \neq 0 \dots\dots\dots\dots\dots\dots (13)$$

At flat Universe
For

$$k = 0 \Longrightarrow q = ½$$

For

$$k = 1 \Longrightarrow q > \frac{1}{2}$$

And for

$$k = -1 \Longrightarrow q < \frac{1}{2}$$

Now by (10), this yields the critical density

$$\rho_{cr} = \frac{3H^2}{8\pi G} \dots\dots\dots\dots\dots\dots (14)$$

the density needed to yield the flat Universe. Currently, it is

$$\rho_{cr} = \frac{3H^2}{8\pi G}$$

$$= \frac{3\left(\frac{h}{0.98\times10^{10}\ years}\right)^2\left(\frac{1\ years}{3600\times24\times365\ sec}\right)^2}{8\pi(6.67\times10^{-8}\ cm^3 g^{-1} s^{-2})}$$

$$= 1.87 \times 10^{-29} h^2 g\ cm^{-3} \approx 10^{-29} g\ cm^{-3}$$

Here we take

$$h \approx 0.72 \pm 0.02$$

Now by (10) and (14), we get

$$q = \frac{\rho}{2\rho_{cr}} \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

The second Friedmann equation for the matter-dominated Universe ($p = 0$) by (4)

$$\dot{\rho} + \frac{3\rho\dot{a}}{a} = 0$$

$$\Longrightarrow a^3\dot{\rho} + 3\rho\dot{a}a^2 = 0$$

$$\Rightarrow \frac{d}{dt}(a^3\rho) = 0$$

$$\Rightarrow a^3\rho = a_0^3\rho_0 = const \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Radiation-dominated Universe is modeled by perfect fluid approximation with $P = \frac{1}{3}\rho$
The second Friedmann equation becomes

$$\dot{\rho} + 3(\rho + \frac{1}{3}\rho)\frac{\dot{a}}{a} = 0$$

$$\Rightarrow \dot{\rho} + 4\rho\frac{\dot{a}}{a} = 0$$

$$\Rightarrow a^4\dot{\rho} + 4\rho\dot{a}a^3 = 0$$

$$\Rightarrow \frac{d}{dt}(a^4\rho) = 0$$

$$\Rightarrow a^4\rho = a_0^4\rho_0 = const \ldots\ldots\ldots\ldots\ldots\ldots (17)$$

From (16) we have

$$\rho = \left(\frac{a_0}{a}\right)^3 \rho_0 \ldots\ldots\ldots\ldots\ldots (18)$$

3.1. Flat Universe$\left(k = 0,\ q_0 = \frac{1}{2}\right)$:

For matter-dominated: $P = 0, a^3\rho = const.$
Using (11) and (18) the first Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\frac{1}{a^{\frac{1}{2}}}$$

$$\Rightarrow \int a^{\frac{1}{2}}\,da = \frac{2}{3}a^{\frac{3}{2}} + k = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\ t \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

At the Big Bang, $t = 0, a = 0$
So, $K = 0$ and $a_0 = 1$, and the fact that the Universe is flat $\rho_0 = \rho_{cr}$ we finally have

$$\frac{2}{3}a^{\frac{3}{2}} = \left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t$$

$$\Longrightarrow a^{\frac{3}{2}} = \frac{3}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t$$

$$\Rightarrow a = \left(\frac{3}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t\right)^{\frac{2}{3}}$$

$$\Rightarrow a = \left(\frac{9}{4}\left(\frac{8\pi G\rho_0}{3}\right)\right)^{\frac{1}{3}} t^{\frac{2}{3}}$$

$$\Rightarrow a = (6\pi G\rho_0)^{\frac{1}{3}} t^{\frac{2}{3}} = (6\pi G\rho_{cr})^{\frac{1}{3}} t^{\frac{2}{3}}$$

$$= \left(6\pi G\,\frac{3H_0^2}{8\pi G}\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{9H_0^2}{4}\right)^{\frac{1}{3}} t^{\frac{2}{3}}$$

$$= \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}} \dots\dots\dots\dots\dots\dots (20)$$

From here we compute the age of the universe t_0, which corresponds to the Hubble rate H_0 and the scale factor $a = a_0 = 1$ to be:

$$t_0 = \frac{2}{3H_0} \dots\dots\dots\dots\dots\dots (21)$$

Taking

$$H_0 = \frac{h}{0.98 \times 10^{10} years}, and\ \ h \approx 72$$

We get

$$t_0 = \frac{2 \times 0.98 \times 10^{10} years}{3 \times 0.72} \approx 9.1 \times 10^{10} years \equiv 9.1\ \mathcal{A}(aeon) \dots\dots\dots\dots\dots\dots (22)$$

For radiation-dominated: $P = \frac{1}{3}\rho, a^4\rho = const.$

The first Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3}}\frac{1}{a}$$

$$\Rightarrow \int a da = 2a^2 + k = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3}}\ t \dots\dots\dots\dots\dots\dots (23)$$

Again at the Big Bang, $t = 0, a = 0,$

So $K = 0.$ and $a_0 = 1$, and the fact that the Universe is flat $\rho_0 = \rho_{cr}$ we finally have

$$2a^2 = \left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t$$

$$\Rightarrow a^2 = \frac{1}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t$$

$$\Rightarrow a = \left(\frac{1}{2}\left(\frac{8\pi G\rho_0}{3}\right)^{\frac{1}{2}} t\right)^{\frac{1}{2}}$$

$$\Rightarrow a = \left(\frac{1}{4} \times \frac{8\pi G\rho_0}{3}\right)^{\frac{1}{4}} t^{\frac{1}{2}}$$

$$\Rightarrow a = \left(\frac{2}{3}\pi G\rho_0\right)^{\frac{1}{4}} t^{\frac{1}{2}} = \left(\frac{2}{3}\pi G\rho_{cr}\right)^{\frac{1}{4}} t^{\frac{1}{2}}$$

$$= \left(\frac{2}{3}\pi G\frac{3H_0^2}{8\pi G}\right)^{\frac{1}{4}} t^{\frac{1}{2}} = \left(\frac{H_0}{2}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

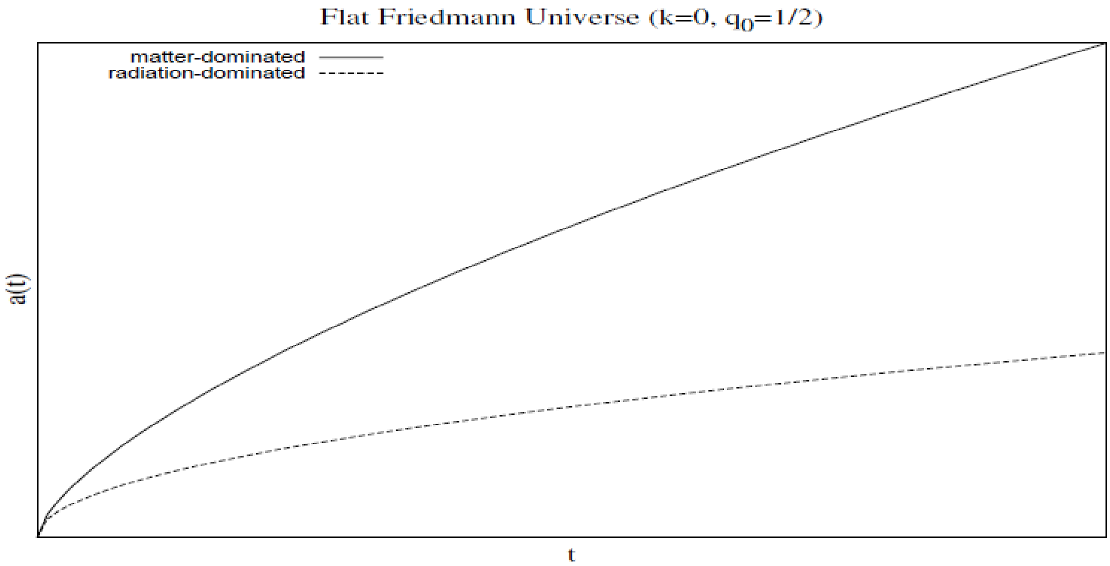

Figure 1: Evolution of the scale factor $a(t)$ for the flat Friedmann Universe.

3.2. Closed Universe($k = 1, q_0 > \frac{1}{2}$):

For matter-dominated: $P = 0, a^3\rho = const.$

By (11) and (18) the first Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 - \frac{1}{a^2}$$

$$\Rightarrow \dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3a} - 1$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} - 1}$$

$$\Rightarrow dt = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} - 1}}$$

$$\Rightarrow \frac{dt}{a} = \frac{da}{a\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} - 1}}$$

$$\Rightarrow \frac{dt}{a} = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a - a^2}}$$

The Conformal time is given in $d\eta = \frac{dt}{a}$ then

$$\Rightarrow d\eta = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a - a^2}}$$

$$\Rightarrow \int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a - a^2}} \dots\dots\dots\dots\dots\dots (25)$$

Finally, we get,

$$a = \frac{q_0}{2q_0 - 1}(1 - cos\eta)$$
$$t = \frac{q_0}{2q_0 - 1}(\eta - sin\eta)$$

For radiation-dominated: $P = \frac{1}{3}\rho, a^4\rho = const.$
Now the first Friedmann equation is

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4 - \frac{1}{a^2}$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3a^2} - 1}$$

$$\Rightarrow dt = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} - 1}}$$

$$\Rightarrow \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} - 1}} \dots\dots\dots\dots\dots (26)$$

By simplified we have,

$$a = \sqrt{\frac{2q_0}{2q_0 - 1}}sin\eta$$

$$t = \sqrt{\frac{2q_0}{2q_0 - 1}}(1 - cos\eta)$$

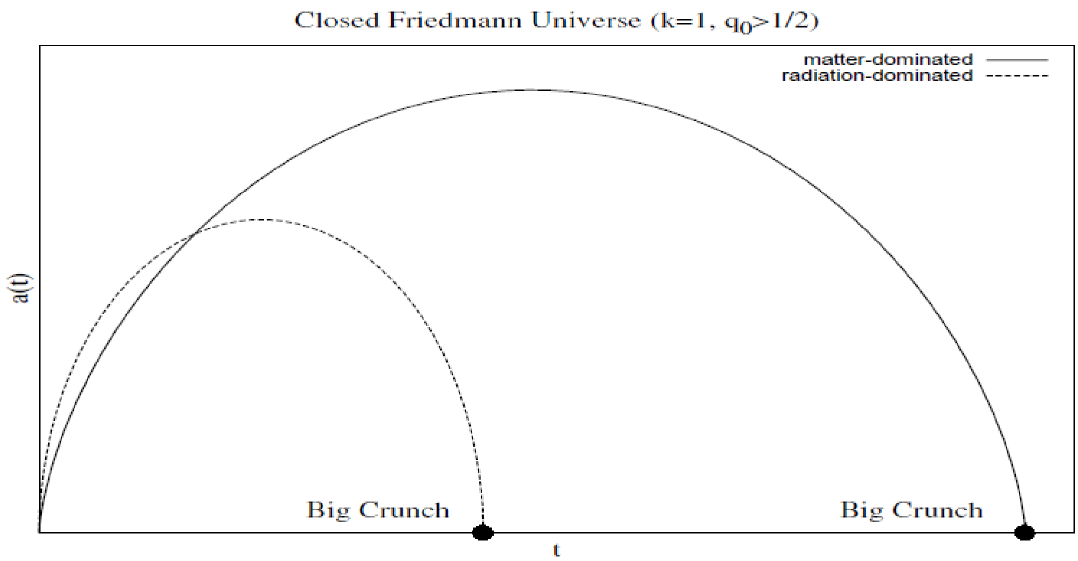

Figure 2: Evolution of $a(t)$ for the closed Friedmann Universe.

3.3. Open Universe($k = -1, q_0 < \frac{1}{2}$):

For Matter-dominated: $P = 0, a^3\rho = const.$
By using (11) and (18) the first Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 + \frac{1}{a^2}$$

$$\Rightarrow \dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3a} + 1$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} + 1}$$

$$\Rightarrow dt = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} + 1}}$$

$$\Rightarrow \frac{dt}{a} = \frac{da}{a\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a} + 1}}$$

$$\Rightarrow \frac{dt}{a} = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a + a^2}}$$

The Conformal time is given in $d\eta = \frac{dt}{a}$ then

$$\Longrightarrow d\eta = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a + a^2}}$$

$$\Longrightarrow \int d\eta = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3}a + a^2}} \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

We finally have that

$$a = \frac{q_0}{2q_0 - 1}(cosh\eta - 1)$$

$$t = \frac{q_0}{2q_0 - 1}(sinh\eta - \eta)$$

For radiation-dominated: $P = \frac{1}{3}\rho, a^4\rho = const.$
The first Friedmann equation becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^4 + \frac{1}{a^2}$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G\rho_0 a_0^4}{3a^2} + 1}$$

$$\Rightarrow dt = \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} + 1}}$$

$$\Rightarrow \int dt = \int \frac{da}{\sqrt{\frac{8\pi G\rho_0 a_0^3}{3a^2} + 1}} \ldots\ldots\ldots\ldots\ldots\ldots (28)$$

So we finally have,

$$a = \sqrt{\frac{2q_0}{1 - 2q_0}} sinh\eta$$

$$t = \sqrt{\frac{2q_0}{1-2q_0}}(cosh\eta - 1)$$

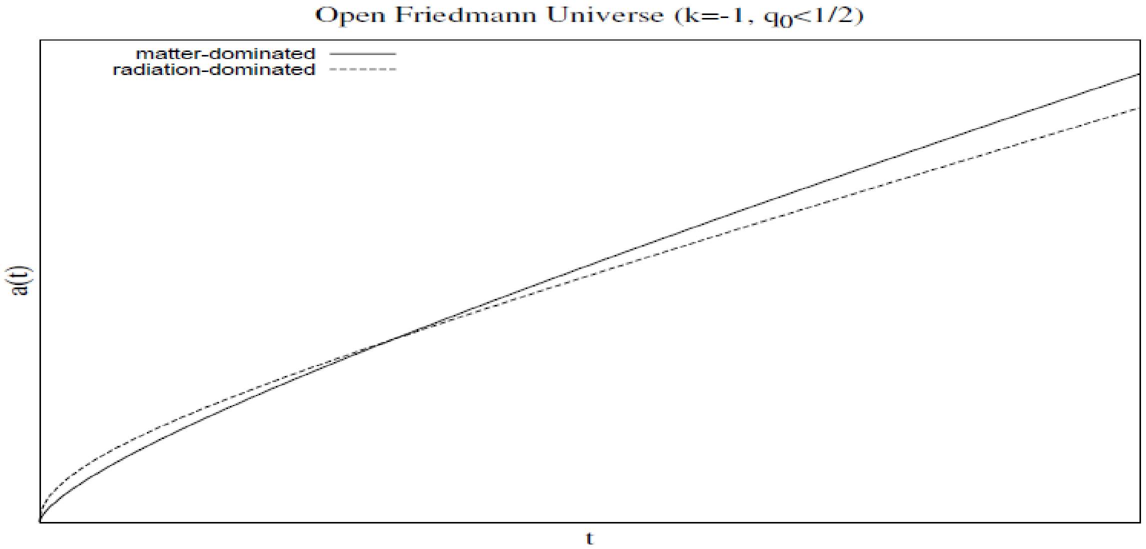

Figure 3: Evolution of $a(t)$ for the open Friedmann Universe.

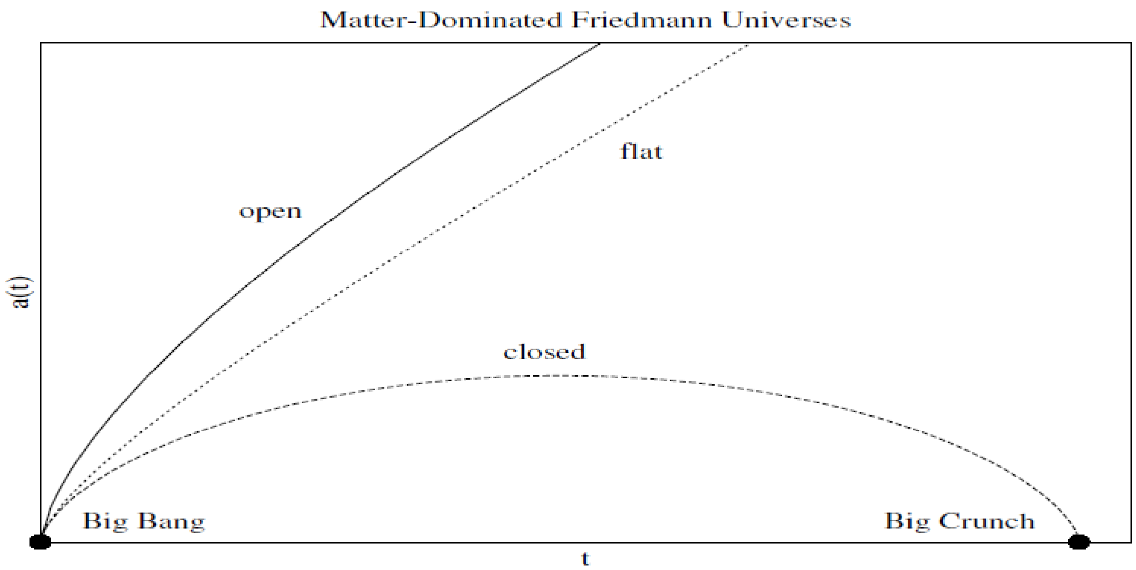

Figure 4: Evolution of $a(t)$ for the flat, closed and open matter-dominated Friedmann Universes.

IV. Discussion of Findings

In our paper, we try to show the complete view of Friedmann equations. For that, we deduced the Friedmann equations and exhibited a solution for it involving the Hubble parameter. We also calculate the density that needed to yield the flat universe and the age of the universe. The solution of the Friedmann equations are presented here for flat, the close and open universe which also shown in matter-dominated and radiation dominated universe. Friedmann equations play vital rules to understanding the cosmological models which help us to enrich the knowledge about our expanding universe.

V. Conclusion

To conclude, in this paper, we have briefly reviewed the Friedmann equations obtained from the Einstein general relativity field equation. We have also explored the simplification form of the solution of Friedmannequations for matter and radiation dominatethe universe base on the lecture of Balsa Terzic. We apply it for the flat, closed and open universe.

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