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Extension of Sin-Gordon Equation: Thickness of Superconductor Greater than London Penetration Depth

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Abstract: *The extended conventional Sine-Gordon equation (perturbed and unperturbed) and its solution for superconductor with thickness greater than London penetration depth is the main objective of this work. The extended Sine-Gordon equation contain mixed penetration depth (London and Josephson penetration depth) while conventional has only Josephson penetration depth. The extended equation contain larger number of parameters which gives deeper and detailed study of IV characteristic of superconductor with greater accuracy. In addition the extended equation give more detail information about thicker superconductors, which is certainly be one of the relevant and emerging field in physics.*

Keywords: *Superconductor, London penetration depth, Josephson penetration depth, Sine-Gordon equation, IV-Characteristics*

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I. Introduction

Superconductivity is the disappearance of electrical resistance in solid below a particular temperature called the transition temperature. Also on the basis of temperature, superconductors are divided into two groups. They are high and low temperature superconductors which are based on boiling point of nitrogen. The relation of temperature and magnetic field, according to Tuyn's law of superconductor is given as

$$H_C(T) = \left(1 - \frac{T^2}{T_C^2}\right) H_C(0) \tag{1}$$

The demonstration of equation (1) is shown in Figure 1 below. The Figure 1 shows variation of the instant temperature (T) for fixed magnetic field ($H_c(0)$) and transition temperature (T_c). The instant magnetic field decrease and become zero at $T = T_c$.

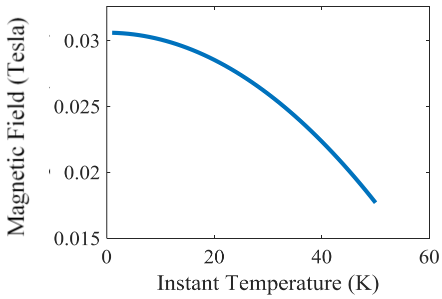

Figure 1: Instant temperature vs Instant magnetic field

When two superconductors are separated by thin insulator (≥ 30Å), a spontaneous current arises and passes through the insulators called Josephson Effect. The current is due to Cooper pair tunneling across the barrier region. There are two type of Josephson Effect, they are AC and DC. When current is passes from one superconductor to another through the barrier (insulator between them) without applying magnetic and electric field is called DC Josephson Effect (Kittel 1996) [1].

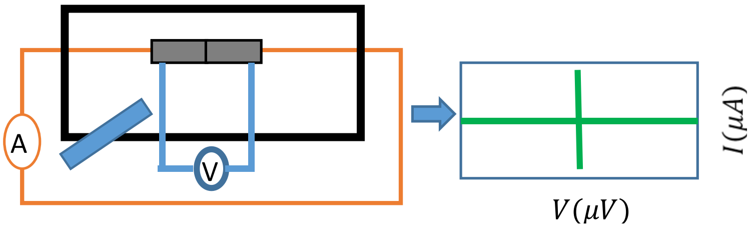

Figure 2: DC Josephson Effect at $T \leq 4.2K$, Liquid Helium

Let $\Psi_1 \approx \sqrt{n_1}\, e^{i\theta_1}$ and $\Psi_2 \approx \sqrt{n_2}\, e^{i\theta_2}$ be the wave function two coupled superconductor, then the phase difference for two ideal superconductors [2] is

$$\frac{\partial\theta_2}{\partial t} = \frac{\partial\theta_1}{\partial t} = \frac{\partial}{\partial t}(\theta_2 - \theta_1) = 0 \tag{2}$$

The result of equation (2) shows is shown in Figure 2 and the physical meaning that the rate of change in phase at the boundary without biasing is zero and current at the junction is $I = I_0 \sin(\theta_2 - \theta_1)$. This relation shows that the current is maximum and minimum at 90° and 0° respectively. The AC Josephson effect occurs when the Josephson junction has an applied DC bias (V). The cooper pairs has charge of $-2e$ and *potential* of $-2eV$.

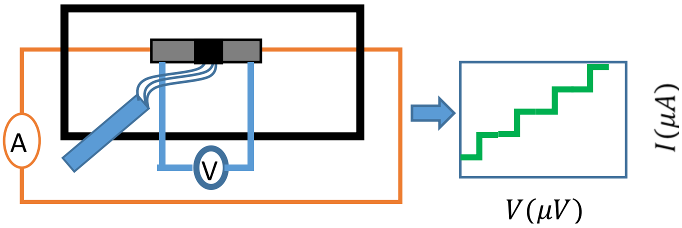

Figure 3: AC Josephson Effect at T ≤ 4.2K, Liquid Helium

The rate of change of phase difference at boundary with biasing, for AC Josephson effect is $-\frac{2eV}{\hbar}$ and current at the junction is $I = I_0 \sin\left(\delta(0) - \frac{2eVt}{\hbar}\right)$. The oscillation of frequency of cooper pair is $\frac{2eV}{\hbar}$ and ratio of frequency and voltage for this system is $483.6\frac{MHz}{\mu V}$. The AC Josephson effect has been visualized in Figure 3. The nature in Figure 3 is known as Shapiron graphs**,** first studies by Shapiro in 1963 [3].

II. Method and Material

The BCS theory was developed by Bardeen, Cooper and Schrieffer initially and extended by different scientist in 1957 using concept of cooper pairs. Bardeen formulate electron-phonon interaction without any physical significant but latter Frohlich developed and explained using exponential term as $exp\left(-\frac{1}{\lambda}\right)$, where λ is the dimensionless electron-phonon coupling constant. There are serval models to solve the problem of electron-phonon coupling constant but well known and widely accepted models are Frohlich Hamiltonian model, the Holstein model, and the BLF (Barisic-Labbe-Friedel) model. McMillan solved the Eliashberg equations numerically using electron-phonon spectral density ($\alpha^2F(\omega)$) (McMillan, 1986) as

$$T_c = \frac{\Theta}{1.45}\exp\left[-\frac{1.04(1+\lambda)}{\lambda-\mu(1+0.62\lambda)}\right] \tag{3}$$

where T_c is transition temperature, μ is Coulomb pseudopotential, λ is coupling constant and θ is Debye temperature and this equation is fitted with Coulomb pseudopotential 0.13 by Pickett et al in 2019. Equation (3) is latter modified by Allen and Dynes [4] to obtain accuracy for electron-phonon less than 1.5 as

$$T_c = \frac{\Theta}{1.45}\exp\left[-\frac{1.04(1+\lambda)}{\lambda-\mu(1+0.62\lambda)}\right] f_1 f_2 \tag{4}$$

Where the numerical value of $f_1 f_2 \sim 1$ for small λ, yielding equation (3). On the basis of electron-phonon coupling constant superconductors are characterized as strong($\lambda_{ep} \gg 1$), weak($\lambda_{ep} \ll 1$) and intermediate($\lambda_{ep} \sim 1$). The T_c relation between Debye frequency(ω_D), Debye temperature (T_θ), electron-phonon coupling constant and Coulomb pseudopotential purposed by BCS theory is

$$T_c = T_\theta \exp\left[-\frac{1}{\lambda_{e-ph} - \mu}\right] \tag{5}$$

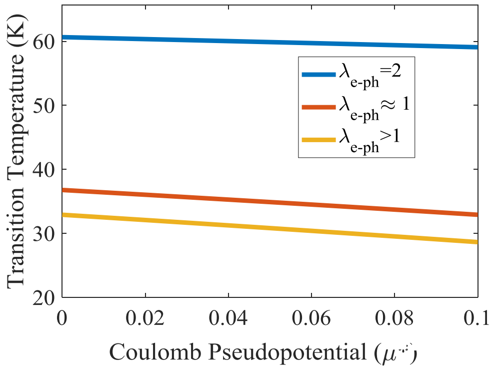

Figure 4: Representation of μ vs T_c of superconductor on the basis of λ_{ep}

The Figure 4 represent the variation of the μ vs T_c at 100K Debye temperature. The blue line represent transition temperature for strong type superconductors, red for intermediate type superconductor and yellow for weak type superconductors. Allen and Dynes also derived an asymptote for higher value of λ_{ep} or $\lambda_{e\text{-}ph}$ or λ as, $T_c = \frac{\lambda_{ep}}{2} T_\theta$ and developed a model [5] as,

$$T_c = \frac{T_\theta}{1.45} \exp\left[-\frac{1.04(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right] \tag{6}$$

The visualization of equation (6) is represented at $\mu = 1.5$ and Debye temperature 100K.

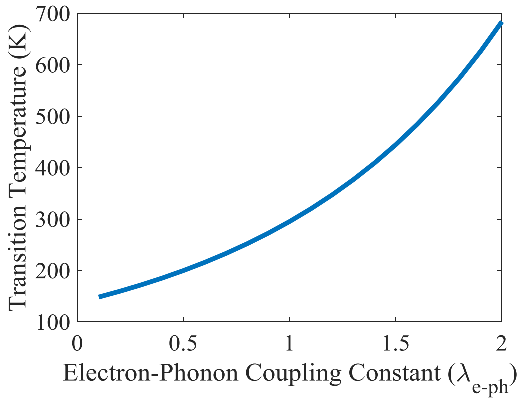

Figure 5: Variation of $\boldsymbol{\lambda_{ep}}$ vs $\boldsymbol{T_c}$ (Allen and Dynes model)

The accuracy of the model ranges as $\mu \le \lambda_{ep} \le 1.5$ [6]. It mean the μ is less than λ_{ep} and $\lambda_{ep} < 1.5$ numerically [7]. The latest and widely accepted model developed by Allen and Dynes from Eliashberg theory as,

$$T_c = \left(\frac{1}{1.20}\right)\left(\frac{\hbar}{k_B}\right)\omega_{ln} \exp\left[-\frac{1.04(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right] f_1 f_2 \tag{7}$$

Here, k_B is the Boltzmann constant, $\hbar$ is reduced Planck constant, f_1 is strong-coupling correction function, f_2 is shape correction function and have more details these term parameters by Talantsev [8]. The theoretical calculation of λ_{ep} ranges from $0 \le \lambda_{ep} \le 10^6$ [9] *while* experimental range was verified up to $\lambda_{e\text{-}ph} = 2.59$ (this value is still one of the largest experimentally measured $\lambda_{e\text{-}ph}$ to date) [10]. The Sine-Gordon equation in 1D for Josephson junction without biasing is given as,

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_J^2\frac{\partial^2\varphi}{\partial x^2} + sin\varphi = 0 \tag{8a}$$

Here, $\lambda_J = \left(\frac{\Phi_0}{2\pi\mu_0 d J_c}\right)^{\frac{1}{2}}$is Josephson penetration depth, $\omega_{pl} = \left(\frac{\Phi_0\epsilon}{2\pi J_c t}\right)^{\frac{1}{2}}$ is Josephson plasma period with flux of quanta (Φ_0), λ_J *is length* and ω_p^{-1} is time (8). The solution of (8) for small amplitude $(\varphi \ll 2\pi)$ is $\varphi \approx e^{i(kx-\omega t)}$. In general, Sine-Gordon equation does not have exact solution yet, but some researchers obtained solution of above equation using some special *cases* [11-13]. Sine-Gordon equation with biasing current for Josephson junction is given as,

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + sin\varphi + \alpha\frac{\partial\varphi}{\partial t} - \gamma = 0 \tag{8b}$$

For small amplitude $sin\varphi = \varphi,$ the solution of the Sine-Gordon equation in x-direction is obtained as $\varphi \approx e^{i(k_x x - \omega t)}$. For an applied static magnetic field in equation (8) the unperturbed Sine-Gordon equation is related as

$$-\lambda_j^2\frac{\partial^2\varphi}{\partial x^2} + sin\varphi = 0 \tag{8c}$$

For small phase ($sin\varphi \approx \varphi$) the solution of Sine-Gordon equation as e^{-x/λ_J}. Also the λ_J for thickness of tunneling barrier (L) and London penetration depth is given as $\lambda_J = \sqrt{\frac{\Phi_0}{2\pi\mu_0 J_c(L+2\lambda_L)}}$. This relation is valid only if London penetration depth is less than superconducting electrode thickness [14]. We have from Katterwe [15]; $\lambda_J^2 = \frac{\Phi_0}{2\pi\mu_0 t J_c}$ and $\lambda_L(T) = \left(\frac{m}{ne^2\mu_0}\right)^{\frac{1}{2}}$, the relation between $\lambda_L$and λ_J^2 become

$$\lambda_J^2(\lambda_L) = \frac{\Phi_0}{2\pi\mu_0 J_c\left[L+2\left(\frac{m}{ne^2\mu_0}\right)^{\frac{1}{2}}\right]} \tag{9}$$

Also we have coherence length $\xi(T) = \frac{\hbar v_f}{\pi\nabla}$ and magnetic field at a temperature $H_c(T) = \frac{\Phi_0}{2\sqrt{2}\lambda_L\xi}$ and equation (9) become,

$$\lambda_J^2(\lambda_L, H_c) = \frac{\Phi_0}{2\pi\mu_0 J_c\left(L+\frac{\pi\Phi_0\nabla}{\sqrt{2}H_C(T)\hbar v_f}\right)} \tag{10}$$

On substituting value of critical magnetic field from equation (1) and transition temperature from equation (7) in equation (10) then after solving we obtain an expression (11a) for $T =\neq 0K$ as

$\lambda_{J(e-ph)}^2(\lambda_L, H_c)_{T\neq 0} =$

$$\frac{\Phi_0\sqrt{2}H_C(0)\hbar v_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{e-ph})}{\lambda_{e-ph}-\mu(1+0.62\lambda_{e-ph})}\right]-T^2\right)}{2\pi\mu_0 J_c\left(L\sqrt{2}H_C(0)\hbar v_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{e-ph})}{\lambda_{e-ph}-\mu(1+0.62\lambda_{e-ph})}\right]-T^2\right)+\pi\Phi_0\nabla\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{e-ph})}{\lambda_{e-ph}-\mu(1+0.62\lambda_{e-ph})}\right]\right)\right)}$$

(11a)

And for T=0K as,

$$\lambda_{J(e-ph)}^2(\lambda_L, H_c)_{T=0} = \frac{\Phi_0}{2\pi\mu_0 J_c\left(L+\frac{\pi\Phi_0\nabla}{\sqrt{2}H_C(0)\hbar v_f}\right)} \tag{11b}$$

Here $\lambda_L = \frac{\pi\Phi_0\nabla}{2\sqrt{2}H_C(0)\hbar v_f}$ when compare with expression $\lambda_J = \sqrt{\frac{\Phi_0}{2\pi\mu_0 J_c(L+2\lambda_L)}}$ developed by Li et al. [14]. Now the unperturbed Sine-Gordon equation is modified with transition temperature and electron-photon coupling constant at certain temperature with thickness of superconductor electrode greater than London penetration depth **is**

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_{J(ep)}^2(\lambda_L, H_c)_{T\neq 0}\frac{\partial^2\varphi}{\partial x^2} + sin\varphi = 0 \tag{12a}$$

And $T = 0K$,

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_{J(ep)}^2(\lambda_L, H_c)_{T=0}\frac{\partial^2\varphi}{\partial x^2} + sin\varphi = 0 \tag{12b}$$

Similarly, the perturbed Sine-Gordon equation modified with transition temperature and electron-photon coupling constant at certain temperature with thickness of superconductor electrode greater than London penetration depth **is**

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_{J(ep)}^2(\lambda_L, H_c)_{T\neq 0}\frac{\partial^2\varphi}{\partial x^2} + sin\varphi + \alpha\frac{\partial\varphi}{\partial t} - \gamma = 0 \tag{13a}$$

And at $T = 0K$,

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_{J(ep)}^2(\lambda_L, H_c)_{T=0}\frac{\partial^2\varphi}{\partial x^2} + sin\varphi + \alpha\frac{\partial\varphi}{\partial t} - \gamma = 0 \tag{13b}$$

On taking the ratio of $\lambda_{J(ep)}^2(\lambda_L, H_c)_{T=0}$ with λ_J^2 and solving for $T \neq 0K$ we get,

$$\lambda_{J(ep)}^2(\lambda_L, H_c)_{T\neq 0} = \frac{\lambda_J^2 t\sqrt{2}H_C(0)\hbar v_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]-T^2\right)}{\left(L\sqrt{2}H_C(0)\hbar v_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]-T^2\right)+\pi\Phi_0\nabla\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1 f_2\right)^2\exp\left[-\frac{2.08(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]\right)\right)}$$

(14)

To preserve the equality we have from equation (14),

$$\lambda_{J(ep)}^2(\lambda_L, H_c)_{T\neq 0} = \lambda^2\lambda_J^2 \tag{15}$$

Here λ^2 is defined as,

$$\lambda^2 = \frac{t\sqrt{2}H_C(0)\hbar\nu_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1f_2\right)^2 \exp\left[-\frac{2.08(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]-T^2\right)}{\left(L\sqrt{2}H_C(0)\hbar\nu_f\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1f_2\right)^2 \exp\left[-\frac{2.08(1+\lambda_{ep})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]-T^2\right)+ \pi\Phi_0\nabla\left(\left(\frac{\hbar\omega_{ln}}{1.20k_B}f_1f_2\right)^2 \exp\left[-\frac{2.08(1+\lambda_{e-ph})}{\lambda_{ep}-\mu(1+0.62\lambda_{ep})}\right]\right)\right)}$$

Now Sine-Gordon equation (perturbed) for the case when thickness of superconductor is greater than London penetration depth with mixed penetration depth as

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda^2\lambda_J^2\frac{\partial^2\varphi}{\partial x^2} + sin\varphi + \alpha\frac{\partial\varphi}{\partial t} - \gamma = 0 \tag{16}$$

The solution of equation (16) for small amplitude is obtained as $e^{-\frac{x}{\lambda_{LJ}}}$ where $\lambda^2\lambda_J^2 = \lambda_{LJ}^2$ is mixed penetration depth. Also the Sine-Gordon equation (unperturbed) for same case (thickness of superconductor is greater than London penetration depth) is obtained as

$$\frac{1}{\omega_p^2}\frac{\partial^2\varphi}{\partial t^2} - \lambda_{LJ}^2\frac{\partial^2\varphi}{\partial x^2} + sin\varphi = 0 \tag{17}$$

III. Result and Discussion

McMillan developed equation (3) to study the transition temperature with electron-phonon coupling constant but equation (3) is modified by Allen and Dynes given as in equation (4). The continuity of Allen and Dynes develop a popular equation widely acceptable now a days is equation (6) and (7), and relates with electron-phonon coupling constant, Coulomb pseudopotential, Debye temperature, strong-coupling correction function and the shape correction function.

The conventional Sine-Gordon equation (8a) and (8b) for normal superconductors are used to study the IV characteristic of superconductors. Sine-Gordon equations (perturbed and unperturbed) of a superconductor have been developed as shown in equation (16) and (17) with their solution. The equation (16) represent perturbed Sine-Gordon equation which contain two extra terms than that of unperturbed Sine-Gordon equation (17), the two additional term is due to biasing current and voltage. The extension of conventional Sine-Gordon equation of a superconductor is possible for those superconductor circuit whose electrode thickness (supercomputer thickness) is greater than the London penetration depth. This extended Sine-Gordon equation is used to study the IV characteristic of superconductor whose thickness is greater than London penetration depth.

Equation (9) interrelates Josephson penetration depth and London penetration depth (Li et al., 2019), the developed equation (11a) represent he Josephson penetration depth at $T \neq 0K$ and (11b) at $T = 0K$. The unperturbed Sine-Gordon equation with at $T \neq 0K$ and T=0K is obtained in (12a) and (12b) respectively. The perturbed Sine-Gordon equation in (13a) and (13b) for $T \neq 0K$ and $T = 0K$, respectively. The developed equation (15) relate the Josephson penetration depth with several parameters which give deeper study of superconductor.

IV. Conclusion

In this work, the transition temperature and conventional Sine-Gordon equation of superconductor was studies. The extension of Sine-Gordon equation was developed for the superconductor with thickness greater than London penetration depth shown in equation (16) and (17). These equations are valid for those superconductor whose thickness is greater than London penetration depth. The penetration depth used in equation (16) and (17) are mixed penetration depth (Josephson and London penetration depth) which is developed in equation (15). Since equation (15) contain numbers of parameters which obviously very useful relation in coming days for the study both experimental and theoretical to study the IV characters the extension equation is helpful than conventional.

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