

Nonlinear analysis of a classical double oscillator model

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Abstract: A classical double oscillator model, that includes in certain parameter limits, the standard harmonic oscillator and the inverse oscillator, is interpreted as a dynamical system. We study its essential features and make a qualitative analysis of orbits around the equilibrium points, period-doubling bifurcation, time series curves, surfaces of section and Poincaré maps. An interesting outcome of our findings is the emergence of chaotic behaviour when the system is confronted with a periodic force term like $f \cos \omega t$.

Keywords: double oscillator, period doubling bifurcation, surfaces of section, Poincaré map, chaotic behaviour

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I. Introduction

Many years ago, Mathews and Lakshmanan [1] proposed a nonlinear deformation of the harmonic oscillator that is guided by a certain parameter λ and which accounts for a non-polynomial oscillator term. They considered the Lagrangian to be given by

$$L = \frac{1}{2} (1 + \lambda x^2)^{-1} (\dot{x}^2 - \alpha^2 x^2), \quad -\frac{1}{\sqrt{|\lambda|}} < x < \frac{1}{\sqrt{|\lambda|}} \quad (1)$$

where α is a non-negative real constant. Without λ in L , the harmonic oscillator is recovered.

The above Lagrangian also characterizes a mechanical system endowed with a position dependent-mass (PDM) [2]. The subject of PDM [3] has evinced a lot of interest in the literature and has developed into a separate branch of research over the past few decades [4–11]. In this connection, the relevance of PDM has been noteworthy in such areas as that of compositionally graded crystals [12], quantum dots [13], liquid crystals [14] etc. and some classical problems possessing quantum analogs, for example, in branched Hamiltonian systems [15].

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Returning to the Lagrangian L , apart from the presence of a rational non-oscillator potential.

$$V_{ML}(x) = \frac{1}{2} \frac{\alpha^2 x^2}{1 + \lambda x^2} \quad (2)$$

it also supports a PDM defined by

$$M(x) = \frac{1}{2} \frac{1}{1 + \lambda x^2} \quad (3)$$

Evidently, with the restriction of the configuration space specified in (1) the mass function $M(x)$ is free from any singularity. Such a form for $M(x)$ has been studied in different contexts [16]. One notices that $M(x)$ asymptotically approaches a zero-value with respect to x .

The equation of motion resulting from (1) reads

$$(1 + \lambda x^2)\ddot{x} - \lambda x \dot{x}^2 + \alpha^2 x = 0 \quad (4)$$

which belongs to the quadratic Liénard class represented by the form

$$\ddot{x} + r(x)\dot{x}^2 + s(x) = 0 \quad (5)$$

where $r(x)$ and $s(x)$ stand for

$$r(x) = -\frac{\lambda x}{1 + \lambda x^2}, \quad s(x) = \frac{\alpha^2 x}{1 + \lambda x^2} \quad (6)$$

Among the properties of the above Lagrangian include the observation of Cariñena et al [17, 18] that the kinetic term is invariant under the tangent lift of the vector field

$$X_x(\lambda) = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x},$$

the setting up of a quantum analogue in the form of a PDM problem and solving it for all possible orderings and including the effects of a linear dissipative term and demonstrating that in the presence of a two-parameter deformation [19] of the guiding potential, one runs into a chaotic behaviour of the whole system if the ranges of the coupling constants are suitably adjusted [20]. It may be remarked that in most of the investigations (see, for instance, [21, 22]) of dynamical systems, including seeking solitary wave solutions for standard systems [23], the prototype of a second-order nonlinear autonomous differential equations is employed even for bifurcation analysis for complicated systems under localized perturbations [24].

Recently, Schulze-Halberg and Wang [25] have considered modifications to the Mathews-Lakshmanan Lagrangian (1) by bringing in additional terms that not only account for the interactions present in (1) but feature a one-parameter harmonic or inverse harmonic oscillator contribution. Their two-parameter Lagrangian reads¹

$$L_{SW} = \frac{1}{2} (1 + \lambda x^2)^{-1} [\dot{x}^2 - (1 + \lambda) x^2] + \frac{\beta}{2} x^2, \lambda \in \mathfrak{R} \quad (7)$$

Here the mass-function is same as that in (3) but the potential is modified to

$$V_{SW} = \frac{1}{2} \frac{(1+\lambda)x^2}{1+\lambda x^2} - \frac{\beta}{2} x^2, \beta \in \mathfrak{R} \quad (8)$$

which represents a double oscillator expressed as a combination of the harmonic oscillator ($\beta = -1, \lambda \rightarrow 0$) and the inverted harmonic oscillator ($\beta = 3, \lambda \rightarrow 0$). Mathews-Lakshmanan nonlinear oscillator is recovered when $\beta = 0$. Specifically, when $\beta = 0$ and α^2 is set equal to $1+\lambda$ i.e. $\alpha^2 = 1+\lambda$, one gets back the Lagrangian (1) which is well known to be exactly solvable. However this is not the case when $\alpha^2 \neq 1 + \lambda$ and $\beta = 0$. The classical dynamics for the system has been studied in detail in [25] in connection with the existence of series solutions along with seeking suitable quantum analogs. Another paper by Blum and Elze [26] also investigated a double-well system semi-classically and pointed out the tendency of a chaotic behaviour. In this paper we pursue a different line of approach in that although we use the same Lagrangian as given by (7), we treat the equation of motion resulting from it as a coupled pair to look upon them from the perspective of a dynamical system [2, 27]. Subsequently we perform a qualitative analysis for the determination of local properties following from them and then include an external periodic forcing term to study its influence as an additional damping effect. As we show explicitly, we run into a chaotic behavior apart from obtaining period doubling bifurcation, time series curves, and surfaces of section and Poincaré map. Estimates of Lyaunov exponents are also made.

II. Existence of fixed points and a qualitative analysis

The equation of motion following from L_{SW} reads²

$$(1 + \lambda x^2)\ddot{x} - \lambda x \dot{x}^2 + (1 - \beta + \lambda)x - 2\beta\lambda x^3 - \beta\lambda^2 x^5 = 0 \quad (9)$$

It is of quintic type in the presence of a dissipative term which is present in the Mathews-Lakshmanan model as well.

By introducing an auxiliary variable y let us re-write (9) in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{\lambda x y^2}{1 + \lambda x^2} - (1 - \beta + \lambda) \frac{x}{1 + \lambda x^2} + 2\beta\lambda \frac{x^3}{1 + \lambda x^2} + \beta\lambda^2 \frac{x^5}{1 + \lambda x^2} \end{aligned} \quad (10)$$

The solutions of $\dot{x} = 0$ and $\dot{y} = 0$ furnish the fixed points. These are easily seen to be located on the x -axis with the squares of the abscissa satisfying

¹In L_{SW} we changed the sign of λ and β without loss of generality. ²The second term in the equation of motion in (9) has been corrected.

$$x_{1,2}^2 = \frac{-\beta \pm [(1+\lambda)\beta]^{\frac{1}{2}}}{\beta\lambda}, \quad \beta, \lambda \neq 0 \quad (11)$$

The positivity of the discriminant leads to the restriction that $(1 + \lambda)\beta > 0$. As equilibrium points are real, the sign of $(1 + \lambda)$ and that of β has to be the same. Taking both to be positive, one obtains

$$\lambda > -1, \quad \beta \geq 0 \quad (12)$$

Specifically, taking $\beta = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$, the three equilibrium points are found to be placed at

$$P_0^* = (0, 0), \quad P_1^* = (2, 0), \quad P_2^* = (-2, 0) \quad (13)$$

We now proceed to write down the Jacobian matrix which is to be evaluated at the critical points x_c

$$J = \begin{pmatrix} 0 & 1 \\ J_{21} & 0 \end{pmatrix} \quad (14)$$

Note that is off-diagonal with the expression for the element J_{21} being

$$J_{21} = \frac{-\lambda(1+\lambda) + 6\beta\lambda x_c^2 + 5\beta\lambda^2 x_c^4}{1 + \lambda x_c^2} + \frac{2\lambda(1-\beta+\lambda)x_c^2 - 4\beta\lambda^2 x_c^4 - 2\beta\lambda^3 x_c^6}{(1 + \lambda x_c^2)^2} \quad (15)$$

where x_c corresponds to the solutions $x_{1,2}$ of (11). Further J_{21} is an even function in x_c . Among the equilibrium points $P_0^* = (0, 0)$ is neutrally stable but both P_1^* and P_2^* are stable and of elliptic type. Orbits and phase portrait about the equilibrium point $P_1^* = (2, 0)$ are presented in Figure 1.

Before we embark on the inclusion of a periodic force term in (9), it is interesting to inquire into the possibility of what happens if the parameters β and λ satisfied the constraint

$$1 - \beta + \lambda = 0 \tag{16}$$

In such a case a great simplification occurs with the fixed points $x_{1,2}^2$ emerging as $x_{1,2}^2 = \frac{1}{1-\beta}$ with J_{21} acquiring the form

$$J_{21} = \frac{\beta \lambda x_c^2 (6+5 \lambda x_c^2)}{1 + \lambda x_c^2} - \frac{2 \beta \lambda^2 x_c^4 (2+2 \lambda x_c^2)}{(1 + \lambda x_c^2)^2} \tag{17}$$

With a trial value of $\beta = \frac{1}{2}$ or equivalently $\lambda = -\frac{1}{2}$, plausible solutions of x_c are ± 2 . In both the cases $J_{21} = -4$. It signals that the eigenvalues of the matrix J correspond to a pair of imaginary quantities. A linear stability analysis for different choices of the parameter values β and λ reveals, that for the reality of fixed points, one has to conform to either the conditions $1 + \lambda > 0$ and $\beta > 0$ or the pair $1 + \lambda < 0$ and $\beta < 0$. Taking the former case without any loss of generality and choosing for illustration the numerical values $\beta = 0.5$ and $\lambda = -0.5$, the fixed points turn out to be $e(0, 0), (2, 0)$ and $(-2, 0)$. Thus the stability analysis points to the following results:

- (i) The point $(0, 0)$ is neutrally stable as for the case (12).
- (ii) Further, similar to the previous case, the fixed points $(2, 0)$ and $(-2, 0)$ point to the eigenvalues $\pm 2i$ and imply that these are of elliptic (or centre) type. Hence the fixed points may be identified as stable points.

III. Inclusion Of A Periodic Forcing Term

The equation of motion when a periodic forcing term $f \cos \omega t$ is included is of the form

$$(1 + \lambda x^2)\ddot{x} - \lambda x \dot{x}^2 + (1 - \beta + \lambda)x - 2\beta\lambda x^3 - \beta\lambda^2 x^5 = f \cos \omega t \tag{18}$$

The above nonautonomous systems can be re-expressed in terms of a set of three autonomous nonlinear systems as arranged below

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{\lambda x y^2}{1 + \lambda x^2} - (1 - \beta + \lambda) \frac{x}{1 + \lambda x^2} + 2\beta\lambda \frac{x^3}{1 + \lambda x^2} + \beta\lambda^2 \frac{x^5}{1 + \lambda x^2} + \frac{f \cos z}{1 + \lambda x^2} \\ \dot{z} &= \omega \end{aligned} \tag{19}$$

In the following section the numerical simulation of the above scheme is carried out.

IV. Simulation Results

The behavior of the coupled system (19) changes drastically when the periodic forcing term is included. In particular, the character of bifurcation phenomena is profoundly changed. Keeping parameters value $\beta = 0.15$, $\lambda = -0.1$, $\omega = 1$ fixed and varying forcing amplitude f , the period doubling phenomena (Figure 2), following chaos, is observed for the system (19), as shown in Figure 3.

To observe the influence of the forcing term $f \cos \omega t$ in equation (18), we keep in mind the fact that the Poincaré map serves as the basic tool for the understanding of the stability and bifurcations of periodic orbits. The time series curves, surfaces of sections as well as the Poincaré maps are drawn by keeping fixed the values of the parameters at $\beta = 0.5$, $\lambda = -0.5$, $\omega = 1$ and different increasing values of amplitude f as shown through various plots in Figure 4. One notices the transformation of regular periodic evolution to a chaotic evolution as the amplitude is increased from the value 0.1 to 0.71. To be specific, orbits start to disintegrate around the value of $f = 0.55$ when the motion is of quasi-periodic type. However, at the values $\beta = 0.5$, $\lambda = 0.01$, $\omega = 1$ and $f = 0.65$ and $f = 0.71$ the motion turns completely chaotic. Surfaces of sections and Poincaré maps at each instant are significant and interesting. The plots of Lyapunov exponents for these two cases are given in Figure 5

V. Summary

We have investigated in this paper a classical double oscillator model that includes in certain parameter limits, the standard harmonic oscillator and the inverse oscillator. By interpreting it as a dynamical system we have made a qualitative analysis of orbits around the equilibrium points, period-doubling bifurcation, time series curves, surfaces of section and Poincaré maps. The main highlight of our results is that a chaotic behaviour is noticed when a periodic force term like $f \cos \omega t$ is imposed upon the system.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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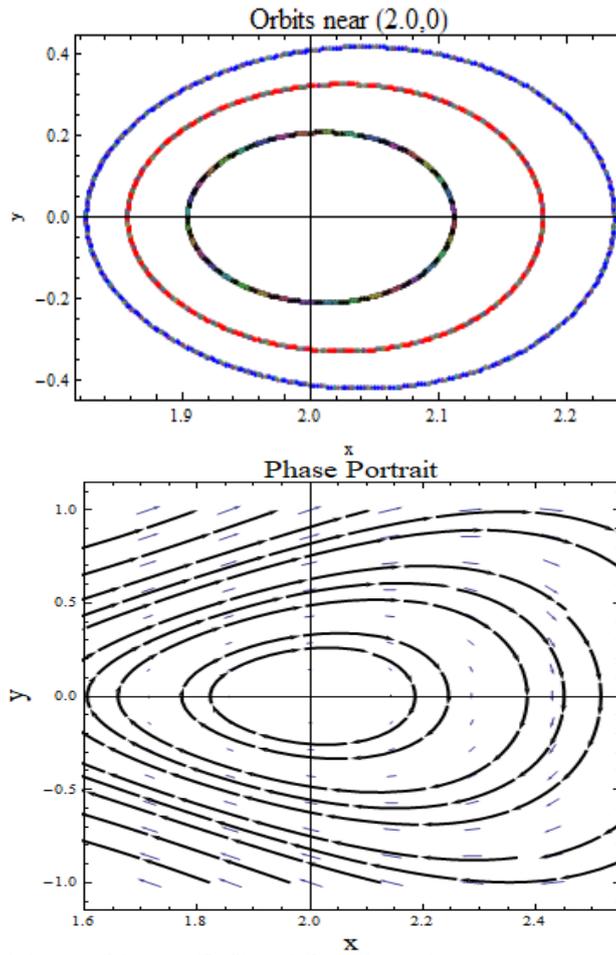


Figure 1: Orbits around the equilibrium $(2, 0)$ and flow there shown, respectively, in left and right plots

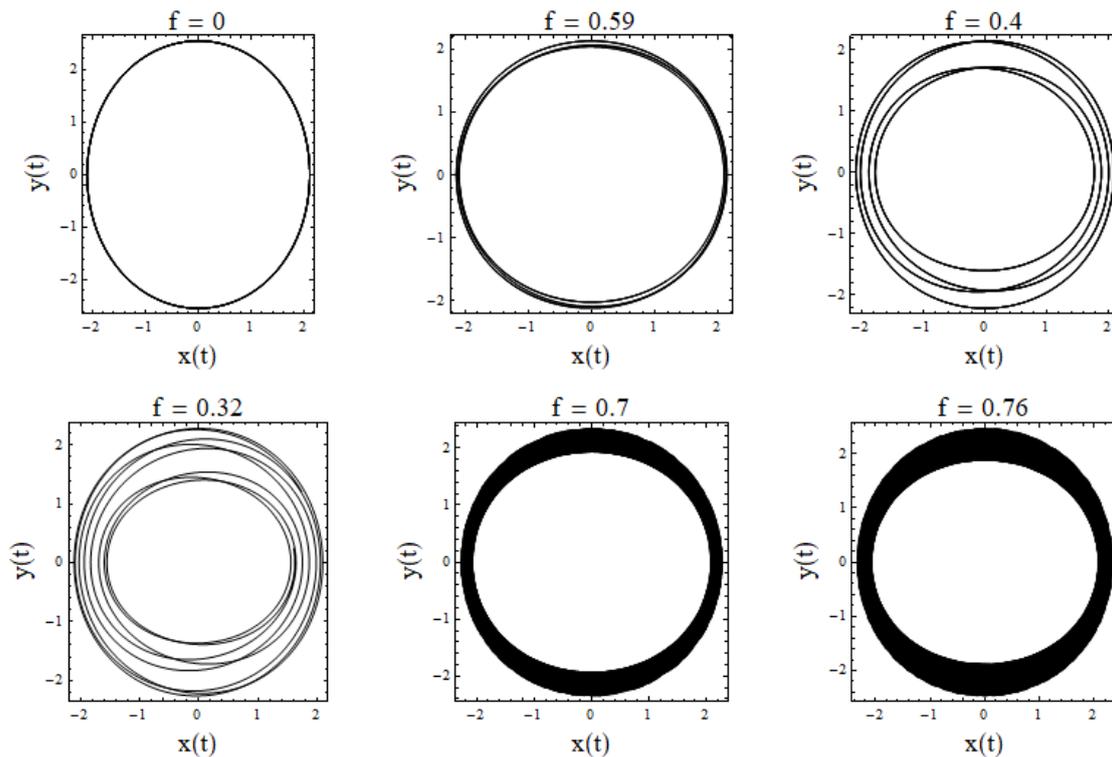


Figure 2: Period doubling bifurcation followed by chaos for system (19). Values of other parameters are $\beta = 0.15$, $\lambda = -0.1$, $\omega = 1$.

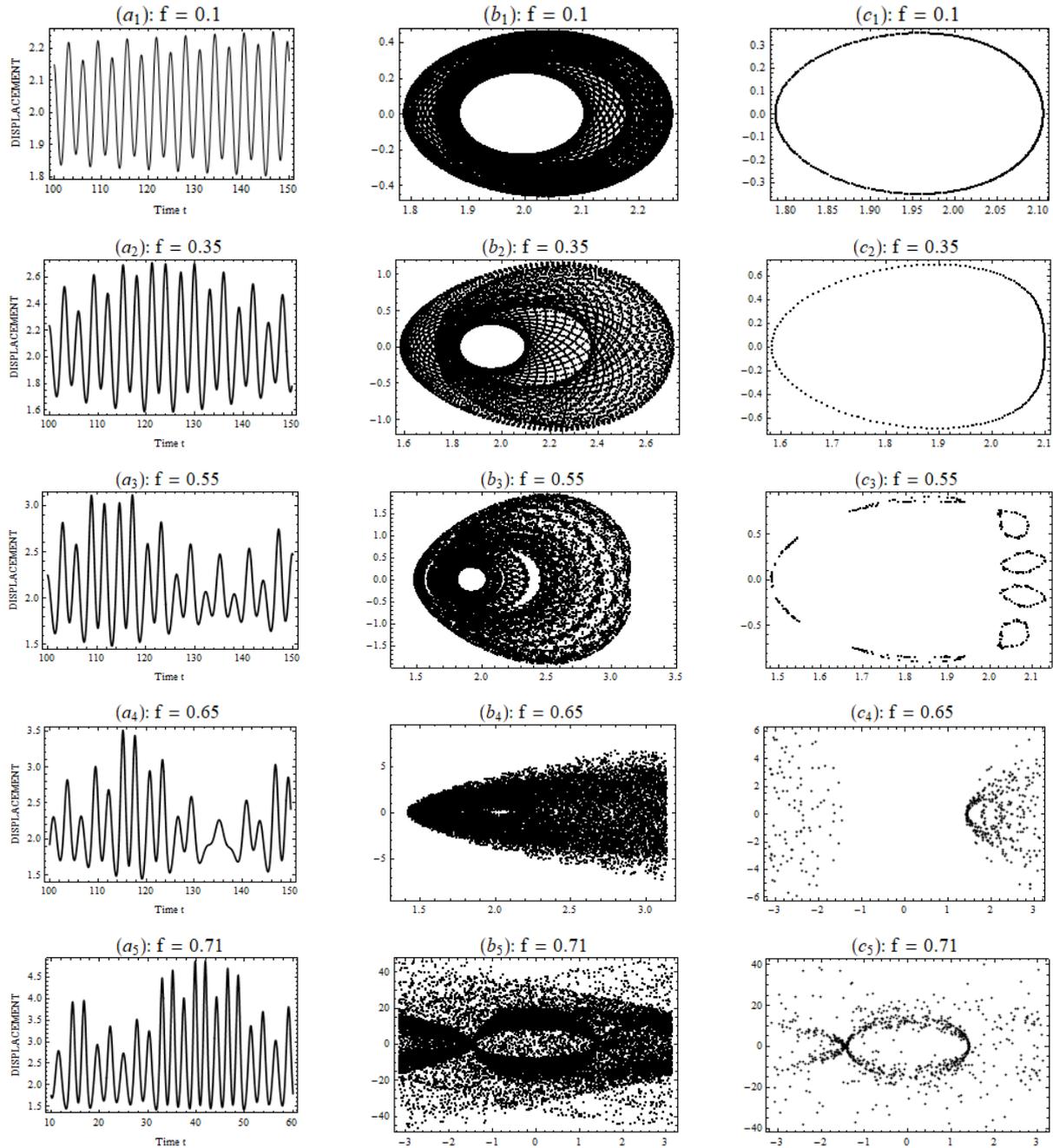


Figure 3: Plots of (i) Left column: Time series curves, (ii) Middle column: surfaces of section and (iii) Right Column: Poincaré maps, of system (19).

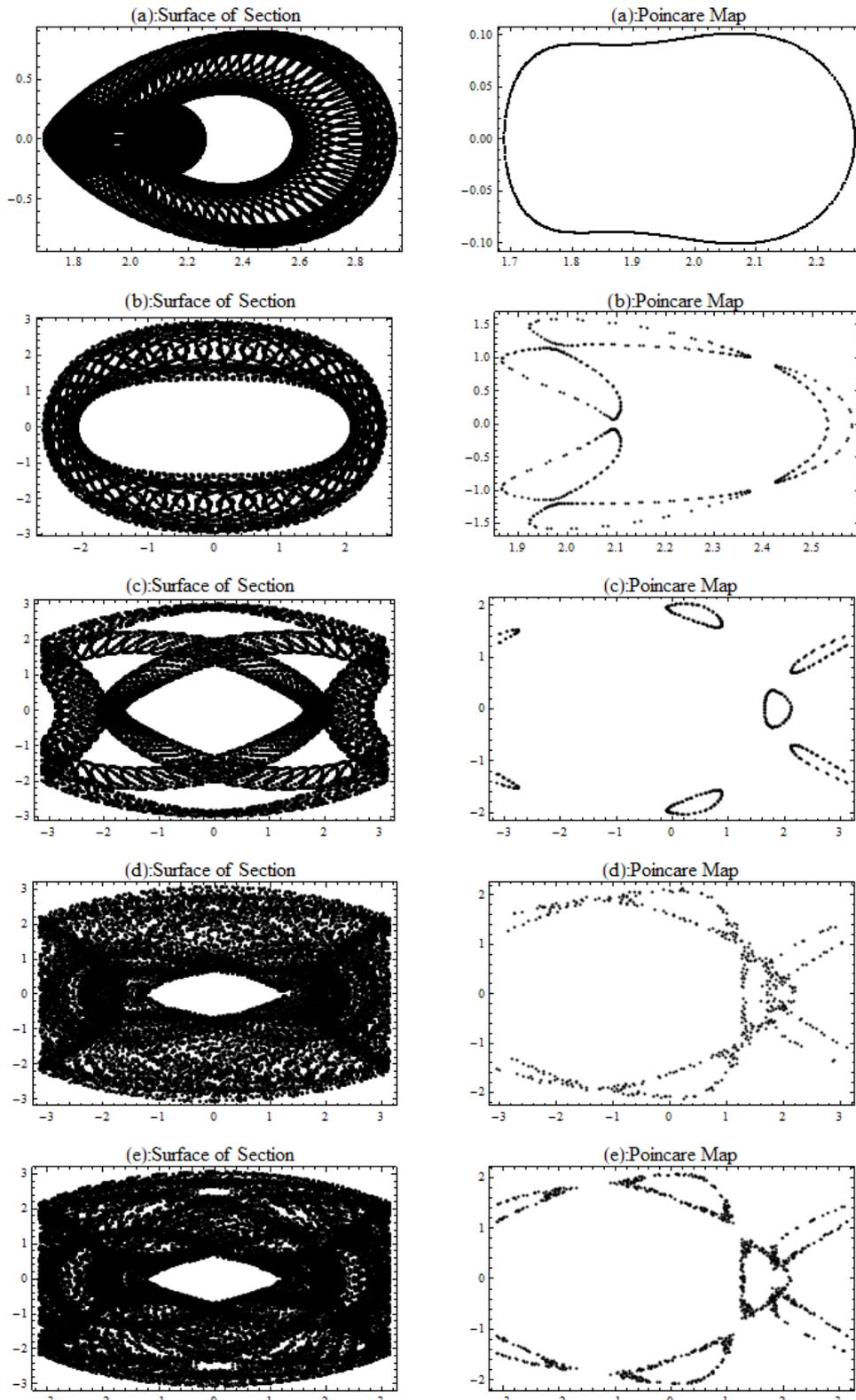


Figure 4: Plots of surface of section and Poincaré map at different parameter spaces. For all cases, $\omega = 1$ and (a) $\beta = 0.25, \lambda = -0.5, f = 0.65$, (b) $\beta = 0.5, \beta = -0.1, f = 0.7$, (c) $\beta = 0.5, \lambda = 0.01, f = 0.7$ (d) $\beta = 0.5, \lambda = 0.01, f = 0.75$, (e) $\beta = 0.5, \lambda = 0.01, f = 0.76$

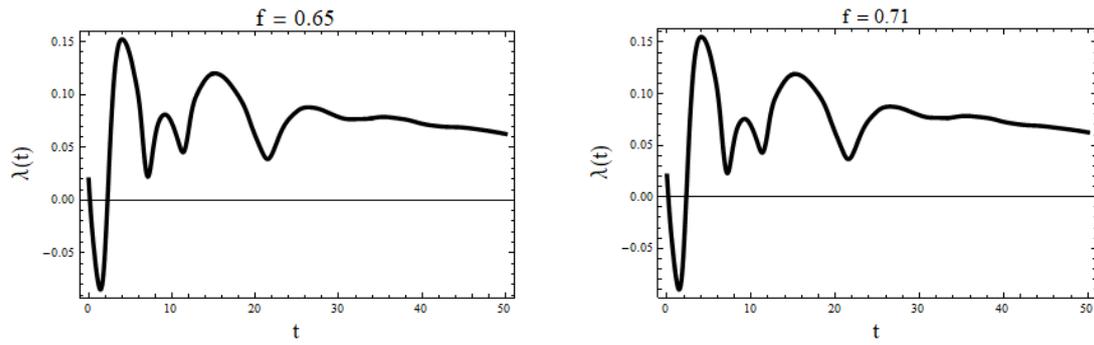


Figure 5: Plots of Lyapunov exponents for two chaotic cases

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