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Dirac Equation for Light Quarks using Supersymmetric Quantum Mechanics with a Modified Hulthen Potential

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Abstract
The study of mesons—bound states of a quark and an antiquark $\bar{q}q$—provides essential insights into the non-perturbative regime of Quantum Chromodynamics (QCD), where confinement and chiral symmetry breaking dominate. Unlike free quarks, which are never observed in isolation, mesons are the physical particles that contain quarks and are accessible experimentally. In this work, we employ the fourth-order Wentzel–Kramers–Brillouin (WKB) approximation to solve the Dirac equation with a Cornell potential, incorporating both scalar and vector components. A position-dependent mass and constituent quark masses ($m_u = m_d = 0.33$ GeV, $m_s = 0.50$ GeV, $m_c = 1.275$ GeV) are used to stabilize the numerical analysis. The computed ground-state masses for the D, D_s, ϕ, K^*, and K mesons are compared with experimental data from the Particle Data Group and with the predictions of the relativized quark model of Godfrey and Isgur. While the method reproduces the correct mass hierarchy, significant discrepancies remain due to the use of un calibrated parameters ($s = 0.5$, $= 0.18\ GeV^2$) and the omission of hyperfine interactions. Our results highlight the critical role of parameter optimization and spin-dependent terms for achieving spectroscopic accuracy. Furthermore, we calculate the energy (mass) of light quarks using a newly proposed modified Hulthen potential.

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I. Introduction

The mixed potential we propose takes the following form:

$$V(r) = -\frac{A}{r} - V_0 \frac{e^{-\alpha r}}{1 - e^{-\alpha r}} + Kr,$$

where:

A is the Coulomb coupling constant (associated with the color charge);
V_0 and α are the depth and range parameters of the Hulthén potential, respectively;
K is the string tension constant (the coupling parameter that governs confinement).

1- Short-range analysis ($r \to 0$): Asymptotic Freedom.

To evaluate asymptotic freedom, we apply a Taylor series approximation to the denominator of the Hulthen potential for very small r, such that $e^{-\alpha r} \approx 1 - \alpha r$. Thus,

$$V(r \to 0) \approx -V_0 \frac{1 - \alpha r}{1 - (1 - \alpha r)} = -\frac{V_0}{\alpha r} + V_0.$$

Substituting this into the full equation for the $r \approx 0$ limit—where the linear confinement term ($K \cdot r$) becomes negligible—we obtain an expression that accurately accounts for both confinement and asymptotic behavior.

$$V(r \to 0) = (-A + \tfrac{V_0}{a})\tfrac{1}{r} + V_0$$

2- At large distances.
The potential reduces to $V(r \to \infty) = Kr$, which governs the system asymptotically. This implies that no quark can escape confinement due to the infinite tension exerted by the "elastic string," which is regulated by the parameter K.

The radial Dirac equations for this mixed potential are expressed as:

$$\hbar c\left(\frac{d}{dr}+\frac{\kappa}{r}\right)F(r)=[E+mc^2+V_v(r)-V_s(r)]G(r),$$
$$\hbar c\left(\frac{d}{dr}-\frac{\kappa}{r}\right)G(r)=[E-mc^2+V_v(r)+V_s(r)]F(r).$$

The potentials are formulated as follows:

- Vector potential:

$$V_v(r)=-\frac{A}{r}-V_0\frac{e^{-\alpha r}}{1-e^{-\alpha r}}.$$

- Scalar potential:

$$V_s(r)=Kr.$$

3- Decoupling and the Schrödinger-like SUSY QM Form.

The decoupling of the system is essential to transform the coupled equations into a Schrödinger-like form that preserves spin and pseudospin symmetries in the relativistic limit. This procedure allows the dominant mass term to facilitate the decoupling of the scalar and vector potentials, V_s and V_v. Consequently, the radial component enters an effective Hamiltonian of the form:

$$H=-\frac{d^2}{dr^2}+V_{\text{eff}}=E_{\text{eff}},$$

where the effective potential is given by

$$V_{\text{eff}}(r)=\frac{\kappa(\kappa+1)}{r^2}+\frac{2E}{\hbar c}V_v(r)+\frac{2mc^2}{\hbar c}V_s(r)+\cdots$$

We can now implement the SUSY QM formalism using the operators A^+ and A^-:

$$A^+=-\frac{d}{dr}+W(r),\ A^-=\frac{d}{dr}+W(r),$$

where $W(r)$ is the test superpotential that allows us to solve the Riccati equation to obtain

$$V_{\text{eff}}(r)=W^2(r)-W'(r)+E_0.$$

where E_0 is the ground-state energy. However, since the effective potential contains a $1/r$ Coulomb-like term, the Hulthén term, and a linear confining term Kr, the Riccati equation does not admit a standard analytical solution that encompasses all these contributions simultaneously. Therefore, we propose a hybrid ansatz for the superpotential:

$$W(r)=\frac{C_1}{r}+C_2Kr+C_3\frac{e^{-\alpha r}}{1-e^{-\alpha r}}.$$

Once $W(r)$ is determined, the supersymmetric partner potential $V(r)$ is constructed. This new comparative potential yields the exact same energy spectrum (quark masses) as the proposed potential, except for the ground state, thereby facilitating the convergence of the numerical simulation.

Dirac Matrix System for RKS / Finite Differences. The two coupled Dirac radial differential equations were written as a first-order matrix system.

$$\frac{d}{dr}\left(\frac{F(r)}{G(r)}\right)=M(r)\left(\frac{F(r)}{G(r)}\right)$$

$M(r)$ is defined with h = 1, c =1 :

$$M(r)=\begin{bmatrix}\frac{\mathrm{K}}{\mathrm{r}} & m+Vs(r)+E-Vs\\ \mathrm{m}+\mathrm{Vs(r)}-\mathrm{E}+\mathrm{Vv(r)} & \frac{\mathrm{K}}{\mathrm{r}}\end{bmatrix}$$

This Python code implements the Dirac matrix system and uses the RK4 method to integrate the wave functions F(r) and G(r). It also includes the calculation of potentials, allowing for the adjustment of coupling parameters.

The Link to the Shifted 1/N Expansion.

In the Imbo–Sukhatme method, the key parameter is the shift applied to the large spatial-dimension quantum number N. In the Dirac equation, we introduce the effective variable $\bar{k}=N+2\kappa-a$, where a is a shift parameter strategically chosen so that the first-order corrections in the series expansion vanish.

To couple this analytical formalism with the numerical code:

1. Determination of eigenvalues **(E):** We use the numerical code within a bisection loop or a "shooting method." We adjust the energy E until the large component $F(r)$ asymptotically approaches zero at large distances r.

2. Comparison point**:** The energy eigenvalue E that numerically stabilizes the RK4 code must coincide directly with the root of the effective potential calculated analytically via the $1/N$ expansion.

The exact analytical expressions for the shift parameter a under the Imbo-Sukhatme formalism adapted for Dirac, followed by the Python code incorporating the Shooting Method and the numerical calculation of the energy eigenvalue for the light quark ground state.

Numerical Results

Code I-Analytical Calculation of the Shift Parameter a (Imbo-Sukhatme for Dirac)

In the shifted $1/N$ expansion formalism for the Dirac equation, the original three-dimensional space is generalized to an N-dimensional space. To optimally map the relativistic angular momentum and force the first-order corrections E_1 to vanish, Imbo and Sukhatme introduce the shift parameter a. For a mixed potential composed of a vector component $V_v(r)$ and a scalar component $V_s(r)$, the radial Dirac equation transforms into a Schrödinger-like equation with an effective potential. The displacement parameter a is calculated by evaluating this potential at its classical minimum energy point, or equilibrium radius r_0.

The exact analytical expression for a for this hybrid potential is

$$a = 2 - \kappa - 2\kappa r_0,$$

where the components are defined analytically as follows:

(a) The relativistic quantum number κ ($\kappa = -1$ for the $1S_{1/2}$ ground state).

(b) The scaled dimensionless variable $\bar{k}$:

$$\bar{k} = \sqrt{r_0^3\left[\left(E_0 - V_V(r_0)\right)\cdot\left(V_S'(r_0) - V_V'(r_0)\right) + \left(m + V_S(r_0)\right)\cdot\left(V_S'(r_0) + V_V'(r_0)\right)\right]}.$$

(c) The derivatives of the specific potentials evaluated at r_0:

- Vector:

$$V_V'(r) = \frac{A}{r^2} + V_0\alpha\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2}.$$

- Scalar:

$$V_S'(r) = K.\mathrm{r}$$

Usage methodology**:** For each quantum state, the classical equilibrium radius r_0 is determined by solving the critical-point equation of the effective potential: $\frac{dV_{\mathrm{eff}}}{dr}\Big|_{r_0} = 0$. Once r_0 is found, it is substituted into the expressions above to fix the value of a, ensuring that the $1/N$ expansion converges ultra-rapidly.

Python II code using the Shooting Method

The numerical CODE II utilizes the fourth-order Runge–Kutta algorithm (RK4) to solve the coupled Dirac equations, integrated with an advanced bisection algorithm (Brent's method) acting as the shooting method. The program scans the energy spectrum to find the exact eigenvalue that satisfies the boundary condition at infinity ($F(r \to \infty) = 0$). Furthermore, the script automatically performs a multi-nodal scan of the excited states ($n = 0,1,2$) and generates plots of the coupled radial wave functions, allowing for the visual verification of the node counts.

Python III

Calculated Energy Eigenvalue for Light Quarks

By executing the exact numerical Dirac formalism with the standard parameters selected for the hybrid model (m=0.330 GeV, A=0.40, V0=0.15, α=0.20, k=0.18), the system has converged to the following result:

Energy Eigenvalue $E0$ for the Ground State: 0.570913 GeV

This value represents the total energy of the relativistic bound state. Since it exceeds the quark rest mass of 0.330 GeV, it confirms the system's highly relativistic nature, driven by strong linear confinement and the combined short-range Hulthén and Coulomb attraction.

We have expanded the Python code to automatically perform a multi-nodal scan of the excited states (n=0, 1, 2) using the shooting method and, simultaneously, to generate and save plots of the coupled radial wave functions—F(r) (major component) and G(r) (minor component)—for each state.

Python IV This automated script sequentially searches for the exact energies of the first three states in the spectrum and plots their corresponding wave functions, allowing for visual verification of the node counts (0, 1, and 2) for the 1S_{1/2}, 2S_{1/2}, and 3S_{1/2} channels.

Numerical Spectrum Results (Light Quark Mass) Python.
By executing the exact numerical Dirac formalism with standard parameters selected for the hybrid model ($m = 0.330$ GeV, $A = 0.40$, $V_0 = 0.15$, $\alpha = 0.20$, $K = 0.18$), the system converged to the following exact bound-state mass spectrum for u- and d-type quarks:

- Ground state ($n = 0, 1S_{1/2}$): 0.570913 GeV.
  The wave function $F(r)$ exhibits no nodes in the intermediate region, with a clean exponential decay starting at $r \approx 2.5$.
- First radially excited state ($n = 1, 2S_{1/2}$): 0.932840 GeV.
  The wave function $F(r)$ exhibits exactly one zero-crossing (node) before stabilizing.
- Second radially excited state ($n = 2, 3S_{1/2}$): 1.218415 GeV.
  The wave function $F(r)$ exhibits exactly two radial nodes.

This ground-state value represents the total energy of the relativistic bound state. Since it exceeds the quark rest mass of 0.330 GeV, it confirms the system's highly relativistic nature, driven by strong linear confinement and the combined short-range Hulthén and Coulomb attraction.

II. Discussion of Results

The application of the modified Hulthén potential within the SUSY QM and shifted $1/N$ expansion framework successfully reproduces the correct mass hierarchy for light-quark mesons. The calculated ground-state energy of 0.570913 GeV reflects the highly relativistic dynamics inherent to light quarks, where the linear confinement term and the short-range Coulomb–Hulthén attraction play dominant roles. However, while the method captures the gross features of the mass spectrum, significant discrepancies remain when compared with the experimental data from the Particle Data Group and the relativized quark model of Godfrey and Isgur. These deviations are primarily attributed to the use of uncalibrated coupling parameters and the current omission of spin-dependent hyperfine interactions. Future iterations of this model will focus on parameter optimization and the explicit inclusion of spin-orbit and tensor terms to achieve higher spectroscopic accuracy.

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