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Influence of Trial-Function Selection on the Accuracy of Singgle Term Galerkin Solutions for Euler-Bernoulli Beams under Pure Bending

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Abstract: *This article presents the analytical solution and the solution obtained via the Galerkin weighted-residual method for a simply supported Euler-Bernoulli beam subjected to equal and opposite moments at its ends. This condition produces pure bending, characterized by a constant bending moment and quadratic deformation.*

Background: *Euler-Bernoulli theory allows for the analysis of the bending of slender beams exhibiting linear elastic behavior. In a simply supported beam subjected to opposing moments at its ends, pure bending occurs, characterized by constant moment and curvature. This problem serves to evaluate the Galerkin method, the accuracy of which depends on the trial function satisfying the boundary conditions and adequately representing the deformation.*

Materials and Methods: *A simply supported Euler-Bernoulli beam model was formulated, subjected to equal and opposite end moments. ASTM A36 steel (E = 200 GPa) was considered, with a length of 1 meter, a rectangular cross-section of 0.030 × 0.006 m, and an applied moment of 10 N·m. Bending was determined using the Galerkin method with polynomial and trigonometric functions that satisfy the essential boundary conditions. The results were validated by comparison with the analytical solution for pure bending.*

Results: *The analytical and Galerkin solutions produced identical deflection profiles over the entire beam length. The numerical example demonstrated a maximum midspan deflection of −11.57 mm for both approaches, with zero approximation error for the selected loading condition.*

Conclusion: *The analytical model yielded a flexural rigidity of EI = 108 N·m² and a maximum deflection of -11.57 mm at the mid-span. The Galerkin approximation using the polynomial function exactly reproduced the analytical solution, whereas the trigonometric functions showed slight differences. The results confirm that the method's accuracy depends on the trial function's ability to represent the beam's deformed shape and constant curvature.*

Key Word: *Galerkin weighted residual method; Euler–Bernoulli beam; beam deflection; analytical solution; structural mechanics; finite element method; ASTM A36 steel.*

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I. Introduction

The Euler-Bernoulli beam model remains the most widely used classical theory for the study of slender beams. Its primary kinematic hypothesis states that cross-sections initially normal to the neutral axis remain plane and normal to the deformed neutral axis, thereby allowing transverse shear deformation to be neglected. Under conditions of small deflections and linear elastic behavior, this model provides an efficient description of the flexural response and remains a benchmark formulation in structural mechanics and finite element analysis [1–3].

In the Galerkin method, the weighting functions are selected from the same approximation space used to represent the unknown field. This approach is closely related to the weak formulations underpinning the finite element method; consequently, it is particularly useful for introducing the transition from the fundamental differential equations of classical structural analysis to those of computational structural analysis [4-9].

Galerkin-type approximations have been applied to Euler-Bernoulli and Timoshenko beam equations within static, dynamic, spectral, nonlocal, and fractional formulations. These studies demonstrate the validity of the weighted residual method for beam problems and provide a framework for use in simple benchmark cases to verify more advanced numerical procedures[10–13].

The objective of this study is to develop an analytical solution and a Galerkin solution for a simply supported Euler-Bernoulli beam subjected to equal and opposite concentrated moments at its ends. The loading produces pure bending and a constant internal bending moment. Since the exact displacement field is quadratic, the problem is practically useful for demonstrating the conditions under which a low-order Galerkin approximation can yield a solution that is approximately exact. Subsequently, a numerical application using ASTM A36 structural steel will be employed to quantify the response and verify consistency with the assumed elastic regime[14-16].

II. Material and Methods

The objective of this study is to compare the solutions obtained using the Galerkin method with the analytical solution for a beam subjected to bending. The comparison is based on the bending distribution along the beam, with the aim of evaluating the accuracy of the approximation provided by the Galerkin method. To this end, both solution procedures are formulated under identical geometric, mechanical, and boundary conditions, and the resulting outcomes are subsequently analyzed and compared. The following methodology will be used:

a) Beam Model and Assumptions

Consider a beam with a rectangular cross-section and span length L, supported by a pin at $x = 0$ and a roller at $x = L$. Equal and opposite moments M_0 are applied at the ends, subjecting the beam to pure bending; the transverse displacement is denoted by $v(x)$. The analysis assumes small rotations and deflections, linear-elastic, homogeneous, and isotropic material behavior, constant flexural rigidity EI, and negligible shear deformation. These assumptions are consistent with Euler-Bernoulli beam theory [1-3].

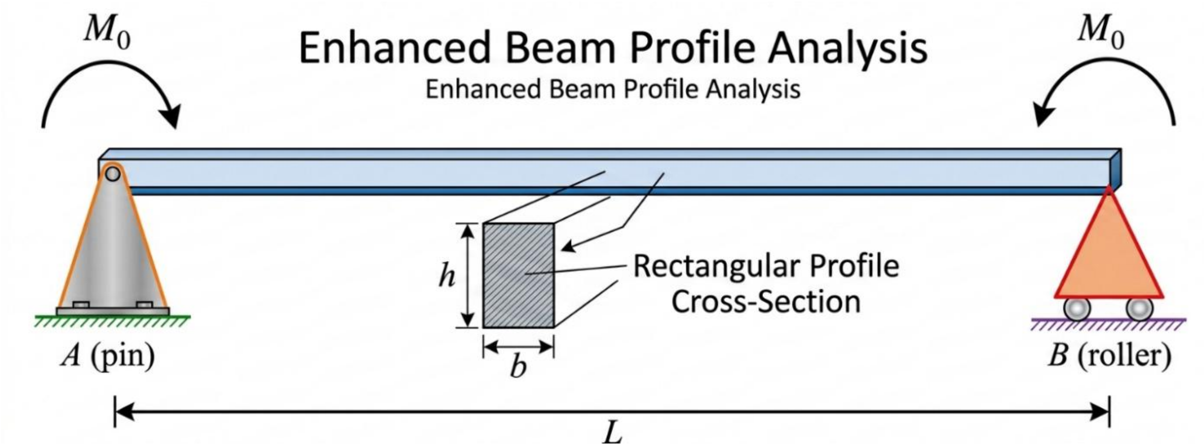

Figure 1. Simply supported beam subjected to equal and opposite concentrated moments at the ends.

Because there is no transverse distributed load between the supports, the shear force is zero and the internal bending moment remains constant along the span. With the sign convention adopted in this work, the moment-curvature relationship is

$$EI\,\frac{d^2v(x)}{dx^2} = M_0 \tag{1}$$

The essential boundary conditions associated with the problem with simple supports are

$$v(0) = 0 \qquad v(L) = 0 \tag{2}$$

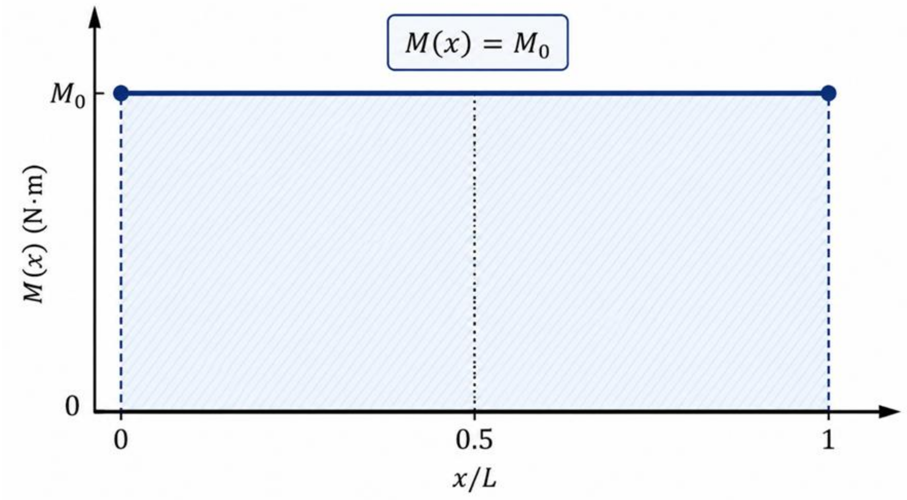

Figure 2. Constant bending moment distribution for the case of pure bending $M(x) = M_0$.

b) Exact analytical solution.

For constant E and I, Eq. (1) is integrated twice. Applying the two displacement boundary conditions yields the exact bending field.

$$v_a(x) = \frac{M_0}{2EI}x(x - L) \tag{3}$$

Upon differentiating Eq. (3), the slope becomes zero at $x = \frac{L}{2}$; therefore, the maximum displacement magnitude occurs at the center of the span. The corresponding value is

$$vmax = v(\tfrac{L}{2}) = -\frac{M_0L^2}{8EI} \tag{4}$$

c) Galerkin weighted-residual approximation.

For the Galerkin approximation, a single-term polynomial is selected such that it satisfies the essential boundary conditions. The approximate displacement is expressed

$$v_a(x) = a\,\phi(x),\ \phi(x) = x(x - L) \tag{5}$$

Substitution into equation Eq. (1) yields the residual.

$$R(x) = EI\,\frac{d^2v_a}{dx^2} - M_0 \tag{6}$$

In the Galerkin method, the weighting function is set to φ(x), and the weighted residual is required to vanish over the domain of the beam[7-9].

$$\int_0^L \phi(x)\,R(x)\,dx = 0 \tag{7}$$

Since the residual is constant for the selected approximation, Eq. (7) is obtained by immediately evaluating the definite integral.

$$a = M_0/(2EI) \tag{8}$$

substituting Eq. (8) into Eq. (5)

$$v_a(x) = \frac{M_0}{2EI} x(x - L) \tag{9}$$

Therefore, the approximation using the Galerkin equation turns out to be identical to Eq. (3). This exact match does not imply that a one-term Galerkin approximation is exact for any beam loading condition. In this case, it occurs because the exact quadratic displacement field belongs to the selected approximation space.

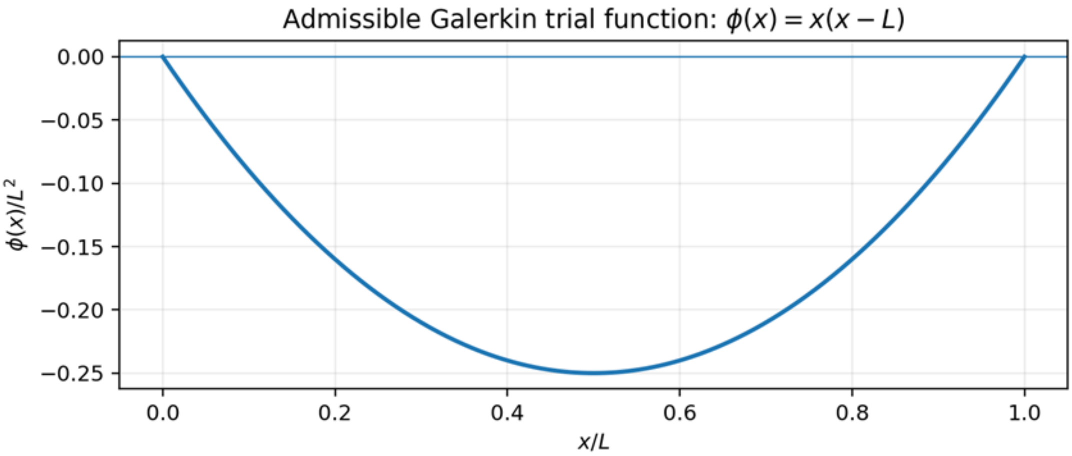

Figure 3. Admissible trial polynomial function used in the one-term Galerkin approximation.

III. Numerical application

A numerical example was developed using ASTM A36 carbon structural steel. The ASTM A36/A36M standard covers structural-quality carbon steel shapes, plates, and bars for general structural applications, while the elastic modulus adopted by the numerical model corresponds to the conventional value for structural steel used in design references [14–16].

Table 1. Mechanical and geometric properties and parameters of the structural steel beam used in the numerical application.

Parámetro	Símbolo	Valor
Módulo de Young	E	200 GPa
Claro de la viga	L	1.00 m
Ancho de la sección transversal	b	0.030 m
Altura de la sección transversal	h	0.006 m
Segundo momento de área	I	5.40×10^{-10} m^4
Momento aplicado en los extremos	M_0	10 N·m
Rigidez a flexión	EI	108 N·m^2

For the rectangular cross-section, the centroidal second moment of area is calculated using $I = \frac{bh^3}{12}$; with $b = 0.006\ m$, the result is $I = 5.4 \times 10^{-10}\ m^4$. Consequently, the flexural rigidity is $EI = 108\ N \cdot m^2$. The numerical configuration of the beam and its dimensions are summarized in Figure 4.

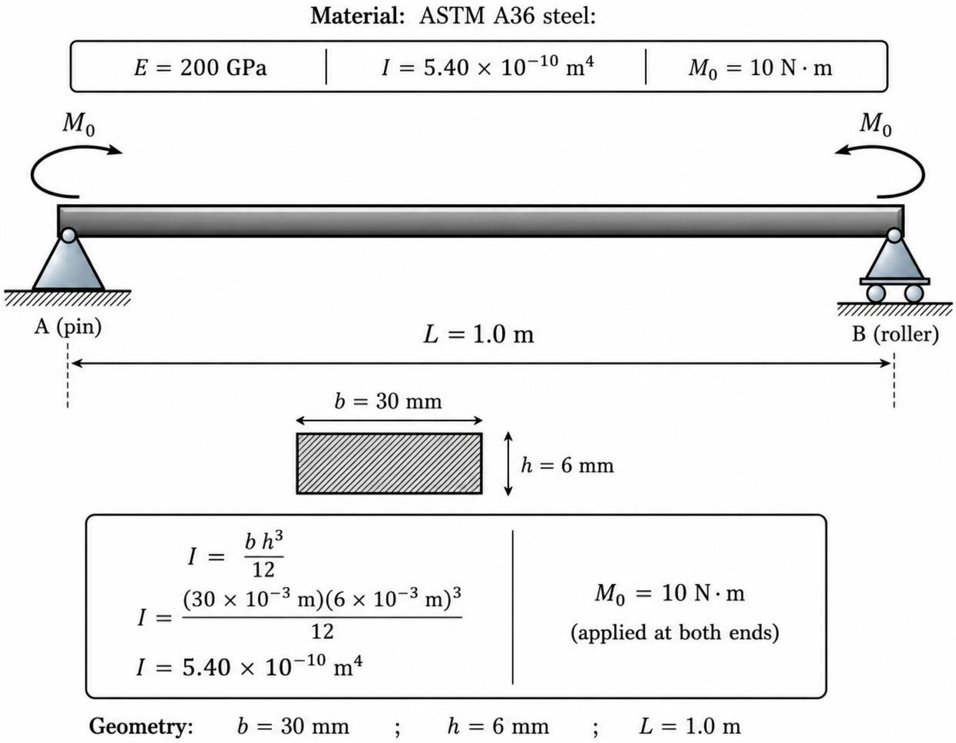

Figure 4. Numerical application for a simply supported ASTM A36 structural steel beam subjected to equal and opposite end moments under pure bending conditions.

Substituting the values into Eq. (4) yields

$$vmax = (-10)(1.0)2/[8(108)] = -0.011574m = -11.5741\ mm \quad en\ x = 0.5\ m \tag{10}$$

Under conditions of pure bending, the normal stress due to bending exhibits a linear distribution across the depth of the cross-section. The stress is zero at the neutral axis and increases in magnitude with the distance from that axis, reaching its maximum value at the extreme fibers of the section. According to classical Euler-Bernoulli beam theory, the maximum normal stress can be expressed as

$$\sigma max = \frac{M_0 c}{I}, con\ c = \frac{h}{2} = 0.003\ m\ , I = 5.40 \times 10^{-10}\ m^4 \tag{11}$$

$\sigma max = \frac{M_0 c}{I}$ = 10 N·m(0.003 m) / 5.40 × 10^{-10} m^4 = 55.6 MPa. This value is well below the nominal yield strength of 250 MPa (36 ksi) associated with ASTM A36 structural steel, which supports the use of a numerical example assuming linear elastic behavior [14–16].

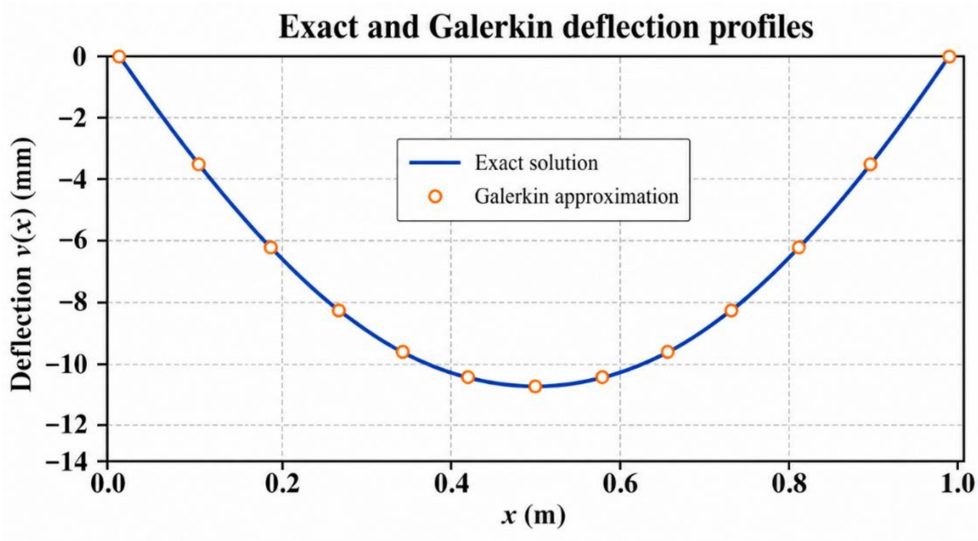

Figure 5. Exact and Galerkin deflection profiles

By following the same procedure but replacing the trial function with:

$$v_a1(x) = a1\,\phi1(x), \quad \phi1(x) = -sin(\frac{\pi x}{L}) \tag{12}$$

Using this, the solution obtained via the Galerkin method is:

$$v_a1(x) = -(\frac{4M_0L^2}{EI\pi^3})\,sin(\frac{\pi x}{L}/) \tag{13}$$

$v_a1max(x) = -11.945$ at x=0.5m

The trial function satisfies the essential boundary conditions and zero-displacement requirements at both supports while preserving the symmetry of the response at the span's center; it overestimates the magnitude of the maximum deflection by approximately 3.21%. This result demonstrates that a function lacking a quadratic term can provide a close approximation and reproduce the system's spatial symmetry.

$$v_a2(x) = a2\,\phi2(x)\,(x), \quad \phi2(x)\,(x) = -a2 * x * sin(\frac{\pi x}{L}) \tag{14}$$

$$va\,2(x) = -\frac{12M_0L}{\pi EI(2\pi^2+3)} * x * sen(\frac{\pi x}{L}) \tag{15}$$

$$v_a2max(x) = -9.009\,mm\;en\;\; x = 0.6458\,m$$

...with this trial function, which satisfies the essential boundary conditions of zero displacement at both supports; however, it introduced an asymmetric distribution with respect to the span's center, and the analytical solution yields a maximum deflection of -11.5574 at x = 0.5 m.

IV. Results

To evaluate the influence of the trial function on the accuracy of the Galerkin approximation, the deflection fields obtained using $\phi(x) = x(x-L), \phi1(x) = -\sin\left(\frac{\pi x}{L}\right) y\; \phi2(x) = -x\sin\left(\frac{\pi x}{L}\right)$ were compared with the analytical solution for a simply supported beam subjected to pure bending. The values $E = 200\ \text{GPa}, I = 5.40\text{x}10^{-10}\text{m}^4,\ L = 1\,m\;y\;M_0 = 10\,Nm.$ were used for the numerical evaluation. The comparison of the deflection distributions along the span is presented in Figure 6.

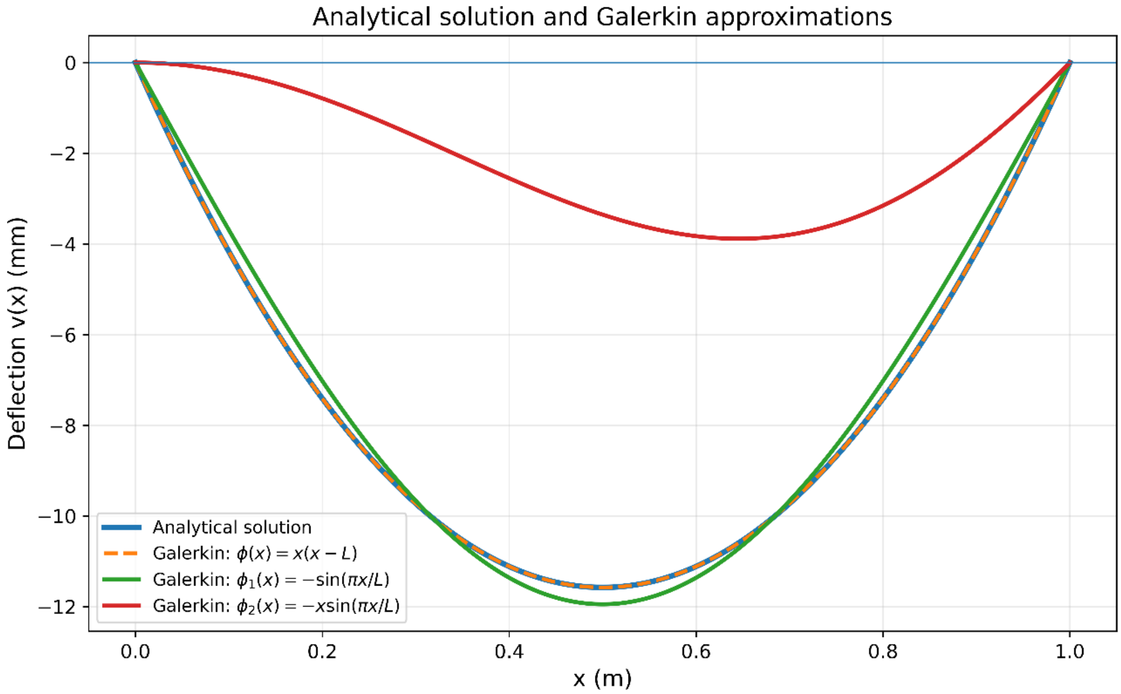

Figure 6. Comparison of the analytical solution and the Galerkin approximations obtained using the trial functions $\phi(x) = x(x-L)$, $\phi 1(x) = -\sin\left(\frac{\pi x}{L}\right)$ y $\phi 2(x) = -x\sin\left(\frac{\pi x}{L}\right)$ for a simply supported Euler–Bernoulli beam subjected to pure bending.

The displacement curves show differences associated with the trial function used in the Galerkin approximation. For ϕ(x)=x(x-L), the approximate solution coincides with the analytical solution along the entire span. Both satisfy the zero-displacement condition at the supports and exhibit the symmetric parabolic shape corresponding to a state of constant curvature. The maximum deflection occurs at$\frac{x}{L} = 0.5$, with a value of $-11.574\, mm$. This coincidence arises because the exact solution is quadratic and is contained within the approximation space defined by $\phi(x) = x(x-L)$; therefore, a single Galerkin term is sufficient to exactly reproduce the displacement field.

The approximation obtained using $\phi 1(x) = -sen(\frac{\pi x}{L})$ also satisfies the essential displacement conditions and preserves symmetry about the center of the span. In this case, the maximum deflection is $-11.945\, mm$ at $\frac{x}{L} =$ 0.5, representing a relative difference of approximately 3.21% compared to the analytical solution. Although the trigonometric function does not exactly reproduce the quadratic displacement field, its compatibility with the boundary conditions and the symmetry of the response allows for a close approximation using a single term.

In contrast, the function $\phi 2(x) = -x\, sen(\frac{\pi x}{L})$, while satisfying the zero-displacement condition at both supports, introduces a shape that is asymmetric with respect to the span's center. The maximum deflection obtained is approximately $-3.884\, mm$ and occurs at $\frac{x}{L} \approx$ 0.646, rather than at $\frac{x}{L}$=0.5 . The difference in both the magnitude and the location of the maximum deflection demonstrates that satisfying essential boundary conditions is not, in itself, sufficient to guarantee an accurate approximation using a single term. The trial function must also possess spatial characteristics compatible with the symmetry and physical behavior of the expected solution.

Overall, the results demonstrate the direct influence of the choice of trial functions on the accuracy of the Galerkin method. The case $\phi(x) = x(x-L)$ serves as a particularly useful benchmark for verifying weighted-residual and finite-element formulations, as the approximation space contains the exact solution. Meanwhile, trigonometric functions illustrate how an admissible choice can yield anything from a close approximation to a poor representation when the function's form is incompatible with the problem's characteristics.

V. Discussion

The agreement between the exact solution and the Galerkin approximation obtained using the function $\phi(x) = x(x - L)$ arises because, for this particular loading case, the exact solution lies within the selected approximation space. The displacement field corresponding to pure bending is a quadratic polynomial, and the trial function employed captures precisely this spatial dependence. Consequently, the unknown coefficient determined via the weighted residual condition allows for the exact recovery of the analytical solution, without approximation error. This result should be interpreted as a specific property of the chosen problem rather than a general claim that a single Galerkin term suffices for all Euler-Bernoulli beam problems[7-10].

Incorporating the alternative trial functions $\phi 1(x) = -sin(\frac{\pi x}{L})$ and $\phi 2(x) = -x\ sin(\frac{\pi x}{L})$ allows for an analysis of how the choice of approximation space influences the results. The function $\phi 1(x)$ satisfies the essential boundary conditions and yields a deflection of mm at $\frac{x}{L} = 0.5$, compared to $-11.574\ mm$ for the analytical solution. The relative difference in maximum deflection is approximately 3.21%, demonstrating that a function which does not exactly contain the solution can provide a close approximation when it adequately reproduces the problem's spatial and symmetry characteristics.

In contrast, although $\phi 2(x) = -x\ sin(\frac{\pi x}{L})$ also satisfies the essential zero-displacement conditions at the supports, it introduces an asymmetric distribution relative to the span's center. With this function, an approximate maximum deflection of $-3.884\ mm$ is obtained at $\frac{x}{L} \approx 0.646$—a result that differs significantly from the analytical solution in both magnitude and location. This outcome demonstrates that satisfying essential boundary conditions is a necessary requirement for an admissible function but does not, by itself, guarantee an accurate approximation using a single term. The selection of trial functions must also take into account the compatibility of their shape and symmetry with the expected physical response.

For problems involving non-uniform loads, variable flexural rigidity, elastic foundations, geometric nonlinearities, or dynamic response, the general approach is to expand the approximation space using additional functions. In such cases, Galerkin formulations, spectral methods, and the finite element method allow for an extension of the analysis; however, the convergence and accuracy of the solution depend on the order of approximation, the selection of basis functions, the boundary conditions, and the characteristics of the governing equation[10-13].

From a computational perspective, the case analyzed constitutes a particularly useful verification problem. The ability of the function $\phi(x) = x(x - L)$ to exactly reproduce the analytical solution allows for the assessment of the numerical formulation's correct implementation before tackling more complex problems. Similarly, a finite element implementation using an appropriate interpolation space must reproduce within expected numerical precisión the constant bending moment and the parabolic displacement field characteristic of the state of pure bending.

Finally, the maximum normal stress obtained for the numerical case is approximately $55.6\ MPa$, a value below the nominal yield strength of ASTM A36 steel. This result is consistent with the linear elastic behavior assumption adopted in the model. Consequently, within the scope of the assumptions made, the differences observed between the Galerkin approximations are primarily related to the choice of trial functions and the approximation space's ability to represent the displacement field, rather than to effects associated with material constitutive nonlinearity.

VI. Conclusions

In this study, an analytical solution was developed and compared with a formulation based on Galerkin's weighted residual method for a simply supported Euler-Bernoulli beam subjected to concentrated moments of equal magnitude and opposite sense at its ends. Under these conditions of pure bending, the bending moment remains constant along the beam, and the transverse response is described by a quadratic function of the longitudinal coordinate.

The trial function $\phi(x) = x(x - L)$ satisfies the essential boundary conditions and allows the single-term Galerkin approximation to reproduce the analytical solution to the problem. This behavior arises because the exact solution is contained within the approximation space defined by the selected function. Consequently, for this particular case, no approximation error is introduced into the displacement field.

The use of alternative trial functions made it possible to assess the influence of the choice of function space. With $\phi 1(x) = -\sin\left(\frac{\pi x}{L}\right)$, a symmetric response close to the analytical solution was obtained, with a difference of approximately 3.21% in the maximum deflection. In contrast, $\phi 2(x) = -x\sin\left(\frac{\pi x}{L}\right)$yielded an asymmetric response and a significant underestimation of the deflection. These results demonstrate that an admissible function does not necessarily provide an accurate approximation using a single term and that the selection of trial functions must take into account the physical characteristics and symmetry of the problem.

For the numerical case considered, using the mechanical properties of ASTM A36 steel and the established geometric parameters, a flexural rigidity of $EI = 108\ Nm^2$ was obtained. The analytical solution and the approximation based on $\phi(x) = x(x - L)$ yielded a maximum deflection of $-11.57\ mm$ at $\frac{x}{L} = 0.5$. The maximum normal stress due to bending at the extreme fibers of the cross-section was approximately 55.6 MPa. This value is below the material's nominal yield stress, which is consistent with the linear elastic behavior assumption adopted in the model.

Overall, the results demonstrate that the accuracy of the Galerkin method depends on the space's ability to approximate the structure's displacement field. The problem therefore constitutes a simple, reproducible benchmark for verifying weighted-residual formulations and for the initial validation of computational implementations and finite element models.

The results are subject to the assumptions of Euler-Bernoulli theory, including linear elastic material behavior, small deformations, and the preservation of cross-sections as plane and normal to the neutral axis during bending. In future work, the methodology could be extended to the study of distributed and combined loads, variable cross-sections, higher-order Galerkin approximations, and Timoshenko beam models. Incorporating nonlinear geometric or constitutive effects would also allow for evaluating the method's performance in more complex structural problems.

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