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Construction of the Real Number System

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Abstract: *In this paper it will be shown that* $\mathbb{R}$ *is the only complete Archimedean ordered field. The two approaches to completeness for Archimedean ordered field are given and then it is concluded that they are equivalent.*
Keywords: *Complete Archimedeanordered field, right end point, open lower segment.*

I. Defect In Rationals

Althoughit has beennoticedthatthesetof rationals $\mathbb{Q}$ formsa richalgebraic systemhavingorderproperties,itisinadequatefor thepurposeofanalysis. It hasalready beennoticed that not every positive rational number has a rational square root. For example,there is norational number whose square is 2. The defect in the rationalsthat has been described here may be described in a variety of ways. One form of the defect that hasalready beennoticed is that a nonempty subset of rational numbers that is bounded above need not have a least upper bound (in the set of rationals). A slightyless standard approach which is more picturesque is that there are open lower segments in $\mathbb{Q}$ without havingrightendpointsin $\mathbb{Q}$,andthis isconsideredtobea defect.Forthis purpose,thenotionofanopenlowersegmentinthesetofrationals is defined.

Definition 1:Aset $J \subset \mathbb{Q}$ iscalledanopenlowersegmentif

1) $J \neq \Phi$
2) $J \neq \mathbb{Q}$
3) Forevery $x \in J$,thereisa $y \in J$ suchthat $y > x$,
4) If $x \in J$ and $y < x$ then $\in J$.

Apoint x willbecalledtherightendpointofanopenlowersegmentJ if

1) forevery $y \in J$, $x > y$,
2) if z issuchthatforevery $y \in J$, $z > y$,then $z \geq x$.

For example, consider $J \subset \mathbb{Q}$ to be the set of all non-positive rational numbers, together withthepositive rational numbers whosesquareisless than2.Then J isanopenlowersegmentof $\mathbb{Q}$ havingnorightendpointin $\mathbb{Q}$.

Note:Ifarightendpointof J exists,itisunique,andisthesmallestrational greaterthaneveryelementof J.

II. The Real Numbers

The set of real numbers is an extension of the set of rational numbers, whichremovesthedefectdescribedabove.It will beseenthattherealnumbers are an Archimedean ordered field in which every open lower segment has a right end point. The real numbers are obtained from the rationalsby what may be described as filling the gaps. In otherwords, the rightendpoints are addedasidealelements,ornewnumbers,tocorrespondtothose openlowersegmentswhichdonothaverightendpointsamongtherationals.

Onewayof doingthisistoconsidertheopenlowersegmentitselftobea substitute foritsownrightendpoint.Onecanseethatthisisanentirely natural approach when one agrees that open lower segments aretobein one-to-onecorrespondencewiththeirrightendpoints.

Inaccordancewiththeabove,theset $\mathbb{R}$ ofrealnumbersisdefinedtobethe setofopenlowersegmentsofrationalnumbers.

Firstaddition is defined in $\mathbb{R}$.
Let I and J berealnumbersi.e.,openlowersegmentsofrationalnumbers. Define $I + J$ as

$$I + J = \{x + y : x \in I, y \in J\}$$

Then $I + J$ isanopenlowersegment.Let $x \in I$ and $y \in J$, so that $x + y \in I + J$, and let $u < x + y$. Then we can write $u = x + z$, where $x \in I$ and $z (= u - x) < y$, so that $z \in J$. So , $x + z > x + y$ and $x + z \in I + J$. Now, it follows that $I + J$ is an open lower segment. Also

$$I + J = J + I$$

andthat

$$(I + J) + K = I + (J + K)$$

follows directly from the definition of $' + '$in $\mathbb{R}$and commutative and associativepropertiesof$\mathbb{Q}$.
Let$\mathcal{O}$betheopen lowersegment ofnegative rational numbers. One canverifythat

$$I + \mathcal{O} = I for all I \in \mathbb{R}$$

Finally,theequation

$$I + X = J$$

hasasolutionforevery$I, J \in \mathbb{R}$.Let$X$consistofall$x$such that$y + x \in J$foreveryy $\in I$,exceptforthelargestsuch$x$ifthereisone.
Thesolutionof$I + X = \mathcal{O}$isdesignated$-I$.

AnorderrelationisintroducedinRbyletting

$$I > J if I \supset J$$

Thus$\mathbb{R}$becomesanorderedset.Clearly$I > \mathcal{O}$ifandonlyif$I$containsa positiverational.
Nowmultiplication is definedin$\mathbb{R}$.If$I > \mathcal{O},\ J > \mathcal{O}$define$IJ$asthe setofallnon-postiverationalstogetherwithall$xy$where$x \in I$,$x > 0$,and $y \in J$,y>0

If$I = \mathcal{O}$ or $J = \mathcal{O}$define$IJ = \mathcal{O}$.Ifboth$I < O$and$J < O$define

$$IJ = (-I)(-J)$$

IfexactlyoneofI, Jislessthan$\mathcal{O}$,say$I < \mathcal{O}$,define

$$IJ = -(-I)(J)$$

Nowonecanshowthat$\mathbb{R}$isanorderedfield.Thecompletedetailscanbe foundin[2].
Moreover,$\mathbb{R}$isArchimedean.Let$I > O$,$J > \mathcal{O}$.Thenthereexistsapositive rational$x \in I$andan$n \in \mathbb{N}$suchthat$x \notin J$,since$\mathbb{Q}$isArchimedeanand I, Jareopenlowersegments.Thus,itfollowsfromthetrichotomyproperty of$\mathbb{R}$that$nI > J$.

Then$J \in \mathbb{R}$forwhich$J$hasarationalrightendpointareinone-to-one correspondence withtherationalnumbers,theassociatedmappingisorder preservingandadditionandmultiplicationpreserving.

Thus,$\mathbb{Q}$isimbeddedin$\mathbb{R}$,orthat$\mathbb{Q}$isisomorphictoanorderedsubfieldof $\mathbb{R}$.Thissubfieldis called therationalnumbersandhencesmallletters will be used forelementsof$\mathbb{R}$inTheorem2.
Nowopenlowersegmentsofrealsandtheirrightendpoints can be defined inthesamewayasforrationals.

Theorem1:Everyopenlowersegmentofrealshasarightendpoint.

Proof: Let$\mathcal{J}$beanopenlowersegmentofreals.Let

$$U = \cup\{J : J \in \mathcal{J}\}$$

Weshowthat$U$isanopen lower segment ofrationalsi.e., $U \in \mathbb{R}$.
Let$x \in U$.Thenthereisa$J \in \mathcal{J}$suchthat$x \in J$. For every$y < x,\ y \in J \subseteq U$.Alsothereisa$y \in J \subseteq U$suchthat$y > x$.
Moreover, thereisan$I \notin \mathcal{J}$ since$\mathcal{J}$ isan open lower segment ofreals. Then $x \notin I$implies $x \notin U$ because if$x \in U$ then $x \in J$ for some $J \in \mathcal{J}$which implies $I \subset J$ and hence $I \in \mathcal{J}$, a contradiction. Thus U is an open lower segment of rationals. Next it is shown that U is the right end point of $\mathcal{J}$.By definition of $U, U > J$ for every $J \in \mathcal{J}$.
Suppose $V > J$for every $J \in \mathcal{J}$.Then $V \geq U$. Hence, $U$is the right end point of $\mathcal{J}$.

NowanotherformofTheorem1 is given whichisreferredtoastheleastupper boundproperty(orcompleteness propertyinthesenseofDedekind) ofthe setofrealnumbers$\mathbb{R}$.

Theorem2:If$S \subset \mathbb{R}$isnonemptyandhasanupperbound,ithasaleast upperbound.

Proof: Ifthegivenupperboundbelongsto$S$thenthatwillonlybetheleast oneandwearedone. Soweassume thatnoupper boundfor$S$isin$S$. Now we define a set U by letting $x \in U$ if and only if there is a$y \in S$ such that $y > x$. We show that U is an open lower segment of reals. If $x \in U$ and $z < x$ then $z \in U$ by definition of U. Also, if $x \in U$ then there is a $z \in U$ such that $z > x$since no upper bound for S is in S. Moreover, every upper bound of S is not in U. Thus, Uisanopenlowersegmentofreals,andsoithasarightendpoint, say$u$byTheorem1. Next it is shownthat$u = \sup S$.Let$x \in S$.Then$y < x$implies$y \in U$sothat$y < u$.Hence$x \leq u$(since u<x⇒u∈Uwhichisacontradiction).Thus,uisanupperboundofS.

Lety<u.Theny∈UsinceUisanopenlowersegmentanduisitsright endpoint.SothereisanxinSwithy<x.Thus,yisnotanupperbound ofS.Henceu=supS.

Thus,itisactuallythepropertyofTheorem1thatisbeingusedtoobtain allfurtherpropertiesof therealnumbers.Thus,thefactthatRisa complete Archimedeanorderedfieldisbasictoallfurtherdevelopments.

The Theorem3 below shows that ℝis the only complete Archimedean orderedfield.

Theorem3:AnytwocompleteArchimedeanorderedfieldsF_1andF_2,with setsofpositiveelementsP_1andP_2,respectively,arealgebraicallyandorder isomorphic, i.e.,thereexists aone-to-one mapping τofF_1ontoF_2such that

$$\tau(x+y)=\tau(x)+\tau(y), \tau(xy)=\tau(x)\tau(y), \tau(x)\in P_2 iff x\in P_1.$$

Proof: Let1_1and1_2betheunitsof$F_1$and$F_2$and0_1and0_2thezeros. Notethateveryorderedfieldcontainsanisomorphofℚ.Sowedefine the mappingτfirstontherationalelementsofF_1asfollows.

$$\tau\left(\frac{m}{n}1_1\right)=\frac{m}{n}1_2$$

where$m$isaninteger,$n$isanonzerointeger

If$x\in F_1$and$x$isnotoftheform ,$\frac{m}{n}1_1$then define

$$\tau(x)=\sup\{\frac{m}{n}1_2:\frac{m}{n}1_1<x\}$$

Onecanprovethatτhasthedesiredproperties.

Nowanotherconstructionof therealnumbersystem is given inwhicha real numberisdefined asanequivalenceclassofCauchysequencesofrational numbers.

Definition 2 : Let$F$ beanorderedfield. A sequence $(a_n)$of elements of F is called bounded if there is an element $b\in F$such that $|a_n|\le b$for each positive integer n.

Definition 3: Asequence (a_n)ofelements of$F$iscalled Cauchy iffor every $e\in F$such that $e>0$, there is a positive integer$N$ such that $|a_p-a_q|<e$for all $p,q\ge N\cdot$

Definition 4:Asequence(a_n)ofelementsof$F$iscallednullifforevery $e\in F$suchthat$e>0$,thereisapositiveinteger$N$suchthat,$|a_p|<e$for all$p\ge N$.

Thefamiliesofsequencessatisfyingtheseconditionswillbedenoted by $\mathcal{B},\mathcal{C}$and $\mathcal{N}$respectively.

Now fewtheorems and lemmas (without proofs) are statedbefore stating thetwoimportantresultswhichconcernthemainthemeof thispaper.The detailsoftheproofscanbefoundin[3].

Theorem4:Theinclusions$\mathcal{N}\subset\mathcal{C}\subset\mathcal{B}$isobtained.

Theorem 5: For(a_n), $(b_n)\in\mathcal{C}$,let $(a_n)+(b_n)=(a_n+b_n)$ and $(a_n)(b_n)=(a_nb_n)$. With these definitions of sum and product, $\mathcal{C}$ is a commutative ring with unity, and $\mathcal{N}$ is an ideal in $\mathcal{C}$such that $\mathcal{N}\subsetneq\mathcal{C}$.

Theorem6:Let$\mathcal{C}/\mathcal{N}$denotethesetwhoseelementsaretheset $(a_n)+\mathcal{N}$ (called cosets of $\mathcal{N}$),where $(a_n)\in\mathcal{C}$. Addition and multiplication in $\mathcal{C}/\mathcal{N}$ are defined by

$((a_n)+\mathcal{N})+((b_n)+\mathcal{N})=(a_n)+(b_n)+\mathcal{N}=(a_n+b_n)+\mathcal{N}$and

$$((a_n)+\mathcal{N})((b_n)+\mathcal{N})=(a_n)(b_n)+\mathcal{N}=(a_nb_n)+\mathcal{N}$$

Thesedefinitionsareunambiguous,andwithadditionandmultiplicationso defined,$\mathcal{C}/\mathcal{N}$isafield.

Notation: The field $\mathcal{C} / \mathcal{N}$ will be written as $\bar{F}$. Henceforth elements $(a_n) + \mathcal{N}$ of $\mathcal{C} / \mathcal{N}$ will be denoted by small greek letters: $\alpha, \beta, \ldots$ .... If $a \in F$ then the element $(a_n) + \mathcal{N}$ of $\bar{F}$ will be written as $\bar{a}$; it is the coset of $\mathcal{N}$ containing the constant sequence all of whose terms are a.

Theorem 7: In $\bar{F}$, let $\bar{P} = \{\alpha \in \bar{F}: \alpha \neq \bar{0} and there exists\ (a_n) \in \alpha$ such that $a_n > 0$ for n= 1,2,…..}. With this set $\bar{P}$, $\bar{F}$ is an ordered field. The mapping τ: $\tau(a) = \bar{a}$ is an order preserving algebraic isomorphism of F into $\bar{F}$.

Definition 5: Given a sequence a_n in an ordered field F and $b \in F$, it is said that limit of a_n is b and we write $\lim_{n\to\infty} a_n = b$ or $a_n \to b$ if for every positive e in F there exists a positive integer L such that $|a_n - b| < e$ for all $n \geq L$. An ordered field is said to be complete (in the sense of Cantor) if every Cauchy sequence in F has a limit in F.

Lemma 1: A sequence with a limit is a Cauchy sequence. If (a_n) is a Cauchy sequence and $\left(a_{n_k}\right)$ is a subsequence with limit b, then (a_n) has limit b.

Lemma 2: For $\alpha > 0, \alpha \in \bar{F}$, there exists $e \in F$ such that $\bar{0} < \bar{e} < \alpha$. If F is Archimedean ordered,then $\bar{F}$ is also Archimedean ordered.

Lemma 3: Let $\alpha \in \bar{F}$ and $(a_n) \in \alpha$. Then $\lim_{n\to\infty} \bar{a}_n = \alpha$.

Now the two important results mentioned above are stated.

Theorem 8: The field $\bar{F}$ is complete (in the sense of Cantor).

ThefollowingTheorem9 belowshowsthatacompleteArchimedeanordered field (completeinthesenseofCantor)isalsocompleteArchimedeanordered field(completeinthesenseofDedekind).

Theorem9:Let$F$beacompleteArchimedean orderedfield(completein thesenseofCantor),andlet$A$beanonemptysubsetof$F$ thatisbounded above.Thensup Aexists.

Note:Theorem3abovealsoholdsforcompleteArchimedeanorderedfields(completeinthesenseofCantor).So,wehavet hefollowingdefinition:

Definition 6: Therealnumber field$\mathbb{R}$isanycomplete ordered field.For example,$\bar{\mathbb{Q}}$.

III. Conclusion:

We see that the two approaches to completeness for Archimedean orderedfieldsareequivalent.So,fortherealsitisentirelya matterof choicewhichapproachoneprefers.However,therearesituationsin whichtheCauchysequenceapproachistheonlyonepossible.Forexample, in the field of complex numbers $\mathbb{C}$which is not ordered $(as i \neq 0\ but i^2 = -1 < 0)$ Theorem 8 which is another versionof the completenessproperty for fields does not require the order relation, <. It is a useful axiom toconsiderforotherfieldsotherthanorderedfields.Allthatisrequiredis thedistancefunction$d(x, y)$tohavemeaninginthatfield.

Reference

Books:

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[4] N.L.Carothers,RealAnalysis