

# On The Fractional Integral Operators Associated With Certain Generalized Hypergeometric Function for Real Positive Definite Symmetric Matrix

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**Abstract:** In the present paper fractional integral operators associated with  $I$ -function for real positive symmetric definite matrix have been discussed. These operators have a wide range of applications in the field of Mathematical Physics and Linear differential equations. A number of special cases of our operators have been mentioned.

**Key words:** Fractional Integral Operators,  $I$ -function,  $H$ -function, Matrix transform, Symmetric matrix

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## I. Introduction

Fractional integration is an immediate generalization of repeated integration. Fractional integral operators occur in the solutions of linear differential equations, partial differential equations and in the integral representations of hypergeometric functions of one or more variables. Riesz [6] and Garding [2] respectively introduced Riemann-Liouville integral of vector and matrix variables and applied them in the solution of differential equation associated with Cauchy's problem.

### (1.1) $I$ -function with matrix argument

Let  $X$  is a  $p \times p$  real symmetric positive definite matrix of functionally independent variables. Let the  $I$ -function of  $X$  be denoted by

The  $I$ -function introduced by Saxena [7] will be represented and defined as follows:

$$I[Z] = I_{p_i, q_i; r}^{m, n} [Z] = I_{p_i, q_i; r}^{m, n} \left[ z \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] = \frac{1}{2\pi\omega} \int_L \chi(\xi) d\xi \quad (1.1)$$

where  $\omega = \sqrt{-1}$

$$\chi(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^r \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} \xi) \right\}} \quad (1.2)$$

$p_i, q_i (i = 1, \dots, r), m, n$  are integers satisfying  $0 \leq n \leq p_i, 0 \leq m \leq q_i, (i = 1, \dots, r), r$  is finite

$\alpha_j, \beta_j, \alpha_{ij}, \beta_{ji}$  are real and  $a_j, b_j, a_{ji}, b_{ji}$  are complex numbers such that

$\alpha_j (b_h + v) \neq \beta_h (a_j - v - k)$  for  $v, k = 0, 1, 2, \dots$

We shall use the following notations:

$$A^* = (a_j, \alpha_j)_{1, n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}; B^* = (b_j, \beta_j)_{1, m}, (b_{ji}, \beta_{ji})_{m+1, q_i}$$

If we take  $r = 1$  in (1.1), the  $I$ -function reduces to the Fox's H-function [1].

It is assumed that  $I(XY) = I(YX)$  for real symmetric  $m \times m$  positive definite matrices  $X$  and  $Y$ ,  $I(X)$  is defined by the following integral equation:

$$\int_{X>0} |X|^{\rho-\frac{m+1}{2}} I(X) dX = \frac{\prod_{j=1}^m \Gamma_m(b_j - \beta_j \xi) \prod_{j=1}^n \Gamma_m(\frac{m+1}{2} - a_j + \alpha_j \xi)}{\sum_{i=1}^r \left\{ \prod_{j=m+1}^{q_i} \Gamma_m(\frac{m+1}{2} - b_{ji} + \beta_{ji} \xi) \prod_{j=n+1}^{p_i} \Gamma_m(a_{ji} - \alpha_{ji} \xi) \right\}} \quad (1.3)$$

Sethi [8] discussed the following fractional integral operators involving  $H$ -function of matrix arguments:

$$R[f(X)] = R \left[ \begin{matrix} (a_j, \alpha_j)_{1,r}, (b_j, \beta_j)_{1,s} \\ \sigma, \rho, \gamma \end{matrix}; f(X) \right] \\ = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0 < U < X} |U|^\sigma |X-U|^{\rho-\frac{m+1}{2}} H_{r,s}^{p,q} \left[ \gamma(I-UX^{-1}) \middle| \begin{matrix} (a_j, \alpha_j)_{1,r} \\ (b_j, \beta_j)_{1,s} \end{matrix} \right] f(U) dU \quad (1.4)$$

$$K[f(X)] = K \left[ \begin{matrix} (a_j, \alpha_j)_{1,r}, (b_j, \beta_j)_{1,s} \\ \delta, \rho, \gamma \end{matrix}; f(X) \right] = \\ \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U > X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H_{r,s}^{p,q} \left[ \gamma(I-XU^{-1}) \middle| \begin{matrix} (a_j, \alpha_j)_{1,r} \\ (b_j, \beta_j)_{1,s} \end{matrix} \right] f(U) dU \quad (1.5)$$

Where  $f(X) = f(x_{11}, \dots, x_{1m}, x_{21}, \dots, x_{mm})$  be a real bounded function of a complex parameter.

### (1.2) Matrix transform

A generalized matrix transform or M-transform of a function  $f(X)$  of a  $m \times m$  real symmetric positive definite or strictly negative definite matrix  $X$  is defined as follows:

$$M_f(s) = \int_{X>0} |X|^{\frac{s-m+1}{2}} f(X) dX \quad (X > 0) \quad (1.6)$$

Whenever  $M_f(s)$  exists. Also  $f(X)$  is assumed to be a symmetric function i.e.

$$f(BX) = f(XB) = f\left(B^{\frac{1}{2}}XB^{\frac{1}{2}}\right) \text{ for } B = B' > 0. \text{ When } X < 0 \text{ replace } X \text{ by } -X \text{ in } M\text{-transform.}$$

### (1.3) Integral operators involving $I$ -function

$$Y[f(X)] = Y \left[ f(X) \middle| \sigma, \rho, \gamma; \begin{matrix} A^* \\ B^* \end{matrix} \right] = \\ \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0 < U < X} |U|^\sigma |X-U|^{\rho-\frac{m+1}{2}} I_{P_i, Q_i; r}^{M, N} \left[ \gamma(1-UX^{-1}) \middle| \begin{matrix} A^* \\ B^* \end{matrix} \right] f(U) dU \quad (1.7)$$

$$N[f(X)] = N \left[ f(X) \middle| \delta, \rho, \gamma; \begin{matrix} A^* \\ B^* \end{matrix} \right] = \\ \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U > X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} I_{P_i, Q_i; r}^{M, N} \left[ \gamma(I-XU^{-1}) \middle| \begin{matrix} A^* \\ B^* \end{matrix} \right] f(U) dU \quad (1.8)$$

The above defined operators exists under the following conditions:

(i)  $P_i \geq 1, Q_i < \infty, \frac{1}{P_i} + \frac{1}{Q_i} = 1, |\arg(I-a)| < \pi$

(ii)  $(\text{Re}(\sigma) > \frac{1}{Q_i}, \text{Re}(\delta) > \frac{1}{P_i}, \text{Re}(\rho) > \frac{m+1}{2})$

(iii)  $\text{Re}(\alpha + \min_{1 \leq j \leq M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$

(iv)  $f(X) \in L_p(0, \infty)$ .

The last condition ensures that  $Y[f(X)]$  and  $N[f(X)]$  both exist and also both belong to  $L_p(0, \infty)$ .

## II. Main Results

The following theorems of the operators defined by (1.7) and (1.8) have been established in the expression of matrix transform:

**Theorem 1:** If  $f(X) \in L_{P_i}(0, \infty); i = 1, 2, \dots, r$   $1 \leq P_i \leq 2$  [ $orf(X) \in M_{P_i}(0, \infty)$  and  $P_i > 2$ ] where

$$\operatorname{Re}\left(\alpha + \min_{1 \leq j \leq M} \frac{b_j}{\beta_j}\right) > \frac{m+1}{2}, \operatorname{Re}(\sigma) > -\frac{1}{Q_i}, \operatorname{Re}(t) > \frac{m+1}{2}, \operatorname{Re}(\sigma - t + 1) > \frac{m+1}{2} \text{ and } |\arg(I - a)| < \pi$$

then

$$M\{Y[f(X)]\} = \frac{\Gamma_m\left(\sigma - t + \frac{m+1}{2}\right)}{\Gamma_m(\rho)} I_{P_i+1, Q_i+1; r}^{M, N+1} \left[ \gamma I \left| \begin{matrix} \left(\frac{m+1}{2}, 1\right), A^* \\ B^*, \left(\frac{m+1}{2} - t - \sigma - \rho, 1\right) \end{matrix} \right. \right] M[f(U)] \quad (2.1)$$

Where  $I$  is  $m \times m$  identity matrix.

**Proof:** Taking the matrix transform of equation (1.7), we get

$$M\{Y[f(X)]\} = \int_{X>0} |X|^{t-\frac{m+1}{2}} \left[ \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |X-U|^{\rho-\frac{m+1}{2}} I_{P_i, Q_i; r}^{M, N} \left[ \gamma(I - UX^{-1}) \Big|_{B^*}^{A^*} \right] f(U) dU \right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\{Y[f(X)]\} = \frac{1}{\Gamma_m(\rho)} \int_{X>0} |U|^\sigma f(U) dU \int_{0<U<X} |X|^{t-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} I_{P_i, Q_i; r}^{M, N} \left[ \gamma(I - UX^{-1}) \Big|_{B^*}^{A^*} \right] dX$$

On evaluating  $X$ -integral with the help of the result given by Mathai and Saxena [5],

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} H_{r,s}^{p,q} \left[ |XZ| \Big|_{(b_j, \beta_j)_{1,s}}^{(a_j, \alpha_j)_{1,r}} \right] dX = \Gamma_m(\rho) H_{r+1, s+1}^{p, q+1} \left[ |Z| \Big|_{(b_j, \beta_j)_{1,s}, \left(\frac{m+1}{2} - \delta - \rho, 1\right)}^{\left(\frac{m+1}{2} - \delta, 1\right), (a_j, \alpha_j)_{1,r}} \right] \quad (2.2)$$

Where  $\operatorname{Re}\left(\alpha + \min_{1 \leq j \leq M} \frac{b_j}{\beta_j}\right) > \frac{m+1}{2}$  and  $\operatorname{Re}(\rho) > \frac{m+1}{2} - 1$ .

We obtain the required result.

**Theorem 2:** If  $f(X) \in L_{P_i}(0, \infty)$   $1 \leq P_i \leq 2$  [ $orf(X) \in M_{P_i}(0, \infty)$  and  $P_i > 2$ ] where

$$\operatorname{Re}\left(\delta + \min_{1 \leq j \leq M} \frac{b_j}{\beta_j}\right) > \frac{m+1}{2}, \operatorname{Re}(\delta) > -\frac{1}{Q_i}, \operatorname{Re}(t) > \frac{m+1}{2}, \operatorname{Re}(\delta + t) > \frac{m+1}{2} \text{ and } |\arg(I - a)| < \pi \text{ then}$$

$$M\{N[f(X)]\} = \frac{\Gamma_m(\delta + 1)}{\Gamma_m(\rho)} I_{P_i+1, Q_i+1}^{M, N+1} \left[ \gamma I \left| \begin{matrix} \left(\frac{m+1}{2} - \delta, 1\right), A^* \\ B^*, \left(\frac{m+1}{2} - \delta - \rho, 1\right) \end{matrix} \right. \right] M[f(U)] \quad (2.3)$$

Where  $I$  is  $m \times m$  identity matrix.

**Proof:** Taking the matrix transform of equation (2.2), we get

$$M\{N[f(X)]\} = \int_{X>0} |X|^{\delta-\frac{m+1}{2}} \left[ \frac{|X|^\delta}{\Gamma_m(\rho)} \right]$$

$$\int_{U>X} |U|^{-\sigma-\rho} |U-X|^{-\frac{m+1}{2}} I_{P_i, Q_i; r}^{M, N} \left[ \gamma(I-XU^{-1}) \Big|_{B^*}^{A^*} \right] f(U) dU \Big] dX$$

And changing the order of integration and evaluation  $X$ -integral with the help of (2.2), we obtain the required result.

**Theorem 3:** If  $f(X) \in L_{P_i}(0, \infty), g(X) \in L_{Q_i}(0, \infty)$  where  $\operatorname{Re}(\delta + \min_{1 \leq j \leq M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$ ,

$\operatorname{Re}(\delta) > -\frac{1}{Q_i}, \operatorname{Re}(\rho) > \frac{m+1}{2}, \operatorname{Re}(\sigma) > \max\left(\frac{1}{P_i} + \frac{1}{Q_i}\right)$  and  $|\arg(I-a)| < \pi$  then

$$\int_{X>0} f(X) Y \left[ g(X) \Big|_{B^*}^{\sigma, \rho, \gamma; A^*} \right] dX = \int_{X>0} g(X) N \left[ f(X) \Big|_{B^*}^{\sigma, \rho, \gamma; A^*} \right] dX \tag{2.4}$$

**Proof:** Equation (2.4) immediately follows on interpreting it with the help of equations (1.7) and (1.8).

**Special Cases**

(i) If we put  $M = 1, N = 1, P_i = 2, Q_i = 2, r = 1, \gamma = 1$ , then the operators (1.7) and (1.8) reduce to their Mellin transforms in the following form:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |X-U|^{-\frac{m+1}{2}} H_{2,2}^{1,2} \left[ (I-UX^{-1}) \right] f(U) dU$$

Here

$$Y[f(X)] = Y \left[ f(X) \Big|_{(b_1, \beta_1), (b_2, \beta_2)}^{\sigma, \rho, 1; (a_1, \alpha_1), (a_2, \alpha_2)} \right]$$

And

$$H_{2,2}^{1,2} \left[ (I-UX^{-1}) \right] = H_{2,2}^{1,2} \left[ (I-UX^{-1}) \Big|_{(b_1, \beta_1), (b_2, \beta_2)}^{(a_1, \alpha_1), (a_2, \alpha_2)} \right]$$

Then

$$Y[f(X)] = \frac{\Gamma_m(\chi_1) |X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |I-UX^{-1}|^{-\rho-\beta_1-\frac{m+1}{2}} {}_2F_1 \left[ -; (I-UX^{-1}) \right] f(U) dU$$

Where

$$\Gamma(\chi_1) = \frac{\Gamma_m\left(\frac{m+1}{2} - \alpha_1 + \beta_1\right) \Gamma_m\left(\frac{m+1}{2} - \alpha_2 + \beta_2\right)}{\Gamma_m\left(\frac{m+1}{2} - \alpha_1 - \alpha_2 + \beta_1 + \beta_2\right)}$$

$${}_2F_1 \left[ -; (I-UX^{-1}) \right] =$$

$${}_2F_1 \left[ \frac{m+1}{2} - \alpha_1 - \beta_1, \frac{m+1}{2} - \alpha_2 - \beta_2; \frac{m+1}{2} - \beta_1 - \beta_2; -(I-UX^{-1}) \right]$$

By virtue of the result [9].

Taking  $M$ -transform on both sides, we get

$$M \{ Y(f(X)) \} = \frac{\Gamma_m(\chi_1) \Gamma_m(\chi_2)}{\Gamma_m(\rho)} {}_3F_2(-; I) M[f(U)]$$

Where  $\Gamma(\chi_2) = \frac{\Gamma_m\left(\frac{m+1}{2} + \sigma\right) \Gamma_m(\rho + \beta_1)}{\Gamma_m\left(\sigma + \rho + \beta_1 + \frac{m+1}{2}\right)}$

And

$${}_3F_2(-; I) = {}_3F_2\left(\frac{m+1}{2} - \alpha_1 + \beta_1, \frac{m+1}{2} - \alpha_2 + \beta_2, \sigma + \frac{m+1}{2}; \frac{m+1}{2} - \beta_2 + \beta_1, \sigma + \rho + \frac{m+1}{2} + \beta_1; I\right),$$

$$N[f(X)] = \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U>X} |U|^\sigma |U-X|^{\rho-\frac{m+1}{2}} H_{2,2}^{1,2}[(I-XU^{-1})] f(U) dU$$

Where

$$N[f(X)] = N\left[f(X) \middle| \sigma, \rho, 1; \begin{matrix} (a_1; \alpha_1), (a_2; \alpha_2), 1, 2 \\ (b_1, \beta_1), (b_2, \beta_2), 1, 2 \end{matrix} \right]$$

And

$$H_{2,2}^{1,2}[(I-XU^{-1})] = H_{2,2}^{1,2}\left[(I-XU^{-1}) \middle| \begin{matrix} (a_1, \alpha_1), (a_2, \alpha_2) \\ (b_1, \beta_1), (b_2, \beta_2) \end{matrix} \right]$$

Then

$$N[f(X)] = \frac{\Gamma_m(\chi_1) |X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{-\delta-\rho} |I-XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}} {}_2F_1[-; (I-XU^{-1})] f(U) dU$$

Where

$${}_2F_1[-; (I-XU^{-1})] = {}_2F_1\left[\frac{m+1}{2} - \alpha_1 + \beta_1, \frac{m+1}{2} - \alpha_2 + \beta_2; \frac{m+1}{2} + \beta_1 - \beta_2; -(I-XU^{-1})\right]$$

Taking  $M$  -transform on both sides, we get

$$M\{N(f(X))\} = \frac{(-1)^{\rho-\frac{m+1}{2}} \Gamma_m(\chi_1) \Gamma_m(\chi_2)}{\Gamma_m(\rho)} {}_2F_1(-; I) M[f(U)]$$

(ii) Putting  $P_i = 0, Q_i = 1, M = 1, N = 0, \gamma = 1, r = 1$ , then operators (1.7) and (1.8) reduce to their Mellin transform in the following forms:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |X-U|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}[(I-UX^{-1})]_{(\beta,1)}^- f(U) dU$$

Where

$$Y[f(X)] = Y\left[f(X) \middle| \sigma, \rho, 1; \begin{matrix} - \\ (\beta, 1) \end{matrix} \right]$$

And

$$\begin{aligned} H_{0,1}^{1,0}[(I-UX^{-1})]_{(\beta,1)}^- &= |I-UX^{-1}|^\beta e^{-tr(i-UX^{-1})} \\ &= \frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |I-UX^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} f(U) dU \end{aligned}$$

By virtue of the result [7]. Taking  $M$  -transform on both sides, we get

$$M\{Y(f(X))\} = \frac{\Gamma_m(\rho + \beta)}{\Gamma_m(\rho)} M[f(U)]$$

Also

$$N[f(X)] = \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}[(I-XU^{-1})]_{(\beta,1)}^- f(U) dU$$

Where

$$N[f(X)] = N\left[f(X) \Big|_{(\beta,1)}^{\delta, \rho, 1; -}\right]$$

And

$$\begin{aligned} H_{0,1}^{1,0} \left[ (I - XU^{-1}) \Big|_{(\beta,1)}^{\delta, \rho, 1; -} \right] &= |I - XU^{-1}|^\beta e^{-tr(i-XU^{-1})} \\ &= \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\frac{m+1}{2}} |I - XU^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-XU^{-1})} f(U) dU \end{aligned}$$

By virtue of the result [3]. Taking  $M$  -transform on both sides, we get

$$M\{N[f(X)]\} = \frac{\Gamma_m\left(\rho - \delta + \beta - \frac{m+1}{2}\right)}{\Gamma_m(\rho)} M[f(U)]$$

If we put  $\alpha_j = \beta_j = 1; (j = 1, \dots, P; j = 1, \dots, Q)$  the operators reduce to  $G$  -function given by Vyas [10].

**Theorem4.** If  $f(X) \in L_{P_i}(0, \infty) 1 \leq P_i \leq 2$  [or  $f(X) \in M_{P_i}(0, \infty)$  and  $P_i > 2$ ] where  $\text{Re}(\alpha) > \frac{m+1}{2}$ ,

$\text{Re}(\sigma) > -\frac{1}{Q_i}, \text{Re}(\rho) > \frac{m+1}{2}, \text{Re}(\alpha - \sigma) > \frac{m+1}{2}$  and  $|\arg(I - a)| < \pi$  then

$$\begin{aligned} M\{R[\sigma, \rho, a; f(X)]\} &= \frac{\Gamma_m\left(\sigma - s + \frac{m+1}{2}\right) \Gamma_m(\rho + \alpha)}{\Gamma_m\left(\sigma - s + \alpha + \rho + \frac{m+1}{2}\right) \Gamma_m(\rho)} \\ &{}_1F_1\left[\rho + \alpha; \sigma - s + \alpha + \rho + \frac{m+1}{2}; I\right] M[f(U)] \end{aligned} \tag{2.5}$$

**Proof:** Using the Mellin transform of

$$\begin{aligned} R[f(X)] &= R\left[\begin{matrix} (a_r), (b_r) \\ \sigma, \rho, a \end{matrix}; f(X)\right] = \\ &\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0 < U < X} |U|^\sigma |X - U|^{\rho-\frac{m+1}{2}} G_{r,s}^{p,q} \left[ a(I - UX^{-1}) \Big|_{(b_r)}^{(a_r)} \right] f(U) dU \end{aligned} \tag{2.6}$$

We get

$$\begin{aligned} M\{R[\sigma, \rho, a; f(X)]\} &= \\ &\int_{X>0} \frac{|X|^{s-\frac{m+1}{2}} |X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \left[ \int_{0 < U < X} |U|^\sigma |X - U|^{\rho-\frac{m+1}{2}} G_{0,1}^{1,0} \left[ a(I - UX^{-1}) \Big|_{(b_1)}^{\delta, \rho, 1; -} \right] f(U) dU \right] dX \end{aligned}$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\begin{aligned} &\int_{0 < U < X} |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X - U|^{\rho-\frac{m+1}{2}} G_{0,1}^{1,0} \left[ a(I - UX^{-1}) \right] dX = \frac{1}{\Gamma_m(\rho)} \int_{U>X} |U|^\sigma f(U) dU \\ &\int_{X>U} |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X - U|^{\rho+\alpha-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} dX \end{aligned}$$

On evaluating  $X$  -integral with the help of result given by Mathai [3]

$$\begin{aligned} &\int_0^1 e^{-tr(XZ)} |X|^{\delta-\frac{m+1}{2}} |I - X|^{\rho-\delta-\frac{m+1}{2}} dX \\ &= \frac{\Gamma_m(\delta) \Gamma_m(\rho - \delta)}{\Gamma_m(\rho)} {}_1F_1[\delta; \rho; -Z] \end{aligned} \tag{2.7}$$

For  $\operatorname{Re}(\delta) > \frac{m+1}{2}, \operatorname{Re}(\rho) > \frac{m+1}{2}, \operatorname{Re}(\rho - \delta) > \frac{m+1}{2}$

We obtain the required result.

**Theorem5.** If  $f(X) \in L_{P_i}(0, \infty) 1 \leq P_i \leq 2$  [or  $f(X) \in M_{P_i}(0, \infty)$  and  $P_i > 2$ ] where  $\operatorname{Re}(\alpha) > \frac{m+1}{2}, \operatorname{Re}(\delta) > -\frac{1}{P_i}, \operatorname{Re}(\rho) > \frac{m+1}{2}, \operatorname{Re}(\alpha - \rho) > \frac{m+1}{2}, \frac{1}{P_i} + \frac{1}{Q_i} = 1$  and  $|\arg(I - a)| < \pi$  then

$$M \left\{ K[\delta, \rho, a; f(X)] \right\} = \frac{\Gamma_m \left( \delta + s + \frac{m+1}{2} \right) \Gamma_m(\rho + \alpha)}{\Gamma_m \left( \delta + s + \alpha + \rho + \frac{m+1}{2} \right) \Gamma_m(\rho)} {}_1F_1 \left[ \rho + \alpha; \delta + s + \alpha + \rho + \frac{m+1}{2}; I \right] M[f(U)] \quad (2.8)$$

**Proof:** Using the Mellin transform of

$$K[f(X)] = K \left[ \begin{matrix} (a_r), (b_r) \\ \sigma, \rho, a \end{matrix}; f(X) \right] = \frac{|X|^\delta}{\Gamma_m(\rho)} \int_{U>X} |U|^\delta |X - U|^{\rho - \frac{m+1}{2}} G_{r,s}^{p,q} \left[ a(I - XU^{-1}) \middle| \begin{matrix} (a_r) \\ (b_r) \end{matrix} \right] f(U) dU \quad (2.9)$$

We get

$$M \left\{ K[\delta, \rho, a; f(X)] \right\} = \int_{X>0} \frac{|X|^{\frac{s-m+1}{2}} |X|^\delta}{\Gamma_m(\rho)} \left[ \int_{U>X} |U|^{-\delta-\rho} |U - X|^{\rho - \frac{m+1}{2}} G_{0,1}^{1,0} \left[ a(I - XU^{-1}) \middle| \begin{matrix} - \\ (b_1) \end{matrix} \right] f(U) dU \right] dX \quad (2.10)$$

Changing the order of integration and evaluating  $X$ -integral with the help of (2.5), we obtain the required result.

When  $M = 1, N = 0, P = 0, Q = 1, a = 1$  in (2.6) and (2.9) reduces to the following form of operators:

$$R[f(X)] = R \left[ \begin{matrix} \alpha, \beta \\ \sigma, \rho, 1 \end{matrix}; f(X) \right] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^\sigma |X - U|^{\rho - \frac{m+1}{2}} G_{0,1}^{1,0} \left[ a(I - UX^{-1}) \middle| \begin{matrix} \alpha \\ \beta \end{matrix} \right] f(U) dU \quad (2.11)$$

And

$$K[f(X)] = K \left[ \begin{matrix} \alpha, \beta \\ \sigma, \rho, 1 \end{matrix}; f(X) \right] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\sigma} |U - X|^{\rho - \frac{m+1}{2}} G_{0,1}^{1,0} \left[ a(I - UX^{-1}) \middle| \begin{matrix} \alpha \\ \beta \end{matrix} \right] f(U) dU \quad (2.12)$$

**Theorem6.** If  $f(X) \in L_{P_i}(0, \infty) 1 \leq P_i \leq 2$  [or  $f(X) \in M_{P_i}(0, \infty)$  and  $P_i > 2$ ] where  $\operatorname{Re}(\alpha) > \frac{m+1}{2},$

$\operatorname{Re}(\sigma) > -\frac{1}{Q_i}, \operatorname{Re}(\rho) > \frac{m+1}{2}, \operatorname{Re}(\alpha - \sigma) > \frac{m+1}{2}, \frac{1}{P_i} + \frac{1}{Q_i} = 1$  and  $|\arg(I - a)| < \pi$  then

$$M \left\{ R[\sigma, \rho, 1; f(X)] \right\} = \frac{\Gamma_m(\sigma + \alpha - \beta - s) \Gamma_m(\rho + \beta)}{\Gamma_m(\sigma - s + \alpha + \rho) \Gamma_m(\rho) \Gamma_m(\alpha - \beta)} M[f(U)] \quad (2.13)$$

**Proof:** Using the Mellin transform of (2.11), we get

$$M \left\{ R[\sigma, \rho, 1; f(X)] \right\} =$$

$$\int_{X>0} \frac{|X|^{-\sigma-\rho} |X|^{\frac{s-m+1}{2}}}{\Gamma_m(\rho)} \left[ \int_{0<U<X} |U|^\alpha |X-U|^{\rho-\frac{m+1}{2}} G_{1,1}^{1,0} \left[ a(I-UX^{-1}) \middle| \begin{matrix} \alpha \\ \beta \end{matrix} \right] f(U) dU \right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M \left\{ R_{\sigma,\rho,1}^\alpha; f(X) \right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^\sigma f(U) dU$$

$$\int G_{1,1}^{1,0} \left[ (I-XU^{-1}) \middle| \begin{matrix} \alpha \\ \beta \end{matrix} \right] |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} dX$$

Using the result given by Mathai [3].

$$G_{1,1}^{1,0} \left[ X \middle| \begin{matrix} \alpha \\ \beta \end{matrix} \right] = \frac{1}{\Gamma_m(\alpha-\beta)} |X|^\beta |I-U|^{\alpha-\beta-\frac{m+1}{2}} \tag{2.14}$$

Provided

$$0 < X < I, \text{Re}(\alpha - \beta) > \frac{m+1}{2}$$

We get

$$M \left\{ R_{\sigma,\rho,1}^{\alpha,\beta}; f(X) \right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma-\alpha-\beta-\frac{m+1}{2}} f(U) dU$$

$$\frac{1}{\Gamma_m(\alpha-\beta)} \int_{X>U} |X|^{s-\sigma-\alpha} |X-U|^{\beta+\rho-\frac{m+1}{2}} dX$$

On evaluating X -integral with the help of the following result

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \tag{2.15}$$

For  $\text{Re}(\delta) > 0, \text{Re}(\rho) > \frac{m+1}{2}$

We arrive at the required result.

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