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Fuzzy Variational Problem

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Abstract: *In this paper, the fuzzy variational formulation will be introduced and derived. This paper is about studying variational problems with fuzzy functions, fuzzy condition and fuzzy boundaries by using different approaches for defuzzification, such as centroid method, α-cut method, centroid point and expected interval in which fuzzy sets have been transformed into crisp sets and finding the necessary conditions for extremizing the fuzzy variational problems with fuzzy function and fuzzy boundaries.*

I. Introduction:

Historically, the accepted birth date of the theory of fuzzy sets returns to 1965, when the first article entitled “fuzzy sets” submitted by Zadeh appeared in the journal of information and control. Also, the term “fuzzy” was introduced and coined by Zadeh for the first time, [2]. In which the original definition of fuzzy sets is to consider a class of objects with a continuum grades of membership, such a set is characterized by a membership (or characteristic) function which assigns to each object a grade of membership value ranging between zero and one. As the membership value approaches unity, the grade of membership of an event in the fuzzy set becomes higher. For example, the unit membership value indicates that the event x is strictly contained in the fuzzy set, and on the other hand, the zero membership value indicate strictly that x is strictly does not belong to the fuzzy set. Any intermediate value would reflects the degree on which x could be a member of the fuzzy set, [2].

In addition, the history of the calculus of variation is tightly interwoven with the history of mathematics. The field has drawn the attention of a remarkable range of mathematical luminaries, beginning with Newton, then initiated as a subject in its own right by the Bernoulli family. The first major developments appeared in the work of Euler, Lagrange and Laplace. In the nineteenth century, Hamilton, Dirichlet and Hilbert are among the outstanding contributors. In modern times, the subject of calculus of variations has continued to occupy center stage, witnessing major theoretical advances, along with wide-ranging applications in physics, engineering and all branches of mathematics, [17].

Calculus of variation is a branch of mathematics dealing with the optimization of physical quantities (such as time, area, or distance). It finds applications in many fields, such as aeronautics (maximizing the lift of an airplane wing), sporting equipment design (minimizing air resistance on a bicycle helmet, optimizing the shape of a ski), mechanical engineering (maximizing the strength of a column, a dam, or an arch), boat design (optimizing the shape of a boat hull), physics (calculating trajectories and geodesics in both classical mechanics and general relativity), [13].

A huge amount of problems in the calculus of variations have their origin in physics where one has to minimize the energy associated to the problem under consideration. Nowadays, many problems come from economics. Here is the main point that the resources are restricted. There is no economy without restricted resources, [13].

Minimization principles form one of the most wide-ranging means of formulating mathematical models governing the equilibrium configurations of physical systems. Moreover, many popular numerical integration schemes such as the powerful finite element method are also founded upon a minimization paradigm, [17].

II. Preliminaries:

Fuzzy set theory is a generalization of abstract set theory; it has a wider scope of applicability than abstract set theory for solving problems that involve to some degree subjective evaluation [2]. Zadeh in 1965 [3], suggested a modified set theoretical approach in which an individual may have a degree of membership value which is ranged over a continuum grade of values ranging between 0 and 1, rather than exactly 0 or 1.

The subjects of calculus of variation is concerned with solving extremal problems for a functionals. That is to say the maximum and minimum problems for functions whose domain contains functions, Y(x) (or $Y(x_0,\cdots x_1)$, or n-tuples of functions). The range of the functional will be the real numbers R, [12].

In this section, the basic concepts, definitions and theorems related to fuzzy set theory and variational problems will be introduced.

Definition (2.1), [3]:
Let Xbe a classical set of objects, called the universal set, whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function μ_A from X into {0, 1}, such that:

$$\mu_A(x) = \begin{cases} 1, \text{if} x \in A \\ 0, \text{if} x \notin A \end{cases}$$

{0, 1} is called a valuation set. If the valuation set is allowed to be the real interval $[0,1]$, then A is called a fuzzy set (which is denoted by $\widetilde{A}$), and $\mu_{\widetilde{A}}(x)$ is the grade of membership of x in$\widetilde{A}$.

Definition (2.2), [10]:
A fuzzy number is a fuzzy set like $\widetilde{M}: R \to I = [0,1]$ which satisfies:

1. $\widetilde{M}$ is upper semi-continous,
2. $\widetilde{M}(x) = 0$ outside some interval [a, d],
3. There are real numbers a, d such that $a \le b \le c \le d$ and

a. $\widetilde{M}(x)$ is monotonic increasing on $[a, b]$,
b. $\widetilde{M}(x)$ is monotonic decreasing on $[c, d]$,
c. $\widetilde{M}(x) = 1, b \le x \le c$.

The membership function of $\widetilde{M}$ may be expressed as:

$$\mu(x) = \begin{cases} \widetilde{M}_L(x), & a \le x \le b, \\ 1, & b \le x \le c, \\ \widetilde{M}_R(x), & c \le x \le d, \\ 0, & \text{otherwise} \end{cases}$$

where$\widetilde{M}_L : [a, b] \to [0,1]$ and $\widetilde{M}_R: [c, d] \to [0,1]$ are left and right membership functions of fuzzy number $\widetilde{M}$.

Definition (2.3), [24]:
Let $f: [a, b] \to R_F$and $x_0 \in (a, b)$ with $\underline{f}(x, \alpha)$ and $\bar{f}(x, \alpha)$ both differentiable at x_0.
$-$fis (i)-gH-differentiable at x_0 if:

$$(i)f'_{gH}(x_0)[\alpha] = \left[(\underline{f})'(x_0, \alpha), (\bar{f})'(x_0, \alpha)\right], \forall \alpha \in [0,1]$$

$-$fis (ii)-gH-differentiable at x_0 if:

$$(ii)f'_{gH}(x_0)[\alpha] = \left[(\bar{f})'(x_0, \alpha), (\underline{f})'(x_0, \alpha)\right], \forall \alpha \in [0,1]$$

It is possible that $f: [a, b] \to R_F$is gH-differentiable at x_0 and not (i)-gHdifferentiable nor (ii)-gH-differentiable.

Definition (2.4), [14]:
Let R be the set of real numbers and Ω a set of functions. Then the function $J: \Omega \to R$ is called a functional.

Lemma (2.1), [12]:
Let $M(x) \in C^n[x_0, x_1], 0 \le n \le \infty$. If $\int_{x_0}^{x_1} M(x)\eta(x)dx = 0$for all $\eta(x)$ such that $\eta(x_0) = \eta(x_1) = 0, \eta(x) \in C^n$, on $[x_0, x_1]$ then $M(x) = 0$ at all points of continuity.

Theorem (2.1), [14]:
A necessary condition for

$$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$$

with$y(x_0) = y_0$and $y(x_1) = y_1$, to have an extremum at y is that y is a solution of

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

with$x_0 < x < x_1$ and $F = F(x, y, y')$. This is known as the Euler-Lagrange equation.

III. Variational Problems with Fuzzy Integrands:

In this section, variational problem with fuzzy function and variational problem with fuzzy boundary conditions are investigated.

3.1 Unconstrained Fuzzy Variational Problem with Fuzzy Function:

Consider the fuzzy variational problem (FVP) with fuzzy function $\tilde{y}: A \to B$, which is to minimize the functional:

$$J(\tilde{y}) = \int_{x_0}^{x_1} F(x, \tilde{y}, \tilde{y}')dx$$

whereA and B are subsets of R, x_0 and $x_1$are crisp fixed points and the boundary conditions $\tilde{y}(x_0) = y_0$ and $\tilde{y}(x_1) = y_1$ are fixed fuzzy number. The goal of FVP is to find an admissible fuzzy curve $\tilde{y}^*$in a fuzzy weak neighborhood, if any exists, such that minimize J. The fuzzy curve $\tilde{y}^* = \tilde{y}^*(x)$ is a minimizing curve for the FVP if for all admissible fuzzy curves $\tilde{y}^*$in the fuzzy weak neighborhood, i.e.,

$$J(\tilde{y}) \geq J(\tilde{y}^*)$$

Corresponding to definition (1.9) and lemma (1.1) the fuzzy Euler-Lagrange conditions occur in the following two cases, [23]:

<u>Case (i)</u>: F is (i)-gH differentiable ((ii)-gH differentiable) with respect to y andy'.

$$\frac{\partial \underline{F}}{\partial \underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial \underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \underline{F}}{\partial \overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial \overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \overline{F}}{\partial \underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \overline{F}}{\partial \overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

<u>Case (ii)</u>:F is (i)-gH differentiable ((ii)-gH differentiable) with respect to y and (ii)-gH differentiable ((i)-gH differentiable) with respect to y'.

$$\frac{\partial \underline{F}}{\partial \underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \underline{F}}{\partial \overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \overline{F}}{\partial \overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \overline{F}}{\partial \underline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial \underline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

$$\frac{\partial \overline{F}}{\partial \overline{y}}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right) - \frac{d}{dx}\left(\frac{\partial \underline{F}}{\partial \overline{y}'}\left(\underline{y}^*,\left(\underline{y}^*\right)', \overline{y}^*, (\overline{y}^*)', x, \alpha\right)\right) = 0$$

3.2 Constrained Fuzzy Variational Problem with Fuzzy Function:

The problem involving minimization of a fuzzy functional with fuzzy integral constraints is called the constrained fuzzy variational problemand it is stated as follows:

$$\text{Minimize} J(\tilde{y}) = \int_{x_0}^{x_1} F(x, \tilde{y}, \tilde{y}')dx$$

$$\text{Subject to } I(\tilde{y}) = \int_{x_0}^{x_1} H(x, \tilde{y}, \tilde{y}')dx \approx c$$

$$\tilde{y}(x_0) \approx y_0, \tilde{y}(x_1) \approx y_1$$

wherec is a given fuzzy number.

According to definition (1.9), and therefore, eight cases can be occur, [23]:

<u>Case (i)</u>: Fand Hare both (i)-gH differentiable ((ii)-gH differentiable) with respect to yand y'. In this case, for all $\alpha \in [0,1]$, we have:

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right)\right) = 0$$

Where$\underline{z} = (\underline{y}^{*}, (\underline{y}^{*})', \overline{y}^{*}, (\overline{y}^{*})', x, \alpha)$ and $\overline{\lambda}_1$,$\underline{\lambda_1}$,$\overline{\lambda}_2$ and $\underline{\lambda}_2$.

<u>Case (ii)</u>:H, F with respect to yand F with respect to with respect to y' are both (i)-gH differentiable ((ii)-gH differentiable) and H is (ii)-gH differentiable ((i)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z) + \underline{\lambda_1}\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z) + \overline{\lambda}_2\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z) + \underline{\lambda}_2\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z) + \overline{\lambda}_1\underline{H}(z)\right)\right) = 0$$

<u>Case (iii)</u>: H with respect to yand y' is (ii)-gH differentiable ((i)-gH differentiable) and F is (i)-gH differentiable ((ii)-gH differentiable) with respect to yand y' .

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z) + \underline{\lambda_1}\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z) + \underline{\lambda_1}\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z) + \overline{\lambda}_2\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z) + \overline{\lambda}_2\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z) + \underline{\lambda}_2\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z) + \underline{\lambda}_2\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z) + \overline{\lambda}_1\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z) + \overline{\lambda}_1\underline{H}(z)\right)\right) = 0$$

<u>Case (iv)</u>:H, F with respect to y' and F with respect to yare both (i)-gH differentiable ((ii)-gH differentiable) and H is (ii)-gH differentiable ((i)-gH differentiable) with respect to y.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z) + \underline{\lambda_1}\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z) + \overline{\lambda}_2\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z) + \underline{\lambda}_2\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z) + \overline{\lambda}_1\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right)\right) = 0$$

<u>Case (v)</u>:H, F with respect to yand H with respect to y' are both (i)-gH differentiable ((ii)-gH differentiable) and F is (ii)-gH differentiable ((i)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z) + \underline{\lambda_1}\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z) + \overline{\lambda}_2\underline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z) + \underline{\lambda}_2\overline{H}(z)\right)\right) = 0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z) + \overline{\lambda}_1\overline{H}(z)\right)\right) = 0$$

Case (vi):H, F with respect to yare both (i)-gH differentiable ((ii)-gH differentiable) and H, F are both (ii)-gH differentiable ((i)-gH differentiable) with respect to y'.

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)\right)=0$$

Case (vii): h with respect to y' and F with respect to y are both (i)-gH differentiable ((ii)-gH differentiable) and Hwith respect to y and F with respect to y' are both (ii)-gH differentiable ((i)-gH differentiable).

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0$$

Case (viii):g with respect to yis (i)-gH differentiable ((ii)-gH differentiable) and H, F with respect to $y'$and H with respect to yare both (ii)-gH differentiable ((i)-gH differentiable).

$$\frac{\partial}{\partial \underline{y}}\left(\underline{F}(z)+\underline{\lambda}_1\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\overline{F}(z)+\underline{\lambda}_1\underline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\underline{F}(z)+\overline{\lambda}_2\overline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\overline{F}(z)+\overline{\lambda}_2\underline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \underline{y}}\left(\overline{F}(z)+\underline{\lambda}_2\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \underline{y}'}\left(\underline{F}(z)+\underline{\lambda}_2\overline{H}(z)\right)\right)=0$$

$$\frac{\partial}{\partial \overline{y}}\left(\overline{F}(z)+\overline{\lambda}_1\underline{H}(z)\right)-\frac{d}{dx}\left(\frac{\partial}{\partial \overline{y}'}\left(\underline{F}(z)+\overline{\lambda}_1\overline{H}(z)\right)\right)=0$$

IV. Variational Problems with Fuzzy Boundary Conditions:

In this section, the variational problems with fuzzy boundary conditions is investigated.

4.1The Centroid Method for Defuzzification:

The centroid method is the most popular methodfor defuzzification, i.e., transforming fuzzy problems into nonfuzzy problems, [27].

This method determines the centre of the area of the combined membership functions [33].

We use this approach when the fuzzy number$\tilde{x}_1$is the union of two or more fuzzy numbers. So, the fuzzy number $\tilde{x}_1=[a_1,a_2,a_3,a_4]$will be transformed into a crisp number x^*, by using the centroid method as follows:

$$x^*=\frac{\int_{\tilde{x}_1}\mu_{\tilde{x}_1}(t)\,t\,dt}{\int_{\tilde{x}_1}\mu_{\tilde{x}_1}(t)\,dt},\qquad a_1\le t\le a_4$$

where $\int_{\tilde{x}_1}$ means that the integration of the membership function is carried over each line segment of the produced union fuzzy number.

Example (4.1):
To find the minimum of the functional:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2 \, dx \qquad (4.1)$$

with

$$y(0) \approx \tilde{2} = \langle 0, 2, 4 \rangle, \quad y(\tilde{x}_1) \approx \tilde{4} = \langle 2, 4, 6 \rangle$$

with $\tilde{x}_1 = A_1 \cup A_2 \cup A_3$, where $A_1 =< 0, 1, 4, 5 >$, $A_2 =< 3, 4, 6, 7 >$, $A_3 =< 5, 6, 7, 8 >$, then:

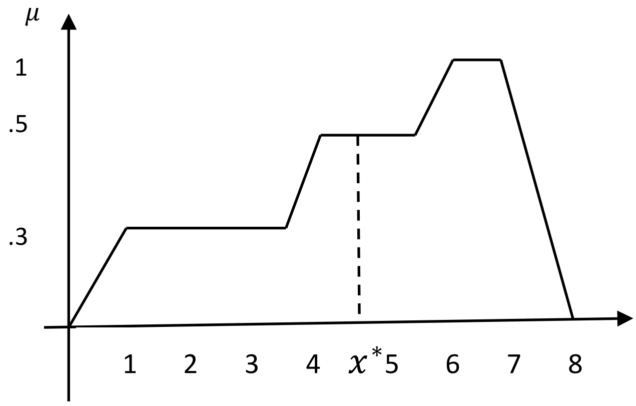

Fig.(3.1) Membership function of example (3.1)

$$x^* = \frac{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, t\, dt}{\int_{\tilde{x}_1} \mu_{\tilde{x}_1}(x)\, dt}, \qquad 0 \le x \le 8$$

$$x^* = \left[\int_0^1 (0.3\,x)xdx + \int_1^{3.6} (0.3\,x)\,dx + \int_{3.6}^4 \left(\frac{x-3}{2}\right)xdx + \int_4^{5.5} (0.5)xdx + \int_{5.5}^6 (x-5)xdx + \int_6^7 xdx + \int_7^8 (8-x)xdx\right]$$

$$\div \left[\int_0^1 (0.3\,x)dx + \int_1^{3.6} (0.3)dx + \int_{3.6}^4 \left(\frac{x-3}{2}\right)dx + \int_4^{5.5} (0.5)dx + \int_{5.5}^6 (x-5)\,dx + \int_6^7 dx + \int_7^8 (8-x)dx\right] = 4.9$$

By representing the endpoint conditions in its α −cut sets:

$2_\alpha = [\alpha\, 2, \frac{1}{\alpha}\, 2]$, $4_\alpha = [\alpha\, 4, \frac{1}{\alpha}\, 4]$

usingx^*which is a crisp number, thenequation (4.1) becomes:

$$J(y) = \int_0^{4.9} -(y')^2 \, dx$$

with

$$y(0) \approx \tilde{2}, \quad y(4.9) \approx \tilde{4}$$

then:

$y(x) = c_1\, x + c_2$

Using the boundary conditions:

$\underline{c}_1 = 0.41\alpha$, $\overline{c}_1 = 0.41\frac{1}{\alpha}$

hence:

$\underline{y}(x,\alpha) = 0.41\alpha\, x + 2\alpha$, $\overline{y}(x,\alpha) = 0.41\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}$

Then:

$$y(x)_\alpha = \left[\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right] = \left[0.41\alpha\, x + 2\alpha\, , 0.41\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}\right]$$

defines the α-level sets of a fuzzy number which minimizes J.

4.2The Expected Interval for Defuzzification:

The interval of defuzification can be used as a crisp approximation set with respect to a fuzzy number or any fuzzy quantity.

The α-cut of a fuzzy number $\tilde{x}_1$ (for simplicity set $A = \tilde{x}_1$) is, [28]:
$A_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)]$,
where $\alpha \in [0,1]$ and :
$\underline{A} = \inf\{x \in R: \mu_A \geq \alpha\}, \overline{A} = \sup\{x \in R: \mu_A \geq \alpha\}$,
The expected interval EI(A) of a fuzzy number A is defined by:

$$EI(A) = [E_*(A), E^*(A)] = [\int_0^1 \underline{A}(\alpha)\, d\alpha, \int_0^1 \overline{A}(\alpha)\, d\alpha]$$

Fuzzy numbers with simple membership functions are preferred in practice. The most used such fuzzy numbers are the trapezoidal fuzzy numbers. A trapezoidal fuzzy number T, $T_\alpha = [\underline{T}(\alpha), \overline{T}(\alpha)]$, $\alpha \in [0,1]$, is given by:
$\underline{T}(\alpha) = x_1 + (x_2 - x_1)\alpha$ and $\overline{T}(\alpha) = x_4 + (x_4 - x_3)\alpha$,
where $x_1, x_2, x_3, x_4 \in R, x_1 \leq x_2 \leq x_3 \leq x_4$. When $x_2 = x_3$, we obtain a triangular fuzzy number. We denote:
$T = [x_1, x_2, x_3, x_4]$,
a trapezoidal fuzzy number and by $F^T(R)$ the set of all trapezoidal fuzzy numbers. The expected interval for a trapezoidal fuzzy number T is:

$$EI(T) = \left[\frac{x_1 + x_2}{2}, \frac{x_3 + x_4}{2}\right]$$

Example (4.2):
Find the minimum of:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2\, dx \quad (4.2)$$

with

$$y(0) \approx \tilde{2} = \langle 0, 2, 4\rangle, \quad y(\tilde{x}_1) \approx \tilde{4} = \langle 2, 4, 6\rangle$$

where $\tilde{x}_1 = [0, 1, 3, 4]$ is a trapezoidal fuzzy number.

$$EI(\tilde{x}_1) = \left[\frac{0+1}{2}, \frac{3+4}{2}\right] = [0.5, 3.5] = x^*$$

using $EI(\tilde{x}_1)$ which is a crisp number, then equation (4.2) becomes:

$$J(y) = \int_0^{x^*} -(y')^2\, dx$$

with

$$y_\alpha(0) \approx 2_\alpha \approx [\alpha\, 2, \frac{1}{\alpha}\, 2], \quad y_\alpha(x^*) \approx 4_\alpha \approx [\alpha\, 4, \frac{1}{\alpha}\, 4]$$

then:
$y(x) = c_1\, x + c_2$
Using the boundary conditions:
$\underline{c}_1 = 4\alpha, \overline{c}_1 = 0.57\, \alpha$
hence:
$\underline{y}(x,\alpha) = 4\alpha\, x + 2\alpha\,, \overline{y}(x,\alpha) = 0.57\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}$
Then:
$y(x)_\alpha = [\underline{y}(x,\alpha), \overline{y}(x,\alpha)] = \left[4\alpha\, x + 2\alpha\,, 0.57\frac{1}{\alpha}\, x + 2\frac{1}{\alpha}\right]$
defines the α-level sets of a fuzzy number which minimizes J.

4.3 Centroid Point Method for Defuzzification:

This method is used for extended fuzzy number. An extended fuzzy number $\tilde{A}$ is described as any fuzzy subset of the universe set U with membership function $\mu_{\tilde{A}}$ defined as follow:

- $\mu_{\tilde{A}}$ is a continuous mapping from U to the closed interval $[0, w], 0 < w \leq 1$.
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a_1]$.
- $\mu_{\tilde{A}}$ is strictly increasing between $[a_1, a_2]$.
- $\mu_{\tilde{A}}(x) = w$, for all $x \in [a_2, a_3]$, w is a constant and $0 < w \leq 1$.
- $\mu_{\tilde{A}}$ is strictly decreasing between $[a_3, a_4]$.
- $\mu_{\tilde{A}}(x) = 0$, for all $x \in [a_1, +\infty]$.

In the above situations a_1, a_2, a_3 and a_4 are real numbers. If $a_1 = a_2 = a_3 = a_4$, $\tilde{A}$ becomes a crisp real number. The membership function $\mu_{\tilde{A}}$ of the extended fuzzy number $\tilde{A}$ may be expressed as:

$$\mu_{\tilde{A}} = \begin{cases} f_{\tilde{A}}^{L}(x), & \text{when } a_1 \le x \le a_2 \\ w, & \text{when } a_2 \le x \le a_3 \\ f_{\tilde{A}}^{R}(x), & \text{when } a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

where$f_{\tilde{A}}^{L}(x): [a_1, a_2] \longrightarrow [0, w]$ and $f_{\tilde{A}}^{R}(x): [a_3, a_4] \longrightarrow [0, w]$. Based on the basic theories of fuzzy numbers, is a normal fuzzy number if $w = 1$, whereas $\widetilde{A}$is a non-normal fuzzy number if $0 < w \le 1$. Therefore, the extended fuzzy number $\widetilde{A}$can be denoted as $[a_1, a_2, a_3, a_4; w]$. The image$-\widetilde{A}$ of$\widetilde{A}$ can be expressed by$[-a_1, -a_2, -a_3, -a_4; w]$.

If $\tilde{x}_1$ is extended fuzzy number, let $g_{\tilde{x}_1}^{L}(y): [0, w] \to [a_1, a_2]$ and $g_{\tilde{x}_1}^{R}(y): [0, w] \to [a_3, a_4]$ be the inverse functions of $f_{\tilde{x}_1}^{L}$and$f_{\tilde{x}_1}^{R}$, respectively. Then $g_{\tilde{x}_1}^{L}(y)$ and $g_{\tilde{x}_1}^{R}(y)$ should be integrable on the closed interval $[0, w]$. In the other words, both $\int_0^w g_{\tilde{x}_1}^{L}(y)\, dy$and $\int_0^w g_{\tilde{x}_1}^{R}(y)\, dy$should exist.

In the case of trapezoidal fuzzy number, the inverse functions $g_{\tilde{x}_1}^{L}(y)$ and $g_{\tilde{x}_1}^{R}(y)$ can be analytically expressed as:

$$g_{\tilde{x}_1}^{L}(y) = a_1 + \frac{(a_2 - a_1)y}{w}, 0 \le y \le w \text{ and } g_{\tilde{x}_1}^{R}(y) = a_4 + \frac{(a_4 - a_3)\, y}{w}, 0 \le y \le w.$$

In order to determine the centroid point $(\bar{x}_0(\tilde{x}_1), \bar{y}_0(\tilde{x}_1))$ of a fuzzy number $\tilde{x}_1$.provided the following centroid formulae, [27]:

$$\bar{x}_0(\tilde{x}_1) = \frac{\int_{a_1}^{a_2} x\, f_{\tilde{x}_1}^{L}(x)\, dx + \int_{a_2}^{a_3} (x\, w)\, dx + \int_{a_3}^{a_4} x\, f_{\tilde{x}_1}^{R}(x)\, dx}{\int_{a_1}^{a_2} f_{\tilde{x}_1}^{L}(x)\, dx + \int_{a_2}^{a_3} (w)\, dx + \int_{a_3}^{a_4} f_{\tilde{x}_1}^{R}(x)\, dx}$$

$$\bar{y}_0(\tilde{x}_1) = \frac{\int_0^w y\left(g_{\tilde{x}_1}^{R}(y) - g_{\tilde{x}_1}^{L}(y)\right) dy}{\int_0^w \left(g_{\tilde{x}_1}^{R}(y) - g_{\tilde{x}_1}^{L}(y)\right) dy}$$

For this trapezoidal fuzzy number, the following results are derived from(3.5) and (3.6),

$$\bar{x}_0 = \frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_{1-}a_2)}\right]$$

$$\bar{y}_0 = w\frac{1}{3}\left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_{1-}a_2)}\right]$$

Example (4.3):

To find the minimum of the functional:

$$J(y) = \int_0^{\tilde{x}_1} -(y')^2\, dx \quad (4.3)$$

with

$$y(0) = \tilde{2} = \langle 0, 2, 4 \rangle, \qquad y(\tilde{x}_1) = \tilde{4} = \langle 2, 4, 6 \rangle$$

where$\tilde{x}_1 = [0, 1, 3, 4; 1]$is a trapezoidal fuzzy number, then the centroid point $(\bar{x}_0(\tilde{x}_1), \bar{y}_0(\tilde{x}_1))$ of a fuzzy number $\tilde{x}_1$ is:

$$\bar{x}_0 = \frac{1}{3}\left[a_1 + a_2 + a_3 + a_4 - \frac{a_4 a_3 - a_1 a_2}{(a_4 + a_3) - (a_{1-}a_2)}\right]$$

$$= \frac{1}{3}\left[0 + 1 + 3 + 4 - \frac{(4)(3) - (0)(1)}{(4 + 3) - (0 + 1)}\right] = 3.33$$

$$\bar{y}_0 = w\frac{1}{3}\left[1 + \frac{a_3 - a_2}{(a_4 + a_3) - (a_{1-}a_2)}\right]$$

$$= (1)\frac{1}{3}\left[1 + \frac{3 - 1}{(4 + 3) - (0 - 1)}\right] = 0.42$$

Now, using the centroid point $(3.33, 0.42)$ of a fuzzy number $\tilde{x}_1$, then equation (4.3) becomes:

$$J(y) = \int_0^{4.9} -(y')^2\, dx$$

with

$$y(0) \approx \tilde{2}, \quad y(3.33) \approx \tilde{4}$$

then:

$y(x) = c_1\, x + c_2$

Using the boundary conditions:

$\underline{c}_1 = 1.2\alpha$and$\bar{c}_1 = 0.6\frac{1}{\alpha}$

hence:

$\underline{y}(x, \alpha) = 1.2\alpha x + 2\alpha$, $\bar{y}(x, \alpha) = 0.6\frac{1}{\alpha}x + 2\frac{1}{\alpha}$

Then:

$$y(x)_\alpha = \left[\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right] = \left[1.2\alpha x + 2\alpha\,, 0.6\frac{1}{\alpha}x + 2\frac{1}{\alpha}\right]$$

defines the α-level sets of a fuzzy number which minimizes J.

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