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Operators on Pythagorean Matrices

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Abstract: *In this paper we introduce Pythagorean matrices having column entries as odd or even, primitive or non-primitive Pythagorean triplets. We have classified these matrices and have shown fundamental characteristics. The second unit introduces certain square matrices which act on such Pythagorean matrices as operators converting their status and order. In the next unit we have represented each column of a matrix of a given class to represent an algebraic polynomial. Using integration, along with boundary condition on Libra values interesting properties of primitive polynomials are derived. Graphs of given matrix polynomial and primitive polynomial show important relations.*
Key Words: *Pythagorean Triplets, Odd Primitives Triplet, Even Primitive Triplet, Non-Primitive Triplet, Operator Matrix, Shift Operator Matrix: PJ1, Primitive matrices and Classes.*

I. Introductions

In this paper, we have two important features to deal with. First we introduce the basic concepts of Pythagorean triplets and classify them into four different categories. We have certain classical matrices which when operated on the matrices of Pythagorean triplets of different classes result into corresponding set of Pythagorean matrices of another classes. These operator matrices demonstrate many classical properties on dealing them in connection to algebraic structural properties. The second part is a contribution to the novel concepts of enjoining successive coefficients of column entries to algebraic polynomials and finding their graphs. In addition, when they are dealt with on finding their primitives under some boundary conditions, they yield giving matrices of higher order which also preserve the Libra value properties.

II. Families of Triplets:

(A) All about Some Pythagorean Triplet:
For a given positive integer '**a**' if there exists positive integers **b** and **h so that the relation**

$$\mathbf{a^2 + b^2 = h^2} \qquad \textbf{(1)}$$

holds true then the integers a, b, and h are known to form a Pythagorean triplets; generally denoted as (a, b, h).

We have, for the given **a,** $\boldsymbol{b} = \frac{a^2 - i^2}{2i}$ and $\boldsymbol{h} = \frac{a^2 + i^2}{2i}$ for some i* ∈ N **(2)**

There are many suggestions and known methods to find **b** and **h** but finally they end up giving the same result that is obtained using the above formula (2) for some **i*** ∈ N

(a) It can be seen that when **a** is an odd integer **b** is even and both casting **h** into an odd integer. In such cases i is an odd positive integer. [In some cases there can exist more than one possible values of **i** which gives different values of **b** and hence **h** too. In any case for **a** >1, i = 1, is guaranteed to give a triplet.]

(b) It can be seen that when **a** is an even integer **b** is odd and both casting **h** into an odd integer. In such cases i is an even positive integer. [In some cases there can exist more than one possible values of **i** which gives different values of **b** and hence **h** too. In any case for **a** ≥ 4, i = 2, is guaranteed to give a triplet.]

For given **a** ≥ **3** in case of odd integers and in case of **a** being even, we consider **a** ≥ 8**

We consider two forms as follows

(1) (a, b, h=b+1), where b = $\frac{a^2 - 1}{2}$, h = b+1 = $\frac{a^2 + 1}{2}$; **a < b**

(2) (a, b, h=b+2) , where b = $\frac{a^2 - 4}{4}$, h = b+2 = $\frac{a^2 + 4}{4}$; **a < b**

(B) **Odd Primitive Pythagorean Triplet:** Let '**a**' be an odd integer and an integer **b** such that (a, b) = 1 (condition forces **b** to be an even integer) then by given form (1) above, we have, for some positive integer **h**, the relation $a^2 + b^2 = h^2$;we call (a, b, h) is an odd primitive Pythagorean triplet. The stated condition (1) above generates triplets of **Plato** family.

(C) **Even Primitive Pythagorean Triplet:** Let '**a**' be an even integer and an integer **b** such that (a, b) = 1 (condition forces **b** to be an odd integer) then by given form (2) above, we have

$a^2 + b^2 = h^2$; we call (a, b, h) is an even primitive Pythagorean triplet.
If we drop the condition (a, b) =1 then it is non-primitive even Pythagorean triplet.
In both cases above, we have
[** For **a** = 4, by condition (2) above, we get **b** = 3 and **h** = 5 ,which contradicts the stated condition that **a < b < h.** Also for **a = 6**, it generates a non-primitive triplet viz. (6, 8, 10) which is a non-primitive triplet as (6, 8) = 2.
At this stage, we note that even integers of the form (4n +2) $\forall n \in N$ cannot generate primitive Pythagorean triplets. (We have mentioned and proved this fact.)

(D) Families of Triplets:
In connection to the above reference of all points stated above, we construct two infinite set P_1 and P_2 defined as follows.
For **a** ≥ (2n + 1) $\forall n \in$ N, we define

P_1 = {(**a, b, h**) | **a, b, h** ∈ **N,** a < b < h, (a, b) = 1, b = $\frac{a^2-1}{2}$, $h = b + 1$, and $a^2 + b^2 = h^2$} **(3)**

This structure defines all odd primitive triplets.*
For **a ≥ (2n)** ∀ n **≥ 3,** we define

P_2 = {(**a, b, h)** | **a, b, h** ∈ **N,** a < b < h, (a, b) = 1, b = $\frac{a^2-4}{4}$, $h = b + 2$, and $a^2 + b^2 = h^2$} **(4)**

This structure defines all even triplets.*

*** Comment:**
(1) To some natural numbers of the form (12 n + 8) and (6n + 27) $\forall n \in$ N, there corresponds more than one triplets but at this stage we focus on the triplets having characteristics shown in the unit A (1) and (2) above.
(2) Positive integers of the 2(2n + 1) $\forall n \in$ N do not possess any primitive triplet.
(3) In addition to the above, we have also Fermat family, and Pythagoras family of triplets but we focus on two sets P_1 and P_2 described above.

III. Pythagorean Matrices

(A) Odd Pythagorean Matrices: We define Pythagorean matrix denoted as, $O_{a,\,3}$ x n where **a** = 2k + 1 $\forall k \in$ N and **3 x n** is the order of the matrix. There are 'n' columns and each column vector represents a Pythagorean triplet. Also, we follow the convention that the absolute difference between the first elements of any two consecutive columns remains constant value **p**, **p ∈ N** for a given matrix under consideration. In this type, we have all odd triplet members of the matrix from set P_1
$O_{2k+1\,,\,3\,x\,n}$ represents a Pythagorean matrix having odd triplets equally spaced. A particular format of the above form is given as follows.

$$O_{2k+1\,,\,3\,x\,3} = \begin{bmatrix} a & a+p & a+2p \\ \frac{a^2-1}{2} & \frac{(a+p)^2-1}{2} & \frac{(a+2p)^2-1}{2} \\ \frac{a^2+1}{2} & \frac{(a+p)^2+1}{2} & \frac{(a+2p)^2+1}{2} \end{bmatrix} \text{ where a = 2k +1; and p = 2r, k, r} \in N \qquad \textbf{(5)}$$

E.G. For k = 1, a = 3, p = 2 and n = 3, $O_{3,3\,X\,3}$ = . $\begin{bmatrix} 3 & 5 & 7 \\ 4 & 12 & 24 \\ 5 & 13 & 25 \end{bmatrix}$

(B) Even Pythagorean Matrices: We define Pythagorean matrix denoted as, $E_{a,\,3}$ x n where **a** = 2k $\forall k \geq 3, k \in N$ and n is the order There are 'n' columns and each column vector represents a Pythagorean triplet. Also, we follow the convention that the absolute difference between the first elements between any two consecutive columns remains constant for a given matrix under consideration. In this type, we have all even triplet members of the matrix are from the set P_2.
Thus $E_{a,\,3}$ x n represents Pythagorean matrix having even triplets equally spaced. A particular format of the above form is given as follows.

$$E_{2k\,,\,3\,x\,3} = \begin{bmatrix} a & a+p & a+2p \\ \frac{a^2-4}{4} & \frac{(a+p)^2-4}{4} & \frac{(a+2p)^2-4}{4} \\ \frac{a^2+4}{4} & \frac{(a+p)^2+4}{4} & \frac{(a+2p)^2+4}{4} \end{bmatrix} \text{ where a = 2k for all k} \geq 3\text{; and p = 2r, k, r} \in N \qquad \textbf{(6)}$$

E.G. For k = 3, a = 6, p = 2 and n = 3, $E_{6,\,3\,X\,3}$ = . $\begin{bmatrix} 6 & 8 & 10 \\ 8 & 15 & 24 \\ 10 & 17 & 26 \end{bmatrix}$

(C) S-Pythagorean Matrices: We define S-Pythagorean matrix denoted as, $P_{a,\,3\,x\,n}$ where $\boldsymbol{a} = 2k,\ \forall\ k\ \geq 3\ or\ a = 2k+1, k \in N - \{1\}$ and **3 x n** is its order. There are 'n' columns and each column vector represents a Pythagorean triplet. Also we set-up a convention that absolute difference between the first element of any two consecutive columns remains unity.

If C_i and C_{i+1} for i = 1 to n-1, are any two consecutive triplet columns of any S- Pythagorean matrix then

$$|\mathbf{a}_{1,\,i+1} - \mathbf{a}_{1,\,i}| = 1 \tag{7}$$

The consecutive columns of the matrix, beginning with the first, represent sequence of either odd triplets and even triplets or even triplets and odd triplets. We have seen that odd triplets are the members of the set P_1 and even triplets of the set P_2.

General format of S-Pythagorean matrices of even and odd nature of the order 3 x 3 is shown below.

$$P_{2k\,,\,3\,x\,3} = \begin{bmatrix} a & a+1 & a+2 \\ \frac{a^2-4}{4} & \frac{(a+1)^2-1}{2} & \frac{(a+2)^2-4}{4} \\ \frac{a^2+4}{4} & \frac{(a+p)^2+1}{2} & \frac{(a+2)^2+4}{4} \end{bmatrix} \tag{8}$$

Where, **a** = 2k for all k ≥ 3; or **a** = 2k +1 for all k ∈ N- {1}

The above matrix $P_{2k\,,\,3\,x\,3}$ represents the most general case in 3 x 3 type even matrices.

The reason to exclude k = 1 is that the next column will begin with **a** + 1 = 4 which generates a non- primitive Pythagorean triplet **(a, b, h)** = (4, 3, 5) in the second column which violates the essential condition that **a < b**

E.G. For k = 3, a = 6, and n = 3, $P_{6,\,3\,X\,3} = \begin{bmatrix} 6 & 7 & 8 \\ 8 & 24 & 15 \\ 10 & 25 & 17 \end{bmatrix}$

Now, we look for another case;

$$P_{2k+1\,,\,3\,x\,3} = \begin{bmatrix} a & a+1 & a+2 \\ \frac{a^2-1}{2} & \frac{(a+1)^2-4}{4} & \frac{(a+2)^2-1}{2} \\ \frac{a^2+1}{2} & \frac{(a+1)^2+4}{4} & \frac{(a+2)^2+1}{2} \end{bmatrix} \text{ where a = 2k+1 for all k} \geq 2;\ k \in N \tag{9}$$

The above matrix $P_{2k+1\,,\,3\,x\,3}$ represents the most general case in 3 x 3 type odd matrices.

E.G. For k = 3, a = 7 and n = 3, $P_{7,\,3\,X\,3} = \begin{bmatrix} 7 & 8 & 9 \\ 24 & 15 & 40 \\ 125 & 17 & 41 \end{bmatrix}$

6 Fundamental Characteristics:

In the following notes we state some important characteristics of

(1) S-Pythagorean class either $P_{2k+1,\,3\,x\,3}$ and $P_{2k,\,3\,x\,3}$

(2) (Odd) matrices of class $O_{2k+1,\,3\,x\,3}$

(3) (Even) matrices of class $E_{2k,\,3\,x\,3}$

(A) Determinant values:

(1) Matrices having column entries as S-Pythagorean triplets start from **odd** entry,

$$P_{2k+1,\,3\,x\,3} = \begin{bmatrix} a & a+1 & a+2 \\ \frac{a^2-1}{2} & \frac{(a+1)^2-4}{4} & \frac{(a+2)^2-1}{2} \\ \frac{a^2+1}{2} & \frac{(a+1)^2+4}{4} & \frac{(a+2)^2+1}{2} \end{bmatrix} \text{ for any } \mathbf{a} = 2k+1;\ k \in N$$

$$|P_{2k+1,\,3\,x\,3}| = \text{det. } P_{2k+1,\,3\,x\,3} = -\frac{(a+3)(a-1)}{2} \tag{10}$$

E.G. For k =2, $|P_{2k+1,\,3\,x\,3}|$ = det. $\begin{bmatrix} 5 & 6 & 7 \\ 12 & 8 & 24 \\ 13 & 10 & 25 \end{bmatrix} = -\frac{(5+3).(5-1)}{2} = -16.$

(2) The Matrices having column entries as S-Pythagorean triplets start from **even** entry

$$P_{2k,\,3\,x3} = \begin{bmatrix} a & a+1 & a+2 \\ \frac{a^2-4}{4} & \frac{(a+1)^2-1}{2} & \frac{(a+2)^2-4}{4} \\ \frac{a^2+4}{4} & \frac{(a+1)^2+1}{2} & \frac{(a+2)^2+4}{4} \end{bmatrix} \text{ For any } \mathbf{a} = 2k, \text{ for } k \geq 3$$

$$|P_{2k,\,3\,x\,3}| = \text{det.}P_{2k,\,3\,x3} = -\frac{a(a+2)}{2} \tag{11}$$

E.G. For k = 3, det. $\begin{bmatrix} 6 & 7 & 8 \\ 8 & 24 & 15 \\ 10 & 25 & 17 \end{bmatrix} = -24.$

(3) Now, we discuss the case (2) as mentioned above. We represent the most general format of odd matrices of class 3x3.

$$O_{2k+1,\,3\times 3} = \begin{bmatrix} a & a+p & a+2p \\ \frac{a^2-1}{2} & \frac{(a+p)^2-1}{2} & \frac{(a+2p)^2-1}{2} \\ \frac{a^2+1}{2} & \frac{(a+p)^2+1}{2} & \frac{(a+2p)^2+1}{2} \end{bmatrix}$$

We note that for any odd integer $\mathbf{a = 2k + 1 \geq 3}$ and $p = 2n$; $n \in N$,

$|O_{2k+1,\,3\times 3}| = \det.(O_{2k+1,\,3\times 3}) = \mathbf{p^3}$ **(12)**

E.G. For **a = 3 and p= 2,** the above matrix

$$O_{3,\,3\times 3} = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 12 & 24 \\ 5 & 13 & 25 \end{bmatrix} \text{ and } |\mathbf{O_{3,\,3\times 3}}| = \mathbf{p^3 = (2)^3 = 8}$$

for **a = 7** and **p = 4,** the matrix is

$$O_{7,\,3\times 3} = \begin{bmatrix} 7 & 11 & 15 \\ 24 & 60 & 112 \\ 25 & 61 & 113 \end{bmatrix} \text{ and } |\mathbf{O_{7,\,3\times 3}}| = \mathbf{p^3 = (4)^3 = 64}$$

(4) Now, we discuss the case (3) as mentioned above. We represent the most general format of even matrices of class 3x3.

$$E_{2k,\,3\times 3} = \begin{bmatrix} a & a+p & a+2p \\ \frac{a^2-4}{4} & \frac{(a+p)^2-4}{4} & \frac{(a+2p)^2-4}{4} \\ \frac{a^2+4}{4} & \frac{(a+p)^2+4}{4} & \frac{(a+2p)^2+4}{4} \end{bmatrix} \quad \textbf{(13)}$$

We note that for any odd integer $\mathbf{a = 2k \geq 6}$ **and** $p = 2n$; $n \in N$,

$|E_{2k,\,3\times 3}| = \det.(E_{2k,\,3\times 3}) = \mathbf{p^3}$ **(14)**

E.G. For **a = 3 and p= 2,** the above matrix is

$$E_{6,\,3\times 3} = \begin{bmatrix} 6 & 8 & 10 \\ 8 & 15 & 24 \\ 10 & 17 & 26 \end{bmatrix} \text{ and } |\mathbf{E_{6,\,3\times 3}}| = \mathbf{p^3 = (2)^3 = 8}$$

For **a = 12** and **p = 4,** the matrix is

$$E_{12,\,3\times 3} = \begin{bmatrix} 12 & 16 & 20 \\ 35 & 63 & 99 \\ 37 & 65 & 101 \end{bmatrix} \text{ and } |\mathbf{E_{12,\,3\times 3}}| = \mathbf{p^3 = (4)^3 = 64}$$

IV. Operator Matrices of class 1:

As a part of our published research work, we have categorized matrices in different classes and derived many algebraic properties; we mention here the calling property of the first class.

Let $A = (a_{ij})_{m\times n}$ be a matrix on the field of real numbers R , ∀ i = 1 to m and j= 1 to n. where m, n ϵ N

If $\sum_{i=1}^{m} a_{ij}$ = Constant for each j = 1, 2,….n.

i.e. If the sum of all the entries of a column for each one of the columns of the given matrix A, remains the same real constant than the matrix is said to satisfy the property P_1.

A set of matrices which observe the property P1 constitutes class1; denoted as CJ1.

CJ1 = { A |A = $(a_{ij})_{m\times n}$, A satisfies P_1 and L(A)= p; p ϵ R for a given matrix A} **(15)**

[L(A) is called Libra value, the constant sum value associated with the algebraic sum of elements of each column of the given matrix under consideration.

(A) A **PJ** matrix of class-1:

We mention here the first matrix in the sequence of class-1 matrices; it is denoted as **PJ.**

$$\mathbf{PJ} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{6} \\ \mathbf{-3} & \mathbf{-8} & \mathbf{-15} \\ \mathbf{3} & \mathbf{6} & \mathbf{10} \end{bmatrix} \in \boldsymbol{CJ1(3\ x\ 3, L(PJ) = 1)} \quad \textbf{(16)}$$

This matrix **PJ** will be known as the first **Pythagorean operator**. It demonstrates many classical properties but at this stage, we cite only two of them.

(1) From PJ we can write its **n^th product** denoted as (PJ)

$$(PJ1)^n = \begin{bmatrix} 3^2(\sum(n-1)) + 1 & \frac{3n(3n-1)}{2} & \frac{3n(3n+1)}{2} \\ -(3n-1)^2 + 1 & -(3n)^2 + 1 & -(3n+1)^2 + 1 \\ \frac{3n(3n-1)}{2} & \frac{3n(3n+1)}{2} & 3^2(\sum(n)) + 1 \end{bmatrix} \quad \text{for } n \in N \quad \textbf{(17)}$$

[The general form can be verified for different values of n∈ N]

(2) The matrix $(PJ)^n \in \boldsymbol{CJ1(3\ x\ 3, L(PJ)^n = 1)}$ **for any n ∈ *N***

(B) Matrix PJ – An Operator:

The matrix $\mathbf{(PJ)^n}$ for any n ∈ N can be established as an Operator on the above mentioned $O_{2k+1,\,3\times 3}$, $E_{2k,\,3\times 3}$, $P_{2k,\,3\times 3}$, and $P_{2k+1,\,3\times 3}$ classes.

(1) **For the matrices of class $O_{2k+1,\,3\times3}$:**

The product ($O_{2k+1,\,3\times3}$) (PJ) for k = p, some positive integer, results into $O_{2k+1,\,3\times3}$ for k = p + 3.

E.G. for k = 1, $O_{2k+1,\,3\times3}$ is $\begin{bmatrix}3 & 5 & 7\\4 & 12 & 24\\5 & 13 & 25\end{bmatrix}$ also for n =1, **PJ =** $\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix}$

and $\begin{bmatrix}3 & 5 & 7\\4 & 12 & 24\\5 & 13 & 25\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix} = \begin{bmatrix}9 & 11 & 13\\40 & 60 & 84\\41 & 61 & 85\end{bmatrix}$ which is $O_{2k+1,\,3\times3}$ for k = 1+3 =4

Using associative property,

The product ($O_{2k+1,\,3\times3}$) (PJ)^2 for some k = p, some positive integer, results into ($O_{2k+1,\,3\times3}$) for k = p + 6

E. G. $\begin{bmatrix}3 & 5 & 7\\4 & 12 & 24\\5 & 13 & 25\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix}^{\mathbf{2}} = \begin{bmatrix}15 & 17 & 19\\112 & 144 & 190\\113 & 145 & 191\end{bmatrix}$

(2) **For the matrices of class $E_{2k,\,3\times3}$:**

Now we consider matrices of the class $E_{2k,\,3\times3}$,

The product ($E_{2k,\,3\times3}$) (PJ) for k = p ≥ 3, some positive integer, results into $E_{2k,\,3\times3}$ for k = p + 3.

E.G. for k = 3, $E_{2k,\,3\times3}$ is $\begin{bmatrix}6 & 8 & 10\\8 & 15 & 24\\10 & 17 & 26\end{bmatrix}$ also for n =1, **PJ =** $\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix}$

and $\begin{bmatrix}6 & 8 & 10\\8 & 15 & 24\\10 & 17 & 26\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix} = \begin{bmatrix}12 & 14 & 16\\35 & 48 & 63\\37 & 50 & 65\end{bmatrix}$ which is $E_{2k,\,3\times3}$ for k = 3+3 =6

Using associative property,

The product ($E_{2k,\,3\times3}$) (PJ)^2 for some k = p ≥ 3, some positive integer, results into ($E_{2k,\,3\times3}$) for k = p+ 6

E. G. $\begin{bmatrix}6 & 8 & 10\\8 & 15 & 24\\10 & 17 & 26\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{3} & \mathbf{6}\\\mathbf{-3} & \mathbf{-8} & \mathbf{-15}\\\mathbf{3} & \mathbf{6} & \mathbf{10}\end{bmatrix}^{\mathbf{2}} = \begin{bmatrix}18 & 20 & 22\\80 & 99 & 120\\82 & 101 & 122\end{bmatrix}$

(3) Generalized Product:

A Pythagorean matrix is of order (3 x n) where n ≥ 3. The matrix PJ discussed above is of order 3x3 and , as seen, has an ability to convert a given matrix – either $O_{2k+1,\,3x3}$ or $E_{2k,\,3x3}$, into the matrix of immediate consecutive sequence of odd or even integer as the case may be of the given sequence. In the discussion to follow, we shall discuss an operator matrix that will operate on a even or odd Pythagorean matrix of the order (3xn).

We introduce a matrix denoted as 'PJN'.

$$\text{PJN} = \begin{bmatrix}1 & -8 & 8\\8 & -31 & 32\\8 & -32 & 33\end{bmatrix} \qquad (18)$$

We mention two Important and interesting properties.

(PJN) $O_{2k+1,\,3xn} = O_{2p+1,\,3xn}$ for p = k +n (19-A)

(PJN) $E_{2k,\,3xn} = E_{2p,\,3xn}$ for p = k +8 (19-B)

E.G.

For k = 2, $O_{5,\,3x4} = \begin{bmatrix}5 & 7 & 9 & 11\\12 & 24 & 40 & 60\\13 & 25 & 41 & 61\end{bmatrix}$

(PJN) $O_{5,\,3x4} = \begin{bmatrix}13 & 15 & 17 & 19\\84 & 112 & 144 & 180\\85 & 113 & 145 & 181\end{bmatrix} = O_{13,\,3x4}$

For k = 3, $E_{6,\,3x5} = \begin{bmatrix}6 & 8 & 10 & 12 & 14\\8 & 15 & 24 & 35 & 48\\10 & 17 & 26 & 37 & 50\end{bmatrix}$

(PJN) $E_{6,\,3x5} = E_{22,\,3x5} = \begin{bmatrix}22 & 24 & 26 & 28 & 30\\120 & 143 & 168 & 195 & 224\\122 & 145 & 170 & 197 & 226\end{bmatrix}$

We conclude that PJN is an operator that converts Odd or Even matrices into Odd and Even matrices of the next higher sequence of members.

At this point we mention the result of product of n$^{th.}$ order of (PJN).

$$(PJN)^n = \begin{bmatrix} n^0 & -8n & 8n \\ 8n & -2^5n^2+1 & 2^5n^2 \\ 8n & -2^5n^2 & 2^5n^2+1 \end{bmatrix} \text{ for } n \in N \qquad (20)$$

We can use associative property and establish results for product of higher order of (PJN) with odd or even matrices of order (3 x n) for $n \in N \; and \; n \geq 3$

V. Matrix—A Set of Polynomials:

In this section, we **r**epresent each column of a given matrix (order m x n) in a well defined pattern of a m^{th} degree polynomial in one variable; say x. This, in turn helps identify each column as an m^{th} degree curve for each column. This convention casts the given matrix of order m x n matrix as a set of n curves each of order m in a single variable.

(A) Matrix—Set of Curves:

As per the above convention, we write a column matrix of order m x 1 in the following pattern.

Let $\mathbf{Y} = \begin{bmatrix} \boldsymbol{a_0} \\ \boldsymbol{a_1} \\ \cdot \\ \cdot \\ \boldsymbol{a_m} \end{bmatrix}$ which is represented as $\mathbf{Y} = \sum_{i=0}^{i=m} \boldsymbol{a_i x^i}$

Thus a m x n matrix is $[Y_1\ Y_2 \ldots\ldots Y_n]$ and each one of the n column vector is an m^{th} degree polynomial in real variable x. In addition, in context to reference to follow, we consider each one of the coefficients ($\mathbf{a_i}$)as real variable.

E.G. A 3 x 3 matrix A = $[Y_1\ Y_2\ Y_3]$ where $Y_1 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$, $Y_2 = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$, $Y_3 = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

Where $Y_1 = a_0 + a_1x + a_2x^2$ $\quad Y_2 = b_0 + b_1x + b_2x^2$ $\quad Y_3 = c_0 + c_1x + c_2x^2$; each one shows quadratic curve with real coefficients.

(B) PJ matrix and Graphs:

As discussed above paragraph and previous sections, a matrix for which if the sum of all column entries remains constant and same for all columns then we call it a class one matrix. The constant value is called the Libra value of the matrix denoted as L(A).

A = $[Y_1\ Y_2\ Y_3]$ where $Y_1 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$, $Y_2 = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$, $Y_3 = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

If $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = c_0 + c_1 + c_2$ = constant = p = L(A) then

A ∈ CJ1(3 x 3, L(A) = p); $p \in R$

Also, by the convention, each column is a quadratic curve,

$Y_1 = a_0 + a_1x + a_2x^2$ $\quad Y_2 = b_0 + b_1x + b_2x^2$ $\quad Y_3 = c_0 + c_1x + c_2x^2$.

Combining both the facts for x = 1,

We have $Y_1 = a_0 + a_1 + a_2$, $Y_2 = b_0 + b_1 + b_2$, $Y_3 = c_0 + c_1 + c_2$; if the underlying matrix is of class one then $Y_1 = Y_2 = Y_3 = P = L(A)$ = Libra Value of the matrix. The fact conveys that all these quadratic curves are concurrent at the point (1, L(A)= P)

E.G. We consider the Pythagorean operator matrix of class one, denoted as PJ and observe their correspondence.

$$\mathbf{PJ} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{6} \\ \mathbf{-3} & \mathbf{-8} & \mathbf{-15} \\ \mathbf{3} & \mathbf{6} & \mathbf{10} \end{bmatrix} \in \mathbf{CJ1\ (3\ x\ 3,\ L\ (PJ) = 1)}$$

Let $Y_1 = 1 - 3x + 3x^2$ $\quad Y_2 = 3 - 8x + 6x^2$ $\quad Y_3 = 6 - 15x + 10x^2$

For x = 1, $Y_1 = Y_2 = Y_3 = 1$, i.e. all these quadratic curves have a common point of intersection = **(1, 1).**

The corresponding graph is as follows.

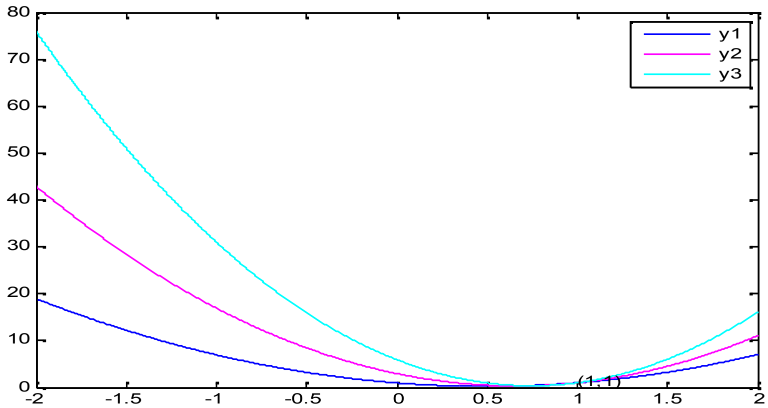

Figure 1

(C)Primitives and Class Preservation on Integration:

One of the most interesting properties of class 1 matrices is that each curve obtained on integration of the given column curves of a matrix preserves the class. i.e. Each corresponding curve obtained on integration , resulting in one higher degree, also passes through **(x=1, Y = L(A)= p)** The resulting matrix is of the increased order **(m +1) x n** from the previous one of order **m x n.**

We tackle the point technically as follows.

For the given set-up as discussed above,

We have, A = [Y_1 Y_2 Y_3] where $Y_1 = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$, $Y_2 = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$, $Y_3 = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$

If $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = c_0 + c_1 + c_2$ = constant = p = L(A) then A ∈ CJ1(3 x 3, L(A) = p); p ∈ R

Also, by the convention, each column is a quadratic curve,

$Y_1 = a_0 + a_1x + a_2 x^2$ $Y_2 = b_0 + b_1x + b_2 x^2$ $Y_3 = c_0 + c_1x + c_2 x^2$

On integration, the set-up of the new system of curves is

$Y_{11} = a_0 x + a_1x^2/2 + a_2 x^3/3 + C_1$, $Y_{21} = b_0x + b_1x^2/2 + b_2 x^3/3 + C_2$, $Y_{31} = c_0x + c_1x^2/2 + c_2 x^3/3 + C_3$

Using the boundary conditions, for x = 1, $Y_{11} = Y_{21} = Y_{31}$ = L(A) = p =1

We have, $C_1 = 1 - a_0 - a_1/2 - a_2/3$ $C_2 = 1 - b_0 - b_1/2 - b_2/3$ $C_3 = 1 - c_0 - c_1/2 - c_2/3$

Then the derived matrix is

$A_1 = \begin{bmatrix} C_1 & C_2 & C_3 \\ a_0 & b_0 & c_0 \\ (a_1)/2 & (b_1)/2 & (c_1)/2 \\ (a_2)/3 & (b_2)/3 & (c_2)/3 \end{bmatrix}$ which is a matrix of order 4 x 3 [From 3 x 3 to 4 x 3]

Also, it can be checked that $C_1 + a_0 + a_1/2 + a_2/3 = 1$, $C_2 + b_0 + b_1/2 + b_2/3 = 1$,
and $C_3 + c_0 + c_1/2 + c_2/3 = 1$

and each equals to the Libra value of the matrix = L(A_1) = p =1.

i.e. A_1 ∈ CJ1 (4 x 3, L(A_1) = 1)

If we continue the same process of integration and finding the constants of integration using boundary condition, each time we get a new matrix of higher order. All such matrices are

(1)Always of class CJ1

(2) Columns of each matrix obtained stage wise represent curves of higher order and all of them, when graphed, are concurrent at a point (x=1, y = L(PJ)=p)

We continue with the above example and obtain PJ1 matrix (order 4 x 3) from the PJ matrix of order 3 x3.

We have **PJ** = $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{6} \\ \mathbf{-3} & \mathbf{-8} & \mathbf{-15} \\ \mathbf{3} & \mathbf{6} & \mathbf{10} \end{bmatrix}$ **∈ CJ1 (3 x 3, L (PJ) = 1)**

Let $Y_1 = 1 - 3x + 3x^2$ $Y_2 = 3 - 8x + 6x^2$ $Y_3 = 6 - 15x + 10x^2$

Now, as described above, on integration we have the set of curves of the primitive generation;

$Y_{11} = x - (3/2)x^2 + x^3 + C_{11}$, $Y_{21} = 3x - 4x^2 + 2x^3 + C_{21}$, $Y_{31} = 6x - (15/2)x^2 + (10/3)x^3 + C_{31}$

In order to determine the values of constants of integration, we use boundary condition,

that all such curves pass through the point (1,1) i.e. the matrix formed by column vectors is of the class CJ1. On letting x =1 and Y_{11} = 1, we get C_{11} = ½ , C_{21} = 0, and C_{31} = -5/6

We have the matrix PJ1 of higher order.

The matrix PJ1 = $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{5}{6} \\ 1 & 3 & 6 \\ -\frac{3}{2} & -4 & -\frac{15}{2} \\ 1 & 2 & \frac{10}{3} \end{pmatrix}$ ϵ CJ1 (4 ×3, L(PJ1) =1)

corresponding to each column vector we have three cubic curves with the following equations.

Y_{11}= (1/2) + 1x – (3/2) x^2+ $1x^3$

Y_{21}= 0+ 3x -$4x^2$ + $2x^3$

Y_{31}= - (5/6) + 6x – (15/2)x^2 + (10/3)x^3

For x =1, Y_{11} = Y_{21} = Y_{31} = L (PJ1) = 1.

These three equations are graphed as follows.

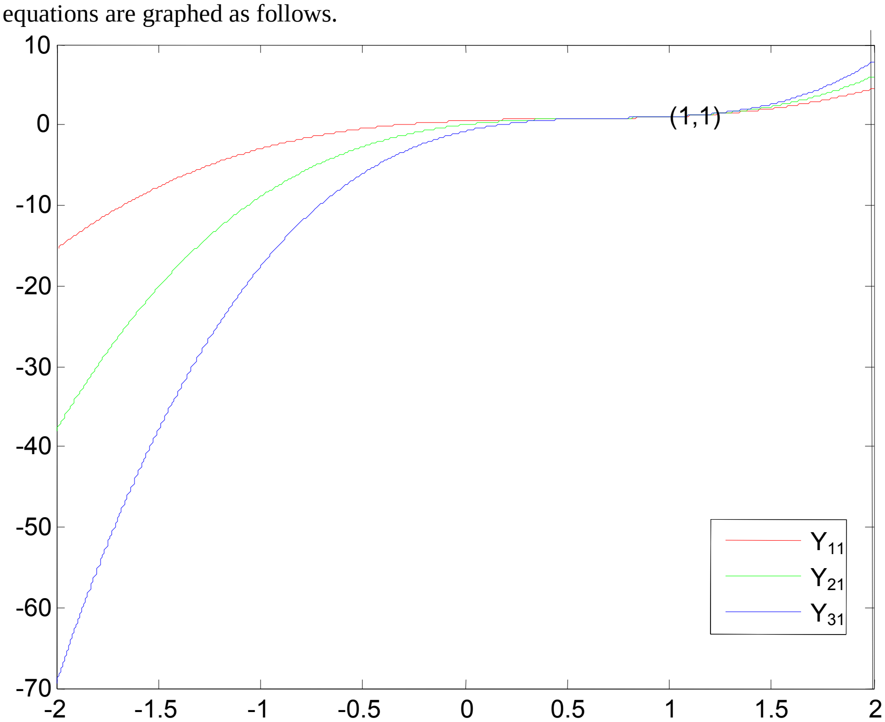

Figure -2

We continue in the same way and get the matrix of the next higher order. On following the same procedure, we get

On integrating Y_{21}, Y_{22}, Y_{23}, we get,

$Y_{21} = \frac{1}{4} + \frac{1}{2} x + \frac{1}{2} x^2 - \frac{1}{2} x^3 + \frac{1}{4} x^4$

$Y_{22} = \frac{1}{3} + \frac{3}{2} x^2 - \frac{4}{3} x^3 + \frac{1}{2} x^4$

$Y_{23} = \frac{1}{2} - \frac{5}{6} x + 3 x^2 - \frac{5}{2} x^3 + \frac{5}{6} x^4$

PJ2=$\begin{pmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{5}{6} \\ \frac{1}{2} & \frac{3}{2} & 3 \\ -\frac{1}{2} & -\frac{4}{3} & -\frac{5}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{5}{6} \end{pmatrix}$ ϵCJ1(5×3,L(PJ2)=1)

Each one of the bi-quadratic curves are graphed below and the point of their concurrence is (1,1)

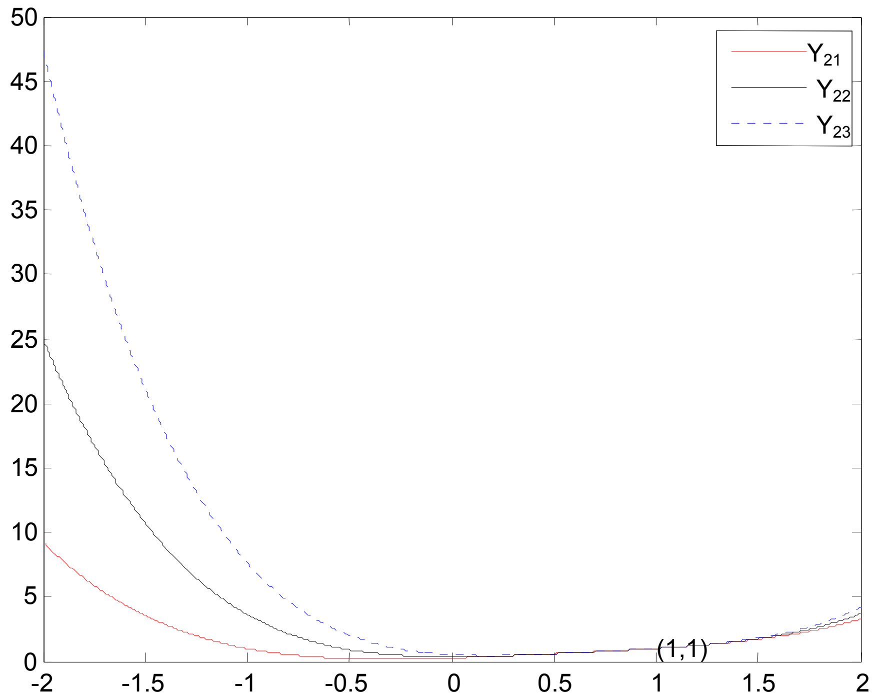

Figure-3

All these curves Y_i , for different values of i are graphed on the same page and it is clearly found that they have the same point of concurrence (1,1)

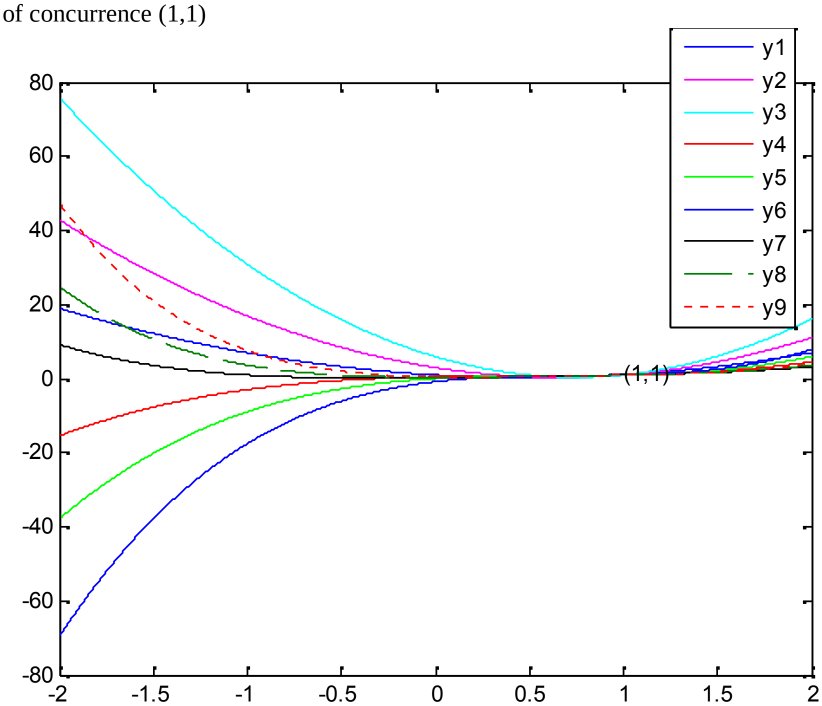

Figure-4

VI. Conclusion

In the above text we have enjoined Pythagorean matrices and their transformations which hold true in all the cases. Transformation of higher order of matrices, as derived, can give Pythagorean triplets of higher order. The next important concept establishes algebraic presentation of successive entries of each column of a matrix on discussion. Libra values of the matrix of a given class and that of higher order matrices continues to remain constant and that too of the same class, is the important feature of the class.

VII. Visions

The same concept of obtaining matrices of higher order through integration of algebraic polynomial representatives of columns of the matrices of different classes, as we have already defined.(e.g. class CJ2, CJ3, CJ4, and CJ5), observing graphical pattern, and checking class- preservation Libra value property is the prime work on hands.

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