

A Study on Some Properties of Poisson Size-Biased Quasi Lindley Distribution

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Abstract : The main aim of this paper is to introduced and study notion of Poisson Size-biased Quasi Lindley distribution. Besides deriving its p.m.f., some of its properties and the expressions for raw and central moments, coefficient of skewness and kurtosis, coefficient of variation, index of dispersion have been obtained. The problem of parameter estimation by using method of moments has been also discussed.

Keywords: Compounding; size-biased distribution; Poisson-Lindley distribution; size-biased Quasi Poisson-Lindley distribution; Estimation of parameters.

I. Introduction

Size-biased distribution is a special case of the more general form known as weighted distribution. These distributions were first introduced by Fisher (1934) to model ascertainment biases which were later formalized by Rao (1965) in a unifying theory. Size-biased distributions arise in practice when observations from a sample are recorded with unequal probabilities having probability proportional to some measure of unit size.

Symbolically, if the random variable X has distribution $f(x; \theta)$, with unknown parameter θ , then a simple size-biased distribution is given by its probability mass function / probability density function as

$$f^*(x; \theta) = \frac{xf(x; \theta)}{\mu}, \text{ where } \mu = E(x), \text{ is the mean of the distribution.}$$

Ghitany and Al-Mutairi (2008) obtained a size-biased Poisson-Lindley (SBPL) distribution given by its probability mass function (pmf)

$$P_1(x; \theta) = \frac{\theta^x}{\theta+2} \frac{x(x+\theta+2)}{(1+\theta)^{x+2}}; \quad x = 1, 2, \dots; \quad \theta > 0 \quad (1.1)$$

by size-biasing the Poisson-Lindley distribution of Sankaran (1970) having pmf

$$P(x; \theta) = \frac{\theta^x (x+\theta+2)}{(1+\theta)^{x+3}}; \quad x = 0, 1, 2, \dots; \quad \theta > 0 \quad (1.2)$$

They have also showed that, size-biased Poisson-Lindley distribution also arises from the size-biased Poisson distribution when its parameter λ follows a size-biased Lindley distribution. Dutta and Borah (2014) reviewed some properties and applications of size-biased Poisson-Lindley distribution.

Adhikari and Srivastava (2013) proposed a new form of size-biased Poisson-Lindley distribution obtained by compounding the size-biased Poisson distribution with Lindley distribution without considering its size-biased form with pmf

$$P_2(x; \theta) = \frac{\theta^x (x+\theta+1)}{(1+\theta)^{x+2}}; \quad x = 1, 2, 3, \dots; \quad \theta > 0 \quad (1.3)$$

The Poisson size-biased Lindley (PSBL) distribution has been obtained in Adhikari and Srivastava (2014) compounding Poisson distribution with size-biased Lindley distribution with pmf

$$P_3(x; \theta) = \frac{\theta^x (x+1)(x+\theta+3)}{(\theta+2)(1+\theta)^{x+3}}; \quad x = 0, 1, 2, \dots; \quad \theta > 0 \quad (1.4)$$

Recently, Dutta and Borah (2015) worked on certain recurrence relations arising in size-biased Poisson-Lindley and Poisson size-biased Lindley distributions of Adhikari and Srivastava (2013, 2014). They also investigated some statistical properties of these distributions including fitting of distributions to some published data sets. Shanker and Mishra (2013) proposed size-biased Quasi Poisson-Lindley distribution, of which the size-biased Poisson-Lindley distribution of Ghitany and Al Mutairi (2008) is a particular case, has been obtained by size biasing the quasi Poisson - Lindley distribution of Shanker and Mishra(2013) as

$$P_4(x; \theta, \alpha) = \frac{xP(x; \theta, \alpha)}{\mu} = \frac{\theta^2}{\alpha+2} \frac{x(\theta x + \theta\alpha + \theta + \alpha)}{(1+\theta)^{x+2}}; x = 1, 2, 3, \dots; \theta > 0, \alpha > -2 \quad (1.5)$$

where $P(x; \theta, \alpha)$ is the pmf of Quasi Poisson-Lindley distribution, i.e.,

$$P(x; \theta, \alpha) = \frac{\theta[\alpha + \theta(1+\alpha) + \theta x]}{(1+\alpha)(1+\theta)^{\alpha+2}}; x = 0, 1, 2, \dots; \theta > 0, \alpha > -1 \quad (1.6)$$

and $\mu = \frac{\alpha+2}{\theta(1+\alpha)}$ be the mean of Quasi Poisson-Lindley distribution [see Shanker and Mishra (2013)]. They

have also derived a general expression for the r^{th} factorial moment of this distribution. They also showed that, the size-biased quasi Poisson-Lindley distribution can also obtained from the size-biased Poisson distribution when its parameter λ follows size-biased Quasi Lindley distribution with probability density function (pdf)

$$f(\lambda; \theta, \alpha) = \frac{\theta^2}{\alpha+2} \lambda(\alpha + \theta\lambda)e^{-\theta\lambda}; \lambda > 0, \theta > 0, \alpha > -2 \quad (1.7)$$

In this paper, the Poisson size-biased Quasi-Lindley (PSBQL) distribution of which the Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014) is a particular case has been obtained. Some distributional properties of this distribution including probability mass function have been investigated. A general expression for the r^{th} order factorial moments of the distribution has been derived and hence its first four moments have been obtained. The estimation of its parameters by using method of moments has been also discussed.

II. Poisson Size Biased Quasi Lindley Distribution

The Poisson size-biased Quasi Lindley (PSBQL) distribution can be obtained without considering its size-biased form by compounding the Poisson distribution with size-biased Quasi Lindley distribution with probability density function (pdf)

$$f(\lambda; \theta, \alpha) = \frac{\theta^2}{\alpha+2} \lambda(\alpha + \theta\lambda)e^{-\theta\lambda}; \lambda > 0, \theta > 0, \alpha > -2 \quad (2.1)$$

The resultant probability mass function (pmf) of PSBQL distribution may be obtained as

$$P(x; \theta, \alpha) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!} \frac{\theta^2}{(\alpha+2)} \lambda(\alpha + \theta\lambda)e^{-\theta\lambda} d\lambda \quad (2.2)$$

$$\begin{aligned} &= \frac{\theta^2}{\alpha+2} \frac{1}{x!} \int_0^\infty e^{-(1+\theta)\lambda} (\alpha\lambda^{x+1} + \theta\lambda^{x+2}) d\lambda \\ &= \frac{\theta^2}{\alpha+2} \left[\frac{\alpha(x+1)}{(1+\theta)^{x+2}} + \frac{\theta(x+1)(x+2)}{(1+\theta)^{x+3}} \right] \\ &= \frac{\theta^2}{(\alpha+2)} \frac{(1+x)(\alpha + \alpha\theta + 2\theta + \theta x)}{(1+\theta)^{x+3}}; x = 0, 1, 2, \dots, \theta > 0, \alpha > -2. \end{aligned} \quad (2.3)$$

While, θ is the scale and α the shape parameters of the distribution. Simply, denote it by PSBQL (θ, α) . It can be seen that, Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014), is a particular case of PSBQL distribution at $\alpha = \theta$.

Since, the ratio

$$\frac{P(x+1; \theta, \alpha)}{P(x; \theta, \alpha)} = \left(1 + \frac{1}{x+1}\right) \left(1 + \frac{\theta}{\alpha + \alpha\theta + 2\theta + \theta x}\right) \quad (2.4)$$

is a decreasing function of x , $P(x; \theta, \alpha)$ is log-concave. Therefore, the PSBQL distribution is unimodal, has an increasing failure rate. [See Johnson et al. (2005)]

III. Statistical Properties

In this section, some properties of the PSBQL distribution has been studied as follows:

Probability generating function

Let X be a random variable with PSBQL (θ, α) distribution. Then the probability generating function (pgf), $g(t)$ of X can be obtained as

$$g(t) = \sum_{x=0}^{\infty} t^x P(x; \theta, \alpha) = \frac{\theta^2}{(\alpha+2)(1+\theta)^3} \sum_{x=0}^{\infty} t^x \frac{(1+x)(\alpha+\alpha\theta+2\theta+\theta x)}{(1+\theta)^x}$$

$$= \frac{\theta^2}{(\alpha+2)} \frac{(\alpha+\alpha\theta+2\theta-\alpha t)}{(1+\theta-t)^3}, \quad \theta > 0, \alpha > -2. \tag{3.1}$$

The pgf, $g(t) = \frac{\theta^3}{(\theta+2)} \frac{(\theta+3-t)}{(1+\theta-t)^3}$; $\theta > 0$ of Poisson size-biased Lindley distribution, may be obtained as a particular case of PSBQL distribution at $\alpha = \theta$.

The expression for probabilities of PSBQL distribution obtained from (2.5) is given as

$$P_r = \frac{1}{(1+\theta)^3} [3(1+\theta)^2 P_{r-1} - 3(1+\theta) P_{r-2} + P_{r-3}]; \quad r > 3 \tag{3.2}$$

where,

$$P_0 = \frac{\theta^2}{(\alpha+2)} \frac{(\alpha+\alpha\theta+2\theta)}{(1+\theta)^3}, \quad P_1 = \frac{\theta^2}{(\alpha+2)} \frac{2(\alpha+\alpha\theta+3\theta)}{(1+\theta)^4}$$

$$P_2 = \frac{\theta^2}{(\alpha+2)} \frac{3(\alpha+\alpha\theta+4\theta)}{(1+\theta)^5}, \quad P_3 = \frac{\theta^2}{(\alpha+2)} \frac{4(\alpha+\alpha\theta+4\theta)}{(1+\theta)^6} \text{ so on.}$$

The r^{th} order probability of PSBQL distribution may also be written as

$$P_r = \frac{\theta^2}{(\alpha+2)} \frac{(r+1)(\alpha+\alpha\theta+(r+2)\theta)}{(1+\theta)^{r+3}}; \quad r = 0, 1, 2, \dots; \quad \theta > 0, \alpha > -2 \tag{3.3}$$

The higher order probabilities may be obtained very easily by using expression (3.3).

Moment generating function

The distribution of a random variable is often characterized in terms of its moment generating function (mgf), a real function whose derivatives at zero are equal to the moments of the random variable. Moment generating functions have great practical relevance not only because they can be used to easily derive moments, but also because a probability distribution is uniquely determined by its moment generating function.

If $X \sim \text{PSBQL}(\theta, \alpha)$, then the mgf of X can be derived as

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} P(x; \alpha, \theta) = \sum_{x=0}^{\infty} e^{tx} \frac{\theta^2}{(\alpha+2)} \frac{(1+x)(\alpha+\alpha\theta+2\theta+\theta x)}{(1+\theta)^{x+3}}$$

$$= \frac{\theta^2}{\alpha+2} \frac{(\alpha+\alpha\theta+2\theta-\alpha e^t)}{(1+\theta-e^t)^3}; \quad \theta > 0, \alpha > -2 \tag{3.4}$$

The mgf of PSBQL distribution is same as that of Poisson size-biased Lindley distribution of Adhikari and Srivastava (2014) at $\alpha = \theta$.

Factorial Moments

The r^{th} order factorial moment of the PSBQL distribution can be obtained as

$$\mu'_{(r)} = E[E(X^{(r)}|\lambda)] \tag{3.5}$$

where, $X^{(r)} = X(X-1)(X-2)\dots(X-r+1)$

From equation (2.2), we get

$$\mu'_{(r)} = \int_0^{\infty} \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} \right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha + \theta\lambda) e^{-\theta\lambda} d\lambda \tag{3.6}$$

$$= \int_0^{\infty} \left[\lambda^r \sum_{x=r}^{\infty} \frac{e^{-\lambda} \lambda^{x-r}}{(x-r)!} \right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha + \theta\lambda) e^{-\theta\lambda} d\lambda \tag{3.7}$$

Taking $(x+r)$ in place of x , we get

$$\mu'_{(r)} = \int_0^{\infty} \lambda^r \left[\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} \right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha + \theta\lambda) e^{-\theta\lambda} d\lambda \tag{3.8}$$

The expression within bracket is equal to one and hence we have

$$\mu'_{(r)} = \frac{\theta^2}{(\alpha+2)} \int_0^{\infty} \lambda^{r+1} (\alpha + \theta\lambda) e^{-\theta\lambda} d\lambda \tag{3.9}$$

Using gamma integral, we get finally, after a little simplification, a general expression for the r^{th} order factorial moment of the PSBQL distribution as

$$\mu'_{(r)} = \frac{\Gamma(r+2)}{(\alpha+2)\theta^r} [(\alpha+r+2)]; \quad r = 1, 2, 3, \dots, \theta > 0, \alpha > -2 \quad (3.10)$$

The factorial moments of the PSBQL distribution is obtained very easily by using (3.10) as follows

$$\mu'_{(1)} = \frac{2!(\alpha+2)}{\theta(\alpha+2)} \quad (\text{Mean}), \mu'_{(2)} = \frac{3!(\alpha+4)}{\theta^2(\alpha+2)}, \mu'_{(3)} = \frac{4!(\alpha+5)}{\theta^3(\alpha+2)}, \mu'_{(4)} = \frac{5!(\alpha+6)}{\theta^4(\alpha+2)} \quad \text{etc.}$$

Raw and Central moments

The r^{th} order moment about the origin (raw moment) of the PSBQL distribution can be obtained as

$$\mu'_r = E[E(X^r|\lambda)] \quad (3.11)$$

From equation (2.2), we get

$$\mu'_r = \int_0^\infty \left[\sum_{x=0}^\infty x^r \frac{e^{-\lambda} \lambda^x}{\Gamma(1+x)} \right] \frac{\theta^2}{(\alpha+2)} \lambda(\alpha+\theta\lambda) e^{-\theta\lambda} d\lambda \quad (3.12)$$

Obviously the expression under bracket is the r^{th} order moment about origin of the Poisson distribution. Taking $r = 1$ in (3.12) and then using the mean of the Poisson distribution, the mean of the PSBQL distribution is obtained as

$$\mu'_1 = \frac{\theta^2}{(\alpha+2)} \int_0^\infty \lambda^2(\alpha+\theta\lambda) e^{-\theta\lambda} d\lambda = \frac{\theta^2}{(\alpha+2)} \left[\frac{\Gamma_3(\alpha+3)}{\theta^3} \right] = \frac{2(\alpha+3)}{\theta(\alpha+2)} \quad (3.13)$$

Taking $r = 2$ in (3.12) and using the second moment about origin of the Poisson distribution, the second moment about origin of the PSBQL distribution becomes

$$\mu'_2 = \frac{\theta^2}{(\alpha+2)} \int_0^\infty (\lambda^2 + \lambda)\lambda(\alpha+\theta\lambda) e^{-\theta\lambda} d\lambda = \frac{\theta^2 \Gamma_3}{(\alpha+2)} \left[\frac{3(\alpha+4)}{\theta^4} + \frac{(\alpha+3)}{\theta^3} \right] = \frac{2\theta(\alpha+3)+6(\alpha+4)}{\theta^2(\alpha+2)} \quad (3.14)$$

Substituting $r = 3$ and $r = 4$ in (3.12) and using respective moments of the Poisson distribution we obtain the third and the fourth moments of the discrete PSBQL distribution as

$$\mu'_3 = \frac{2\theta^2(\alpha+3)+18\theta(\alpha+4)+24(\alpha+5)}{\theta^3(\alpha+2)} \quad (3.15)$$

$$\mu'_4 = \frac{2\theta^3(\alpha+3)+42\theta^2(\alpha+4)+144\theta(\alpha+5)+120(\alpha+6)}{\theta^4(\alpha+2)} \quad (3.16)$$

It can be seen that, at $\alpha = \theta$ these moments reduces to the respective moments of the Poisson size-biased Lindley distribution. The moments about origin of the PSBQL distribution can be also obtained by using the relationship between factorial moments and moments about origin of the distribution.

The first three central moments of the PSBQL distribution may be obtained as

$$\mu_2 = \frac{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)]}{\theta^2(\alpha+2)^2} \quad (\text{Variance}) \quad (3.17)$$

$$\mu_3 = \frac{2[\theta^2(\alpha+3)(\alpha+2)^2+3\theta(\alpha^3+8\alpha^2+18\alpha+12)+2(\alpha^3+9\alpha^2+18\alpha+12)]}{\theta^3(\alpha+2)^3} \quad (3.18)$$

$$\mu_4 = \frac{2[\theta^3(\alpha+3)(\alpha+2)^3+2\theta^2(9\alpha^4+65\alpha^3+230\alpha^2+348\alpha+192)+24\theta(\alpha^4+11\alpha^3+39\alpha^2+42\alpha+30)-3(19\alpha^4+228\alpha^3+119\alpha^2+2106\alpha+1536)]}{\theta^4(\alpha+2)^4} \quad (3.19)$$

Coefficient of Skewness and Kurtosis

A fundamental task in many statistical analyses is to characterize the location and variability of a distribution. Further characterizations of the distribution include skewness and kurtosis. The skewness $\sqrt{\beta_1}$ and kurtosis β_2 coefficients can be derived as

$$\sqrt{\beta_1} = \frac{[\theta^2(\alpha+3)(\alpha+2)^2+3\theta(\alpha^3+8\alpha^2+18\alpha+12)+2(\alpha^3+9\alpha^2+18\alpha+12)]}{\sqrt{2[\theta(\alpha+2)(\alpha+3)+(\alpha^2+6\alpha+6)]^3}}$$

and
$$\beta_2 = \frac{[\theta^3(\alpha+3)(\alpha+2)^3 + 2\theta^2(9\alpha^4 + 65\alpha^3 + 230\alpha^2 + 348\alpha + 192) + 24\theta(\alpha^4 + 11\alpha^3 + 39\alpha^2 + 42\alpha + 30) - 3(19\alpha^4 + 228\alpha^3 + 119\alpha^2 + 2106\alpha + 1536)]}{2[\theta(\alpha+2)(\alpha+3) + (\alpha^2 + 6\alpha + 6)]^2}$$

Both $\sqrt{\beta_1}$ and β_2 are an increasing function in θ when α is fixed.

Coefficient of Variation

The coefficient of variation (CV) is the ratio of the standard deviation to the mean. The higher is the coefficient of variation, the greater the level of dispersion around the mean. The coefficient of variation (CV) of the PSBQL distribution is given by

$$CV = \frac{\sqrt{2[\theta(\alpha+2)(\alpha+3) + (\alpha^2 + 6\alpha + 6)]}}{(\alpha+3)\sqrt{2}} \tag{3.20}$$

The coefficient of variation (CV) of the PSBQL distribution is increased, when θ is increased and the other parameter α is fixed.

Index of dispersion

The index of dispersion for the PSBQL distribution is obtained as

$$\begin{aligned} \gamma &= \frac{\sigma^2}{\mu} = \frac{\theta(\alpha+2)(\alpha+3) + (\alpha^2 + 6\alpha + 6)}{\theta(\alpha+2)(\alpha+3)} \\ &= 1 + \frac{\alpha^2 + 6\alpha + 6}{\theta(\alpha+2)(\alpha+3)} \end{aligned} \tag{3.21}$$

As the values of α increases, the coefficient of dispersion decreases slowly whereas the coefficient of dispersion increases to a certain point then it decreases as θ increases.

IV. Figure 1

The plots of PSBQL distribution for different values of parameters θ and α are shown in Figure 1.

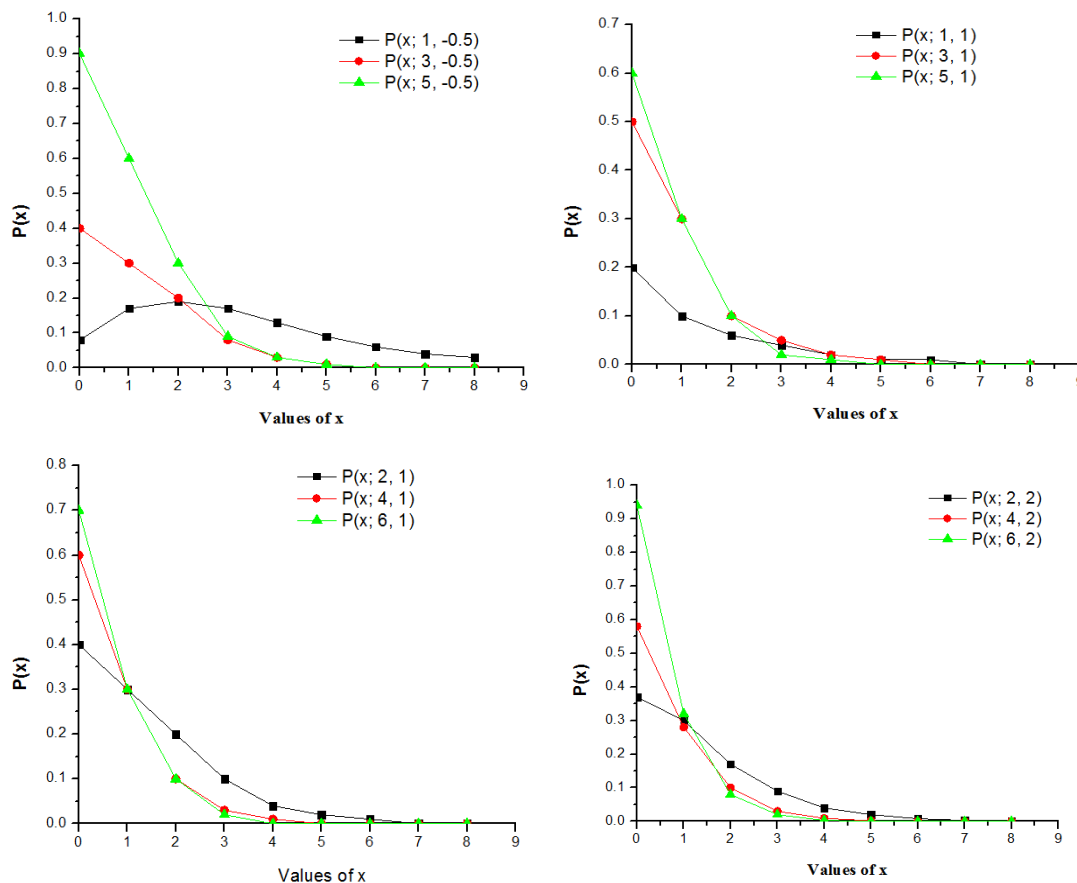


Fig 1: Plots of SBQPL distribution for different values of parameters θ and α .

V. Parameter Estimation

One of the most important properties of a distribution is the problem of estimation of the parameter. One of the most important and oldest methods of estimation is the method of moments. To estimate the parameters of PSBQL distribution, we have been used method of moment estimation procedure as follows:

Estimation based on the method of moments

The PSBQL distribution has two parameters, viz. α and θ . The first two moments are required to get the estimates of these parameters by the method of moments. From equations (3.13) and (3.14), we have

$$\frac{\mu'_2 - \mu'_1}{(\mu'_1)^2} = \frac{6(\alpha+4)(\alpha+2)}{4(\alpha+3)^2} = K \text{ (say)} \tag{5.1}$$

which gives a quadratic equation in α as

$$g(\alpha) = (3 - 2K)\alpha^2 + (18 - 12K)\alpha + (24 - 18K) = 0 \tag{5.2}$$

Now, replacing the first two population moments by the respective sample moments in (5.1) and estimate k of K can be obtained and using it in (5.2), an estimate $\hat{\alpha}$ of α can be obtained.

Again, substituting the value of $\hat{\alpha}$ in (3.13) and replacing the population mean by the sample mean \bar{x} , an estimate of θ is obtained as

$$\hat{\theta} = \frac{2(\hat{\alpha}+3)}{\bar{x}(\hat{\alpha}+2)}; \bar{x} \text{ be the sample mean.} \tag{5.3}$$

VI. Conclusion

Size-biased distributions arise in practice when observations from a sample are recorded with unequal probabilities having probability proportional to some measure of unit size. In this paper, Poisson size-biased Quasi Lindley distribution is obtained by compounding the Poisson distribution with size-biased Quasi Lindley distribution without considering its size-biased form. We have also been studied some distributional properties of the PSBQL distribution including problem of parameter estimation by method of moments in this paper.

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