A Simple Deterministic Susceptible – Exposed – Infected Epidemic Model

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Abstract: *We have extended the susceptible – infected – removed epidemic model to susceptible – exposed – infected – removed epidemic model by dividing into two cases. One is the number of susceptibles decreases and the number of exposed persons increases each by one and another is the number of exposed persons decreases and the number of infected persons increases each by one.*
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I. Introduction

The occurrence of more cases of a disease than would be expected in a community or region during a given time period is called an epidemic. An epidemic is the rapid spread of infectious disease to a large number of people in a given population within a short period of time, usually two weeks or less. An epidemic may be restricted to one location; however, if it spreads to other countries or continents and affects a substantial number of people, it may be termed a pandemic [12].

Epidemiology is one of the most important fields of research. The modeling and analysis of infectious diseases have been done by many authors. The classical theoretical papers on epidemic models by Kermack and Mckendrick [7, 8, 9] have had a major influence in the development of mathematical models. The modeling literature is now extensive and growing very quickly. Bailey [1] gave a good introduction to the variety of problems and models for the spread and control of infectious disease. The article by Hethcote [5] reviewed three basic epidemiological models. Diekmann and Heesterbeek [3] presented a concentrated review to this field and they discussed how to use biological assumptions in constructing models and present applications.

In mathematics and physics, a deterministic system is a system in which no randomness is involved in the development of future states of the system [13]. A deterministic model will thus always produce the same output from a given starting condition or initial state [14]. For example, physical laws that are described by differential equations represent deterministic systems, even though the state of the system at a given point in time may be difficult to describe explicitly. Markov chains and other random walks are not deterministic systems, because their development depends on random choices.

Most of the mathematical models are described by a system of nonlinear differential equations. It is well known that, the nonlinear systems consisting of at least three autonomous differential equations of first order can have several kinds of behavior that are not possible in linear systems including periodic, quasi periodic and even chaos [11]. However, the solution of a nonlinear system consisting of two autonomous differential equations of the first order converges either to a point or to a closed curve [4].

Basic Concepts

The term epidemiology is the study of health-event, health-characteristic, or health-determinant patterns in a society. Now before we go further in this study, here the most commonly used basic concepts are introduced.

Definition 1.1 [2]

An epidemic is an unusually large short – term outbreak of a disease, such as Cholera and Plague etc.
The spread of disease depends on:
- The mode of transmission.
- Susceptibility.
- Infections period.
- Resistance.

And many other factors.

Definition 1.2 [10]

Any group of individuals, usually of a single species, occupying a given area at the same time is known as population.

The population in an epidemiological model can be classified into three types : Susceptible, Infected and Removal which are denoted by $S(t)$, $I(t)$ and $R(t)$ respectively, where $S(t)$ represent the number of individuals in the population who can be infected, while $I(t)$ represent the number of infected individuals in the population, that is, those with the disease who are actively transmitting it, $R(t)$ denotes the number of individuals removed from the population by recovery, death, immunization or other means **[6]**. Consequently, in order to formulate an epidemiological model mathematically the following assumptions can be adopted:

(a) The disease transmitted by contact between the infected individuals and susceptible individuals.

(b) There is no latent period for the disease; hence the disease is transmitted instantaneously on contact.

(c) All susceptible individuals are equally susceptible and all infected individuals are equally infections.

In this paper, we extend the simple susceptible – infected – removed epidemic model to susceptible – exposed – infected – removed epidemic model by including "Exposed" state. In this extended model, an initially susceptible individual who succumbs to the disease is considered as being "exposed to the disease". In this state, the exposed individual is incubating the disease and is not capable of spreading the disease. After the incubation period, the individual moves on to the "infectious" state.

In this susceptible – exposed – infected – removed epidemic model it is assumed that the population is fixed in size that is there is no births or immigration occur and all deaths are taken into account. Hence, if $N(t)$ is the total population size, then

$$S(t) + E(t) + I(t) + R(t) = N = \text{constant,} \tag{1}$$

where $E(t)$ is the number of exposed persons in the population.

II.   A Simple Deterministic Susceptible – Exposed – Infected – Removed Epidemic Model without Removal

In this section, we consider simple epidemic models, i.e., models in which there are no removals.

Let m be the initial number of susceptibles and n be the initial number of exposed persons in the population in which one infected person has been introduced so that

$$S(t) + E(t) + I(t) = m + n + 1, \tag{2}$$

$$S(0) = S_0 = m; \qquad E(0) = E_0 = n; \qquad I(0) = I_0 = 1. \tag{3}$$

Now, due to infection, there are two cases:

(i) The number of susceptibles decreases and the number of exposed persons increases.

(ii) The number of exposed persons decreases and the number of infected persons increases.

Case (i)

In this case, there is no change in the number of infected persons, that is $I(t)$ is a constant and also we assume that the rate of decrease of $S(t)$, or the rate of increase of $E(t)$, is proportional to the product of the number of susceptibles and the number of exposed so that our model gives

$$\frac{dS}{dt} = -\beta SE,$$

$$\frac{dE}{dt} = \beta SE. \tag{4}$$

From **(2)** and **(4)**,

$$\frac{dS}{dt} = -\beta S \left[(m+n+1) - S - I \right]$$

$$\frac{dS}{dt} = \beta S \left[S - (m+n+1-I) \right]$$

$$\frac{dS}{S \left[S - (m+n+1-I) \right]} = \beta dt \tag{5}$$

By using partial fraction method in L.H.S. of **(5)**, we get

$$\frac{1}{S \left[S - (m+n+1-I) \right]} = \frac{1}{(m+n+1-I)} \left[\frac{1}{\left[S - (m+n+1-I) \right]} - \frac{1}{S} \right] \tag{6}$$

By using (**6**) in (**5**), we get

$$\frac{1}{\left(m+n+1-I\right)}\left[\frac{1}{\left[S-\left(m+n+1-I\right)\right]}-\frac{1}{S}\right]dS=\beta dt$$

Integrating on both sides, we get

$$\frac{1}{\left(m+n+1-I\right)}\left[\log\left[S-\left(m+n+1-I\right)\right]-\log S\right]=\beta t+\log C$$

where *C* is an integral constant.

$$\frac{1}{\left(m+n+1-I\right)}\left[\log\left[\frac{S-\left(m+n+1-I\right)}{S}\right]\right]=\beta t+\log C$$

$$\log\left[\frac{S-\left(m+n+1-I\right)}{S}\right]=\left(m+n+1-I\right)\beta t+\log C\ , \qquad\qquad \text{(Since, }C\text{ is a constant)}$$

$$\log\left[\frac{S-\left(m+n+1-I\right)}{CS}\right]=\left(m+n+1-I\right)\beta t \tag{7}$$

Taking exponential on both sides, we get

$$\frac{S-\left(m+n+1-I\right)}{CS}=\exp\left\{\left(m+n+1-I\right)\beta t\right\}$$

$$1-\frac{\left(m+n+1-I\right)}{S}=C\ \exp\left\{\left(m+n+1-I\right)\beta t\right\}$$

That is, $\ S=\dfrac{\left(m+n+1-I\right)}{1-C\ \exp\left\{\left(m+n+1-I\right)\beta t\right\}}$ $\tag{8}$

Let *t* = 0 in (**8**), we get

$$S_0=\frac{\left(m+n+1-I_0\right)}{1-C}$$

That is, $\qquad m=\dfrac{\left(m+n+1-1\right)}{1-C}$ $\qquad\qquad$ (Using (**3**))

$$1-C=\frac{\left(m+n\right)}{m}$$

$$C=1-\frac{\left(m+n\right)}{m}$$

That is, $\qquad C=-\dfrac{n}{m}$ $\tag{9}$

By using (**9**) in (**8**), we get

$$S=\frac{\left(m+n+1-I\right)}{1+\dfrac{n}{m}\exp\left\{\left(m+n+1-I\right)\beta t\right\}}$$

$$=\frac{\left(m+n+1-I\right)}{\dfrac{1}{m}\left\{m+n\exp\left\{\left(m+n+1-I\right)\beta t\right\}\right\}}$$

Therefore, $\ S\left(t\right)=\dfrac{m\left(m+n+1-I\right)}{m+n\exp\left\{\left(m+n+1-I\right)\beta t\right\}}$ $\tag{10}$

In equation (**2**),

$$E(t) = (m+n+1) - I(t) - S(t)$$

$$= (m+n+1-I) - \frac{m(m+n+1-I)}{m+n\exp\{(m+n+1-I)\beta t\}} \quad \text{(Using (10))}$$

$$= \frac{(m+n+1-I)\left[m+n\exp\{(m+n+1-I)\beta t\}\right] - m(m+n+1-I)}{m+n\exp\{(m+n+1-I)\beta t\}}$$

$$= \frac{m(m+n+1-I) + n(m+n+1-I)\exp\{(m+n+1-I)\beta t\} - m(m+n+1-I)}{m+n\exp\{(m+n+1-I)\beta t\}}$$

Therefore, $E(t) = \dfrac{n(m+n+1-I)\exp\{(m+n+1-I)\beta t\}}{m+n\exp\{(m+n+1-I)\beta t\}}$ 　　　　(**11**)

So that,

$$\lim_{t\to\infty} S(t) = \left[\frac{m(m+n+1-I)}{m+n\exp\{(m+n+1-I)\beta t\}}\right]_{t\to\infty} \quad \text{(Using (10))}$$

$$= \frac{m(m+n+1-I)}{m+(\infty)}$$

i.e., $\quad \lim_{t\to\infty} S(t) = 0.$ 　　　　　　　　　　(**12**)

And $\quad \lim_{t\to\infty} E(t) = \left[\dfrac{n(m+n+1-I)\exp\{(m+n+1-I)\beta t\}}{m+n\exp\{(m+n+1-I)\beta t\}}\right]_{t\to\infty}$ 　　(Using (**11**))

$$= \left[\frac{n(m+n+1-I)\exp\{(m+n+1-I)\beta t\}}{\exp\{(m+n+1-I)\beta t\}\left(\dfrac{m}{\exp\{(m+n+1-I)\beta t\}}+n\right)}\right]_{t\to\infty}$$

$$= \frac{n(m+n+1-I)}{(0+n)}$$

i.e., $\quad \lim_{t\to\infty} E(t) = m+n+1-I.$ 　　　　　　　(**13**)
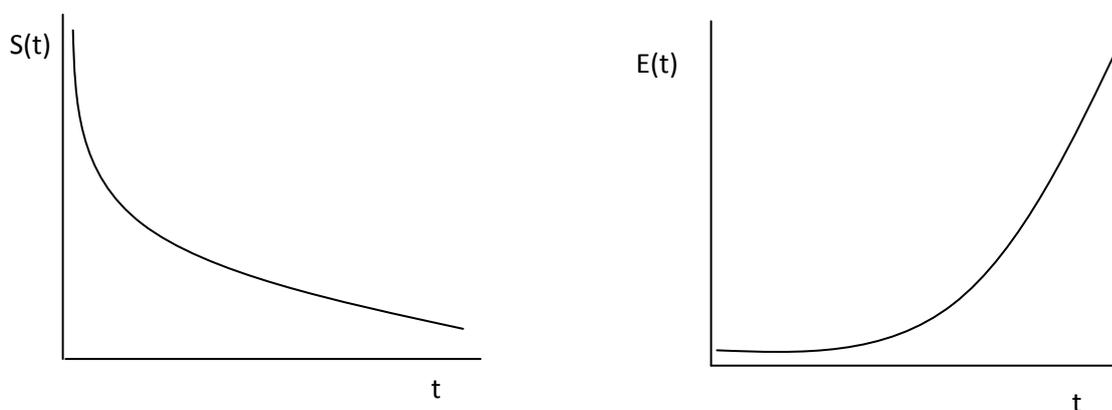
Figure – 1: Decreasing susceptibles and increasing exposed with respect to time

In practice, public records do not show the number of exposed persons; rather they show that the number of new cases reported every day. The corresponding curve gives a relation between $\dfrac{dS}{dt}$ and *t*. This curve is known as the epidemic curve, and the equation for this curve is given by

$$\frac{dS}{dt} = -\beta SE$$

$$= -\beta \left(\frac{m(m+n+1-I)}{m+n\exp\{(m+n+1-I)\beta t\}} \right) \left(\frac{n(m+n+1-I)\exp\{(m+n+1-I)\beta t\}}{m+n\exp\{(m+n+1-I)\beta t\}} \right)$$

$$\frac{dS}{dt} = -\beta \left(\frac{mn(m+n+1-I)^2 \exp\{(m+n+1-I)\beta t\}}{\left(m+n\exp\{(m+n+1-I)\beta t\}\right)^2} \right) \tag{14}$$

which is a symmetrical unimodel curve with a maximum at

$$t_0 = \frac{\log m}{\beta(m+n)}. \tag{15}$$

This shows that the rate of appearance of new cases rises rapidly to maximum at a time, depending on β, *m* and *n*, and then falls to zero.

Case (ii)

In this case, the number of exposed persons decreases, the number of infected persons increases and there is no change in the number of susceptible. That is, *S(t)* is a constant.

We assume that the rate of decrease of *E(t)*, or the rate of increase of *I(t)*, is proportional to the product of the number of exposed persons and the number of infected so that our model gives

$$\frac{dE}{dt} = -\gamma EI,$$

$$\frac{dE}{dt} = \gamma EI. \tag{16}$$

From **(2)**,

$$\frac{dE}{dt} = -\gamma E\left[(m+n+1) - S - E\right]$$

$$\frac{dE}{dt} = \gamma E\left[E - (m+n+1-S)\right]$$

$$\frac{dE}{E\left[E - (m+n+1-S)\right]} = \gamma dt \tag{17}$$

By using partial fraction method in L.H.S. of **(17)**, we get

$$\frac{1}{E\left[E - (m+n+1-S)\right]} = \frac{1}{(m+n+1-S)}\left[\frac{1}{\left[E - (m+n+1-S)\right]} - \frac{1}{E}\right] \tag{18}$$

From **(17)** and **(18)**, we get

$$\frac{1}{(m+n+1-S)}\left[\frac{1}{\left[E - (m+n+1-S)\right]} - \frac{1}{E}\right]dE = \gamma dt$$

Integrating on both sides, we get

$$\frac{1}{(m+n+1-S)}\left[\log\left[E - (m+n+1-S)\right] - \log E\right] = \gamma t + \log C$$

where *C* is an integral constant.

$$\log\left[\frac{E - (m+n+1-S)}{E}\right] = (m+n+1-S)\gamma t + \log C, \qquad \text{(Since, } C \text{ is a constant)}$$

$$\log\left[\frac{E-(m+n+1-S)}{CE}\right]=(m+n+1-S)\gamma t$$

Taking exponential on both sides, we get

$$\frac{E-(m+n+1-S)}{CE}=\exp\{(m+n+1-S)\gamma t\}$$

$$1-\frac{(m+n+1-S)}{E}=C\exp\{(m+n+1-S)\gamma t\}$$

$$1-C\exp\{(m+n+1-S)\gamma t\}=\frac{(m+n+1-S)}{E}$$

That is, $\quad E=\dfrac{(m+n+1-S)}{1-C\exp\{(m+n+1-S)\gamma t\}}$ $\qquad\qquad$ **(19)**

For $t=0$,

$$E_0=\frac{(m+n+1-S_0)}{1-C}$$

That is, $\qquad n=\dfrac{(m+n+1-m)}{1-C}$ $\qquad\qquad$ (Using **(3)**)

$$1-C=\frac{(n+1)}{n}$$

$$C=1-\frac{(n+1)}{n}$$

That is, $\qquad C=-\dfrac{1}{n}$ $\qquad\qquad$ **(20)**

By using **(20)** in **(19)**, we get

$$E=\frac{(m+n+1-S)}{1+\dfrac{1}{n}\exp\{(m+n+1-S)\gamma t\}}$$

$$=\frac{(m+n+1-S)}{\dfrac{1}{n}\{n+\exp\{(m+n+1-S)\gamma t\}\}}$$

Therefore, $\quad E(t)=\dfrac{n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}$ $\qquad\qquad$ **(21)**

In equation **(2)**,

$$I(t)=(m+n+1)-S(t)-E(t)$$

$$=(m+n+1-S)-\frac{n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}\qquad\text{(Using \textbf{(21)})}$$

$$=\frac{(m+n+1-S)\left[n+\exp\{(m+n+1-S)\gamma t\}\right]-n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}$$

$$=\frac{n(m+n+1-S)+(m+n+1-S)\exp\{(m+n+1-S)\gamma t\}-n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}$$

Therefore, $I(t) = \dfrac{(m+n+1-S)\exp\{(m+n+1-S)\gamma t\}}{n+\exp\{(m+n+1-S)\gamma t\}}$ (22)

So that,

$$\lim_{t\to\infty} E(t) = \left[\frac{n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}\right]_{t\to\infty}$$ (Using (21))

$$= \frac{n(m+n+1-S)}{n+(\infty)}$$

i.e., $\lim_{t\to\infty} E(t) = 0.$ (23)

And $\lim_{t\to\infty} I(t) = \left[\dfrac{(m+n+1-S)\exp\{(m+n+1-S)\gamma t\}}{n+\exp\{(m+n+1-S)\lambda t\}}\right]_{t\to\infty}$ (Using (22))

$$= \left[\frac{(m+n+1-S)\exp\{(m+n+1-S)\gamma t\}}{\exp\{(m+n+1-S)\gamma t\}\left(\dfrac{n}{\exp\{(m+n+1-S)\gamma t\}}+1\right)}\right]_{t\to\infty}$$

$$= \frac{(m+n+1-S)}{(0+1)}$$

i.e., $\lim_{t\to\infty} I(t) = m+n+1-S.$ (24)
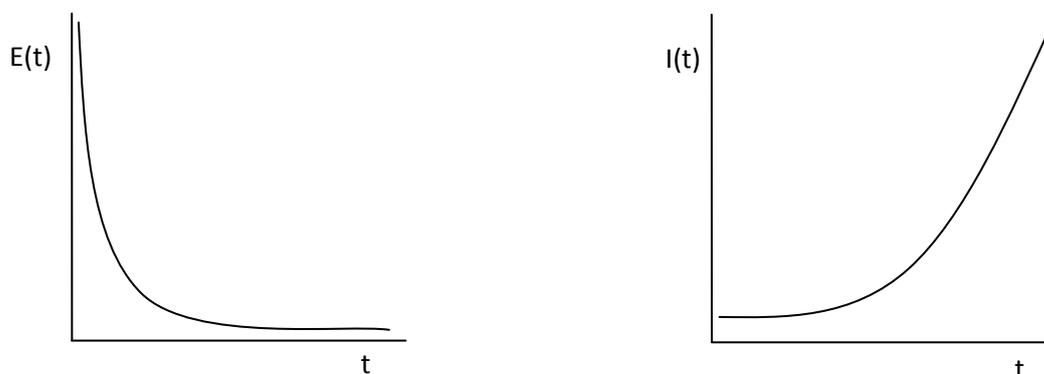
Figure – 2: Decreasing exposed and increasing infectives with respect to time

Now,

$$\frac{dE}{dt} = -\gamma EI$$

$$= -\gamma\left(\frac{n(m+n+1-S)}{n+\exp\{(m+n+1-S)\gamma t\}}\right)\left(\frac{(m+n+1-S)\exp\{(m+n+1-S)\gamma t\}}{n+\exp\{(m+n+1-S)\gamma t\}}\right)$$

$$\frac{dE}{dt} = -\gamma\left(\frac{n(m+n+1-S)^2\exp\{(m+n+1-S)\gamma t\}}{\left(n+\exp\{(m+n+1-S)\gamma t\}\right)^2}\right)$$ (25)

In **(25)**, gives a relation between $\dfrac{dE}{dt}$ and *t*. Also, it is a symmetrical unimodel curve with a maximum at

$$t_0 = \frac{\log n}{\gamma(n+1)}. \tag{26}$$

In this case, the rate of appearance of new cases rises rapidly to maximum at a time, depending on γ and *n*, and then falls to zero.

III.   Conclusion

Here we have discussed the susceptible – exposed – infected – removed epidemic model without removal and two cases are studied with *S(t) = m-1, E(t) = n+1* and *I(t) = 1*; *S(t) = m, E(t) = n-1* and *I(t) = 2*. This model can easily be used for deterministic epidemics.

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