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Orthogonal Reverse Derivations on Semiprime Semiring

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Abstract:*Motivated by some results on Semiprime Gamma Rings with Orthogonal Reverse Derivations, in [4], the authors defined the notion ofReverseDerivations on Gamma Ringsand investigated some results on the Reverse Derivations in Gamma Rings. In this paper, we also introduce the notion of Orthogonal Reverse Derivations of SemiprimeSemirings and derived some interesting results.*

keywords: *Semirings, Reverse Derivations, Orthogonal Reverse derivations.*

I. Introduction

This paper has been inspired by the work of Kalyan Kumar Dey, Akhil ChandraPaul,IsamiddinS.Rakhimov [4]. Bresar and Vukman [1] initiated the notion of orthogonality for two derivations on a semiprime ring, and they obtained several necessary and sufficient conditions for two derivations to be orthogonal. They also obtained a counter part of a result of Posner from [5]. In this paper, we introduce the notion of orthogonality of two reverse derivations on semiprimesemiring and we presented several necessary and sufficient conditions for two derivations to be orthogonal.

II. Preliminaries

Definition: 2.1

A **semiring** (S,+,•) is a non-empty set S together with two binary operations, + and $\cdot$ such that
(1). (S,+) and (S,•) are a Semigroup.
(2). For all a, b, c ϵ S, a. (b + c) = a.b + a .c and (b + c) . a = b. a + c .a

Definition: 2.2

A semiring S is said to be **2- torsionfree** if $2x = 0 \Rightarrow x = 0, \forall x \in S$.

Definition: 2.3

A semiring S is **prime** if $xSy = 0 \Rightarrow x = 0$ or $y = 0$, $\forall x, y \in S$ and S is **semiprime**if $xSx = 0 \Rightarrow x = 0, \forall\ x \in S$.

Definition: 2.4

An additive map $d : S \to S$ is called a **derivation** if $d(xy) = d(x)\, y + x\, d(y), \forall\ x, y \in S$

Definition: 2.5

Let d, g be two additive maps from S to S. They are said to be **orthogonal** if
$d(x)Sg(y) = 0 = g(y)Sd(x), \forall\ x, y \in S$.

We write $[x, y] = x\, y - y\, x$ and note that important identity $[xy, z] = x\, [y, z] + [x, z]\, y$ and
$[x, yz] = y\, [x, z] + [x, y]\, z$

III. Reverse Derivations and Orthogonal Reverse Derivations

Definition: 3.1

In a semiring S, if d is an additive mapping from S into itself satisfying $d(xy) = d(y)\, x + y\, d(x)$, $\forall x, y \in S$, then d is called a **reverse derivation** on S.

Example: 3.2

Let S $=\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} / a, b \in Z^{+} \cup \{0\}\right\}$ be a semiring. Then $d : S \to S$ defined by $d\left[\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right] = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ is a reverse derivation.

Definition: 3.3
An additive mapping d : S → S is called a Jordan derivation, if d(a a) = d(a) a + a d(a), $\forall\ a \in S$.

Definition: 3.4
Let d and g be two reverse derivations on S. If d(x) S g(y) = 0 = g(y) S d(x), $\forall$x, y $\in$S, then d and g are orthogonal to S.

Note:
- If S is Commutative, then both derivation and reverse derivation are the same.
- In general reverse derivation is not a derivation but it is a Jordan derivation.
- A non-zero reverse derivation cannot be orthogonal on itself.

Lemma: 3.5
Let S be a 2-torsionfree semiprimesemiring and a,b$\in$S. Then the following conditions are equivalent
(i) axb = 0, $\forall x \in$S
(ii) bxa = 0, $\forall x \in$S
(iii) axb + bxa = 0, $\forall x \in$S. If one of the conditions is fulfilled, then ab = ba = 0.

Lemma: 3.6
Let S be a semiprimesemiring and suppose that additive mappings d and g on S into itself satisfy d(x) S g(x) = 0, $\forall x \in$S. Then d(x) S g(y) = 0, $\forall x \in$S.

Theorem: 3.7
Let S be a 2-torsionfree semiprimesemiring. Let d and g be reverse derivations on S. Then d(x) g(y) + g(x) d(y) = 0, $\forall x, y \in$S iff d and g are orthogonal.

Proof:
Suppose that d(x) g(y) + g(x) d(y) = 0, $\forall x, y \in$S
Put y = xy, d(x) g(xy) + g(x) d(xy) = 0, $\forall x, y \in$S
⇒d(x) [g(y) x + y g(x)] + g(x) [d(y) x + y d(x)] = 0, $\forall x, y \in$S
⇒d(x) g(y) x + d(x) y g(x) + g(x) d(y)x + g(x) y d(x) = 0, $\forall x, y \in$S
⇒ [d(x) g(y) + g(x) d(y)] x + d(x) y g(x) + g(x) y d(x) = 0, $\forall x, y \in$S
⇒d(x) y g(x) + g(x) y d(x) = 0, $\forall x, y \in$S
By lemma 3.5, d(x) y g(x) = 0, $\forall x, y \in$S
By lemma 3.6, d(x) y g(z) = 0, $\forall x, y, z \in$S
By lemma 3.5, g(z) y d(x) = 0, $\forall x, y, z \in$S
Thus d(x) y g(z) = 0 = g(z) y d(x), $\forall x, y, z \in$S
∴d(x) S g(z) = 0 = g(z) S d(x), $\forall x, z \in$S
∴d and g are Orthogonal.
Conversely, if d and g are Orthogonal.
∴d(x)S g(y) = 0 = g(y) S d(x), $\forall x, y \in$S
ie, d(x) s g(y) = 0 = g(y) s d(x), $\forall x, y, s \in$S
By lemma 3.5, d(x) g(y) = 0 = g(x) d(y), $\forall x, y \in$S
∴d(x) g(y) + g(x) d(y) = 0, $\forall x, y \in$S

Remark:
Let d and g be reverse derivations on a semiring. Then the following results are hold good
1. dg(xy) = d(x) g(y) + x dg(y) + dg(x) y + g(x) d(y)
2. gd(xy) = g(x) d(y) + x gd(y) + gd(x) y + d(x) g(y)

Theorem: 3.8
Let S be a 2-torsion free semiprimesemiring and d and g be reverse derivations on S. Then d and g are orthogonal iff d(x) g(x) = 0, $\forall x \in$S

Proof:
(i)⟺ (ii) Assume that d and g are orthogonal
∴d(x) s g(y) = 0 = g(y) s d(x), $\forall x, y, s \in$S
Now consider d(x) s g(y)=0, $\forall x, y, s \in$S
Using lemma 3.6, we get d(x) s g(x)=0

Using lemma 3.5, d(x) g(x)=0, $\forall x \in S$
Conversely,assume that d(x) g(x) = 0, $\forall x \in S$.
The linearization of d(x + y) g(x + y) = 0 gives d(x) g(y) + d(y) g(x) = 0, $\forall x, y \in S$ (1)
Take y = yz in (1), d(x) g(yz) + d(yz) g(x) = 0, $\forall x, y, z \in S$
$\Rightarrow$d(x) [g(z) y + z g(y)] + [d(z) y + z d(y)] g(x) = 0, $\forall x, y, z \in S$
$\Rightarrow$d(x) g(z) y + d(x) z g(y) + d(z) y g(x) + z d(y) g(x) = 0, $\forall x, y, z \in S$ (2)
Since d(x) g(z) = $-d(z)g(x)$ and d(y) g(x) = $-d(x)g(y)$
(2)$\Rightarrow -d(z)g(x)y + d(x)zg(y) + d(z)yg(x) - zd(x)g(y) = 0$
$\Rightarrow d(z)[yg(x) - g(x)y] + [d(x)z - zd(x)]g(y) = 0$
$\Rightarrow d(z)[y, g(x)] + [d(x), z]g(y) = 0$.Replacing z by d(x).
d(d(x)) [y,g(x)] + [d(x),d(x)] g(y) = 0$\Rightarrow d^2$(x) [y,g(x)] = 0 (3)
Let y = yw, $\forall y, w \in S$ in the above equation d^2(x) [yw,g(x)] = 0
d^2(x) y [w,g(x)] + d^2(x) [y,g(x)] w = 0 $\Rightarrow d^2$(x) y [w,g(x)] = 0, $\forall x, y, w \in S$ [$\because$ by (3)]
Using lemma 3.6, d^2(x) y [w,g(y)] = 0, $\forall x, y, w \in S$ (4)
Replacing x by xu and using the remark, 0 = d^2(xu) y [w,g(y)] , $\forall x, y, u \in S$
= [d^2(x) u + 2 d(x)d(u) + xd^2(u)] y [w,g(y)]
= 2 d(x)d(u) y [w, g(y)] [$\because$ by (4)]
Since S is 2-torsionfree, d(x) d(u) y [w,g(y)] = 0, $\forall x, y, u \in S$ (5)
Take x = xz, d(xz) d(u) y [w, g(y)] = 0
$\Rightarrow$ d(z) x d(u) y [w,g(y)] + z d(x) d(u)y [w, g(y)] $\Rightarrow$ d(z) x d(u) y [w,g(y)] = 0
Inparticular, d(z) x d(x) y [w, g(y)]=0
Take d(z) = d(x) y [w,g(y)], then d(x)y[w,g(y)] x d(x) y [w,g(y)] = 0
Since S is semiprime , d(x) y [w,g(y)] = 0
d(x) g(y) = g(y) d(x), $\forall x, y \in S$. $\therefore$ (1) $\Rightarrow$ g(y) d(x) + d(y) g(x) = 0
Using theorem 3.7 we get d and g are orthogonal.

Theorem: 3.9

Let S be a 2-torsionfree semiprimesemiring and d and g be reverse derivations on S. Then the following conditions are equivalent

(i) d and g are orthogonal
(ii) d(x) g(x) = 0, $\forall x \in S$
(iii) g(x) d(x) = 0, $\forall x \in S$
(iv) d(x) g(x) + g(x) d(x) = 0, $\forall x \in S$

Proof:

(i)$\Longleftrightarrow$ (ii)The proof is immediate from the previous theorem 3.8
The proof of (i) $\Longleftrightarrow$(iii)is the similar proof to that of (i) $\Longleftrightarrow$(ii)
(i)$\Longleftrightarrow$ (iv) The proof is immediate from the theorem 3.7

Theorem: 3.10

Let S be a 2 – torsion free semiprimesemiring. Let d and g be reverse derivations on S. Then the following conditions are equivalent

(i) d and g are Orthogonal
(ii) dg = 0
(iii) gd = 0
(iv) dg + gd = 0
(v) dg is a derivation
(vi) gd is a derivation

Proof:

(i)$\Longleftrightarrow$(ii)Suppose d and g are Orthogonal
d(x) S g(y) = 0 = g(y) S d(x) ,$\forall x, y \in S \Rightarrow$d(x) s g(y) = 0 =g(y) s d(x), $\forall x, y, s \in S$
Now , d(x) s g(y) =0
d[d(x) s g(y)] = d(0)
i.e, d[g(y)] s d(x) + g(y) d(s) d(x) +s g(y) d [d(x)] = 0
Since d and g are orthogonal, dg(y) s d(x) = 0
Replacing x by g(y), dg(y) s dg(y) = 0
Since S is semiprime, dg (y) =0, $\forall y \in S \therefore$ dg = 0

Conversely,Suppose dg = 0
Now by remark dg (xy) = d (x) g (y) + x dg (y) + dg(x) y + g (x) d (y), $\forall x, y, s \in S$ (1)
.Since dg = 0, (1)$\Rightarrow$ 0 = d(x) g (y) + g (x) d (y),$\forall x, y \in S$
By lemma 3.7, d and g are orthogonal
The Proof of (i)$\Longleftrightarrow$(iii) is the similar proof to that of (i)$\Longleftrightarrow$ (ii)
(i) $\Longleftrightarrow$(iv)Suppose d and g are orthogonal
Hence dg = 0 and gd = 0
$\therefore$dg + gd = 0
Conversely, Suppose that dg + gd = 0 (2)
Then, (dg + gd) (xy) = dg (xy) + gd (xy) = 0
By the remark,
d(x) g(y) + x dg(y) + dg (x) y + g(x) d (y) + g (x) d (y) + x gd (y) + gd (x) y + d (x) g(y)=0
2[d(x) g(y) + g(x) d (y)] + (dg + gd) (x) y + x (dg + gd) (y) = 0 [$\because$ by (2)]
$\Rightarrow$2 [d(x) g(y) + g(x) d(y)] = 0
$\Rightarrow$d (x) g (y) + g(x) d(y) = 0 [$\because$ S is 2-torsionfree]
$\Rightarrow$By lemma 3.7, d and g are orthogonal
(i)$\Longleftrightarrow$ (v)Suppose dg is derivation
dg (xy) = dg(x) y + x dg (y), $\forall x, y \in S$ (3)
By remark, dg (xy) = d(x) g(y) + x dg (y) + dg (x) y + g (x) d (y), $\forall x, y \in S$ (4)
Using (3) & (4),dg (xy) = d(x) g(y) + dg(xy) +g (x) d (y), $\forall x, y \in S$
0 = d(x) g(y) + g(x) d(y), $\forall x, y \in S$ [$\because$ S is additively cancellative
$\therefore$D and g are orthogonal.
Conversely, suppose d and g are orthogonal
By lemma 3.7, d(x) g(y) +g (x) d (y) =0
Using this is (4), dg (xy) = dg (x) y + x dg (y)
$\therefore$dg is a derivation
The proof of (i)$\Longleftrightarrow$ (vi) is the similar proof to that of (i) $\Longleftrightarrow$(v)

Corollary: 3.11

Let S be a 2- torsionfreesemiring.Suppose that d and g are orthogonal reverse derivations on S. Then either d = 0 or g = 0
The proof is immediate from the previous theorem.

Theorem: 3.12

Let S be a 2- torsionfreesemiprimesemiring and let d and g be reverse derivations onS.Suppose$d^2 = g^2$, then $d + g$and $d - g$are orthogonal

Proof:

Let d and g be reverse derivations on S.Suppose$d^2 = g^2$ (1)
For all x$\in$ S,[(d+g) (d$-$g) + (d$-$g) (d+g)] x= (d+g) (d$-$g) (x) + (d$-$g) (d+g) (x)
= (d+g) (d (x) $-$ g(x) + (d$-$g) (d(x) + g(x)
= d (d(x))$-$ d g(x) + g d(x) – g (g(x)) + d(d(x) + dg (x) – gd(x) – g(g(x))
=$d^2(x) - dg(x) + gd(x) - d^2(x) + g^2(x) + dg(x) - gd(x) - g^2(x)$
= 0
By the previous theorem, (d+g) and (d$-$g) are orthogonal.

Theorem: 3.13

Let S be a 2- torsionfreesemiprimesemiring and let d and g be reverse derivations of S.
If d(x) d(x) = g(x)g(x), then d+g and d$-$g are orthogonal.

The proof is similar to the previous theorem.

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