*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 4 Ver. II (Jul - Aug. 2015), PP 11-19*
*www.iosrjournals.org*

# Numerical solution of imbibition phenomenon in a homogeneous medium with magnetic fluid

A.V.Patel[1], N.S.Rabari[2], P.H.Bhathawala[3]
[1]*Associate Professor, Pramukh Swami Science College, Kadi-382715, Gujarat, India*
[2]*Assistant Professor, MUIS, Ganpat University, Ganpat Vidyanagar, Mehsana-384012, Gujarat, India*
[3]*Professor of IT, AURO University, Surat – 394510, Gujarat, India*

***Abstract:*** *In this paper, we have discussed imbibition phenomenon in double phase flow through porous media. Numerical solution of non linear partial differential equation governing the phenomenon of imbibition in a homogeneous medium with magnetic fluid has been obtained by finite element method. Finite element method is a numerical method for finding an approximation solution of differential equation in finite region or domain. A Matlab code is developed to solve the problems and the numerical results are obtained at various time levels.*
***Keywords:*** *Porous media, fluid flow, magnetic fluid, finite element method.*

## I. Introduction

It is well known that when a porous medium filled with some fluid is brought into contact with another fluid which preferentially wets the medium, there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid from the medium. Such a phenomena arising due to difference in wetting abilities is called counter-current imbibition. These phenomena have been formally discussed by Brownscombe and Dyes [4], Enright [5], Braham and Richarson [6], Rijik [7], Rijik et al [9], for homogeneous porous medium.

Bokserman, Zhelton and Kocheshkev [10] have described the physics of oil-water flow in a cracked media and Verma [11,12] has investigated two specific oil-water displacement process from analytical point of view. Verma has obtained solution by performing a perturbation technique. He has also investigated this problem in the presence of randomly oriented pores in the fractured medium renders the differential equation highly non-linear due to an additional non-linear term. He has also considered the presence of heterogeneity in the medium marginally.

## II. Statement of the Problem

We consider here a cylindrical mass of porous matrix of length L (=1) containing a viscous oil, is completely surrounded by an impermeable surface except for one end of the cylinder which is labeled as the imbibitions phase and this end is exposed to an adjacent formation of 'injected' water. It is assumed that the injected water and the viscous oil are two immiscible liquids of different salinities with small viscosity difference; the former represents the preferentially wetting and less viscous phase. This arrangement gives rise to the phenomenon of linear counter-current imbibitions, that is, a spontaneous linear flow of water into the porous medium and a linear counter flow of oil from the medium.

## III. Mathematical formulation of the problem

We take the injected liquid containinga thin layer of magnetic fluid where magnetization M is assumed to the directly proportional to the magnetic field intensity H (i.e. M = λH) and the microscopic behavior of fingers is governed by a statistical treatment. Then the additional pressure exerted due to presence of a layer of magnetic fluid in the displacing liquid (w) represented by $\left[\frac{\mu_o M + 16\bar{\lambda}\mu_o M\lambda r^3}{9(l+2)^3}\right]\frac{\partial H}{\partial x}$.Therefore, equation of filteration velocities of injected liquid (w), becomes $V_w = -\left(\frac{K_w}{\delta_w}\right)K\left[\frac{\partial P_w}{\partial x} + \gamma H\frac{\partial H}{\partial x}\right]$ … … … (1)where,

$$\gamma = \mu_o\lambda + \frac{16\bar{\lambda}\mu_o\lambda^2 r^3}{9(l+2)^3}$$

Also, equation of filteration velocity ($V_o$) of native liquid is $V_o = -\left(\frac{K_o}{\delta_o}\right)K\left[\frac{\partial P_o}{\partial x}\right]$ … … …

(2)Where, K = the permeability of the homogeneous medium,
$K_w$ = relative permeability of water, which is functions of $S_w$
$K_o$ = relative permeability of oil, which are functions of $S_o$
$S_w$ = the saturation of water , $S_o$ = the saturation of oil , $P_w$ = pressure of water , $P_o$ = pressure of oil
$\delta_w$ ,$\delta_o$ = constant kinematics viscosities, g = acceleration due to gravity.
Regarding the phase densities to be independent of magnetic field H, and to be constant, the equations of continuity of the two phases can be written as $P\left(\frac{\partial S_w}{\partial t}\right)+\left(\frac{\partial V_w}{\partial x}\right)=0$ … … …(3) where, P is porosity of the medium. The analytical condition (Scheidegger , 1960) governing imbibitions phenomenon is,
$V_o = - V_w$. Also, from the definition of phase saturation, it is obvious that $S_w + S_o = 1$ … … … (4).

The capillary pressure $P_c$ is defined as , $P_c = -\beta_o S_w$ and $P_c = P_o - P_w$ … … … (5)
Where, $P_c$ is a capillary pressure coefficient and $\beta_o$ is a constant quantity. Now, combining equation (1), (2), (3), (4) and (5) we get,

$$P\left(\frac{\partial S_w}{\partial t}\right)+\frac{\partial}{\partial x}\left(\left(\frac{K_o/\delta_o . K_w/\delta_w}{K_o/\delta_o + K_w/\delta_w}\right)K\left[\frac{\partial P_c}{\partial x}-\gamma H\frac{\partial H}{\partial x}\right]\right)=0 \quad \text{… … … (6)}$$

At this state, for definiteness of the mathematical analysis, we assume standard relationship due to Muskat [13] and Jones [14],between phase saturation and relative permeability as,

$$K_w = S_w^3, \quad \text{and } K_o = 1-\alpha S_w, \qquad \alpha = 1.11 \qquad \text{… … … (7)}$$

Assuming $\frac{K_o K_w}{K_o\delta_w + K_w\delta_o} \approx \frac{K_o}{\delta_o}$ , and using equation (5) and (7) in equation (6), weget,

$$P\left(\frac{\partial S_w}{\partial t}\right)+\frac{K}{\delta_o}\frac{\partial}{\partial x}\left((1-\alpha S_w)\left[-\beta_o\frac{\partial S_w}{\partial x}-\gamma H\frac{\partial H}{\partial x}\right]\right)=0 \qquad \text{… … … (8)}$$

This is the desired non linear differential equation of motion which describes the linear counter current imbibition phenomena in a homogeneous porous medium with effect of magnetic fluid. Now considering the magnetic fluid H in the x-direction only, we may write, $H = \frac{\Lambda}{x^n}$ where $\Lambda$ is a constant parameter and n is an integer. Using the value of H for n = -1 in equation (8), we get,

$$C_1\left(\frac{\partial S_w}{\partial t}\right)-C_2\frac{\partial}{\partial x}\left((1-\alpha S_w)\frac{\partial S_w}{\partial x}\right)-C_3\frac{\partial}{\partial x}\left((1-\alpha S_w)x\right)=0 \quad \text{… … … (9)}$$

Where $C_1 = P$ , $C_2 = \frac{K\beta_o}{\delta_o}$, $C_3 = \frac{\gamma K\Lambda^2}{\delta_o}$

A set of suitable boundary conditions associated to problem (9) are,
$S_w(x,0) = 0$ , for all $x > 0$ ; … … … (10)
$S_w(0,t) = S_{w0}$ ; $S_w(L,t) = S_{w1}$ for all $t \geq 0$ ; … … … (11)

Equation (9) is reduced to dimensionless form by setting

$$x^* = x/L, \qquad t^* = \frac{t}{L^2(C_1/C_2)}, \qquad (1-\alpha S_w(x,t)) = S_w^*(x^*,t^*)$$

so that $\frac{\partial S_w}{\partial t}-\frac{\partial}{\partial x}\left(S_w\frac{\partial S_w}{\partial x}\right)-C_o\frac{\partial}{\partial x}(S_w x)=0$ …… … (12)where, $C_o = \frac{-C_3 L^2\alpha}{C_2}$

Asterisks are dropped for simplicity.
The initial and boundary conditions (10) and (11) now becomes,

$S_w(x,0) = 1$ , for all $x > 0$ …… … (13)

$S_w(0,t) = 1-\alpha S_{w0}$ ; $S_w(L,t) = 1-\alpha S_{w1}$ for all $t \geq 0$ … … … (14)

Equation (12) is desired nonlinear differential equation of motion for the flow of two immiscible liquids in homogeneous medium wit effect of magnetic fluid.
A Matlab Code is prepared and executed with $\mathbf{C_o = 4.046 \times 10^{-11}}$ **, h =1/15** , **k = 0.002223** for **225** time levels, $S_{w0} = 0.5$ and $S_{w1} = 0$ .The numerical value are shown by table .Curves indicating the behavior of Saturation of injected fluid with respect to various time period.

**4. Finite Element Method:** For the problem under consideration, the length variable x varies between 0 and L (figure 1(a)). The domain is divided in to set of linear elements (figure 1(b)).


**Figure 1(a).**

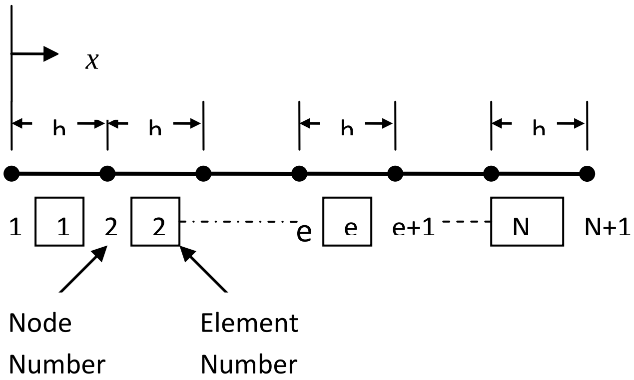


**Figure 1(b).**

Now, the variational form of given partial differential equation (12) is**,**

$$J(S_w) = \frac{1}{2}\int_R \left[ S_w \left( \frac{\partial S_w}{\partial x} \right)^2 + 2S_w \frac{\partial S_w}{\partial t} + C_o x S_w \frac{\partial S_w}{\partial x} \right] dx \qquad \text{… … … (15)}$$

Choose an arbitrary linear element $R^{(e)} = [S_1^{(e)} ,\ S_2^{(e)} ]$ and obtain interpolation function for $R^{(e)}$using Lagrange interpolation method such as,

$$S^{(e)}(x) = \sum_{j=1}^{2} N_j(x) S_j^{(e)} = N^{(e)}\phi^{(e)} = \phi^{(e)T} N^{(e)T} \qquad \text{…… … (16)}$$

$$where \qquad N^{(e)} = [N_1\ N_2] \quad \& \quad \phi^{(e)} = [S_1\ \ S_2]^T$$

and $N_1$ , $N_2$ are shape function for linear element.

Now, apply Variational Method to $R^{(e)}$, therefore equation (15) becomes,

$$J(S^{(e)}) = \frac{1}{2}\int_{R^{(e)}} \left[ S^{(e)} \left( \frac{\partial S^{(e)}}{\partial x} \right)^2 + 2S^{(e)} \frac{\partial S^{(e)}}{\partial t} + C_o x S^{(e)} \frac{\partial S^{(e)}}{\partial x} \right] dx \qquad \text{………(17)}$$

$$As, \quad S^{(e)}(x) = N^{(e)}\varphi^{(e)} = \varphi^{(e)T} N^{(e)T}$$

$$\therefore \quad \frac{\partial S^{(e)}}{\partial x} = \frac{\partial N^{(e)}}{\partial x}\varphi^{(e)} = \varphi^{(e)T}\frac{\partial N^{(e)T}}{\partial x} \quad and \quad \left(\frac{\partial S^{(e)}}{\partial x}\right)^2 = \varphi^{(e)T}\frac{\partial N^{(e)T}}{\partial x}\frac{\partial N^{(e)}}{\partial x}\varphi^{(e)}$$

Therefore Equation $(17)$ becomes,

$$J\left(S^{(e)}\right) = \frac{1}{2}\int_{R^{(e)}} \varphi^{(e)T}\left[N^{(e)}\varphi^{(e)}\left(\frac{\partial N^{(e)T}}{\partial x}\frac{\partial N^{(e)}}{\partial x}\right)\varphi^{(e)} + 2\left(N^{(e)T}N^{(e)}\right)\frac{\partial \varphi^{(e)}}{\partial t} + C_0 x N^{(e)T}\frac{\partial N^{(e)}}{\partial x}\varphi^{(e)}\right]dx$$

… … … (18)

For minimization, first differentiate equation (18) with respect to $\phi^{(e)}$

$$\frac{\partial J^{(e)}}{\partial \phi^{(e)}} = \int_{R^{(e)}}\left[N^{(e)}\phi^{(e)}\left(\frac{\partial N^{(e)T}}{\partial x}\frac{\partial N^{(e)}}{\partial x}\right)\phi^{(e)} + 2\left(N^{(e)T}N^{(e)}\right)\frac{\partial \phi^{(e)}}{\partial t} + C_0 x N^{(e)T}\frac{\partial N^{(e)}}{\partial x}\phi^{(e)}\right]dx$$

Now, $\frac{\partial J^{(e)}}{\partial \phi^{(e)}} = 0$.

The Element equation is $A^{(e)}\frac{\partial \phi^{(e)}}{\partial T} + B^{(e)}(\phi^{(e)})\phi^{(e)} + C^{(e)}\phi^{(e)} = 0$ … … ... (19)

Where,

$$A^{(e)} = \int_{S_1^{(e)}}^{S_2^{(e)}}\left(N^{(e)T}N^{(e)}\right)dx; \quad B^{(e)}(\phi^{(e)}) = \int_{S_1^{(e)}}^{S_2^{(e)}} N^{(e)}\phi^{(e)}\left(\frac{\partial N^{(e)T}}{\partial x}\frac{\partial N^{(e)}}{\partial x}\right)dx;$$

$$C^{(e)} = C_o\int_{S_1^{(e)}}^{S_2^{(e)}} x\left(N^{(e)T}\frac{\partial N^{(e)}}{\partial x}\right)dx$$

From Gauss Legendre Quadrature Method, we evaluate these integral. Thus element matrix transform to

$$A^{(e)} = \int_{-1}^{1}\left(N^{(e)T}N^{(e)}\right)Jdz \approx \sum_{I=1}^{r} A^{(e)}(z_I)W_I$$

$$B^{(e)}(\phi^{(e)}) = \int_{-1}^{1} N^{(e)}\phi^{(e)}\left(\frac{1}{J}\frac{\partial N^{(e)T}}{\partial X}\frac{1}{J}\frac{\partial N^{(e)}}{\partial X}\right)Jdz \approx \sum_{I=1}^{r} B^{(e)}(z_I)W_I$$

$$C^{(e)} = C_0\int_{-1}^{1}\left(\frac{h}{2}(1+z_I)\right)\left(N^{(e)T}\frac{1}{J}\frac{\partial N^{(e)}}{\partial z}\right)Jdz \approx \sum_{I=1}^{r} C^{(e)}(z_I)W_I$$

Where, $z_I$ and $W_I$ are corresponding gauss points and gauss weights.Then, the element matrix becomes,

$$A^{(e)} = \frac{h^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}, \quad B^{(e)}(\phi^{(e)}) = \frac{1}{2h^{(e)}}\begin{bmatrix}S_1+S_2 & -S_1-S_2\\ -S_1-S_2 & S_1+S_2\end{bmatrix}, C^{(e)} = \frac{C_o h^{(e)}}{6}\begin{bmatrix}-1 & -2\\ 1 & 2\end{bmatrix}$$

… … … (20)

**4.1.Assembling of elements:** The assembly of linear elements is carried out by imposing the following two conditions:

(a) The continuity of primary variable requires $S_n^e = S_1^{e+1}$

(b) The balance of secondary variables at connecting nodes requires

$$S_n^e + S_1^{e+1} = \begin{cases} 0 & \text{if no external point surce is applied} \\ S_1 & \text{if an external point surce of magnitude } S_1 \text{ is applied} \end{cases}$$

The inter-element continuity of primary variable can be imposed by simply renaming the variables of all elements connected to common node. For example, for a mesh of N linear finite element connected in series, we have

For a uniform mesh of N elements, by equation (19) and (20), the assembled equation becomes

$$S_1^1 = S_1, \quad S_2^1 = S_1^2 = S_2, \quad S_2^2 = S_1^3 = S_3, \cdots\cdots, S_2^{N-1} = S_1^N = S_N, \quad S_2^N = S_{N+1}$$

$$A\frac{\partial \varphi}{\partial T} - B(\phi)\phi + C\ \ \phi = 0 \qquad \text{…. …. …. (21)}$$

where, $$A = \frac{h}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & (2+2) & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & (2+2) & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & (2+2) & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}$$

$$B(\phi) = \frac{1}{2h}\begin{bmatrix} S_1+S_2 & -S_1-S_2 & 0 & 0 & \cdots & 0 & 0 \\ -S_1-S_2 & S_1+2S_2+S_3 & -S_2-S_3 & 0 & \cdots & 0 & 0 \\ 0 & -S_2-S_3 & S_2+2S_3+S_4 & -S_3-S_4 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & S_{14}+2S_{15}+S_{16} & -S_{15}-S_{16} \\ 0 & 0 & 0 & 0 & \cdots & -S_{15}-S_{16} & S_{15}+S_{16} \end{bmatrix}$$

$$C = \frac{C_o h}{6}\begin{bmatrix} -1 & -2 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}; \quad \varphi = \begin{bmatrix} S_1 & S_2 & \cdots & \cdots & S_{16} \end{bmatrix}^T \quad \text{.........(22)}$$

Equation (21) represents the assembled equation.

**4.2.Time approximation:** We introduced $\delta$ family of approximations which approximates weighted average of a dependent variable of two consecutive time steps by linear interpolation of the values of the variable at the two time steps such as, $S_j = \delta S_j^{n+1} + (1-\delta) S_j^n$ ................. (23).The time derivatives $\dot{\theta}_j$ are replaced by Forward finite difference formula such as $\dot{S}_j = \dfrac{S_j^{n+1} - S_j^n}{k}$ ................. (24).

In view of (23) and (24), equation (21) written as,
$[\, A + \delta\, k(B(\phi^{(n+1)}) + C)\, ]\phi^{(n+1)} = [\, A - (1-\delta)\, k(B(\phi^{(n)}) + C)\, ]\phi^{(n)}$ where, **$\delta$ =1/2** and n = 0,1,2,……

For a uniform mesh of N elements, by equation (22) and above global equation takes the form,

$$\left[K(\phi^{(n+1)})\right]\phi^{(n+1)} = \left[F_1(\phi^{(n)})\right]\phi^{(n)} = F(\phi^{(n)}) \qquad \ldots \ldots \ldots \ldots \ldots (25)$$

where,

$K(\phi^{(n+1)})=$

$$\begin{bmatrix}
\frac{h}{3}+\delta k\left(\frac{1}{2h}\left(S_1^{(n+1)}+S_2^{(n+1)}\right)+\frac{C_oh}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_1^{(n+1)}-S_2^{(n+1)})+\frac{C_oh}{3}\right) & 0 & \cdots\ \cdots & 0 & 0 \\
\frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_1^{(n+1)}-S_2^{(n+1)})-\frac{C_oh}{6}\right) & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_1^{(n+1)}+2S_2^{(n+1)}+S_3^{(n+1)})-\frac{C_oh}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_2^{(n+1)}-S_3^{(n+1)})+\frac{C_oh}{3}\right) & 0\ \cdots & 0 & 0 \\
0 & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_2^{(n+1)}-S_3^{(n+1)})-\frac{C_oh}{6}\right) & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_2^{(n+1)}+2S_3^{(n+1)}+S_4^{(n+1)})-\frac{C_oh}{6}\right) & \cdots\ \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0\ \cdots & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_{14}^{(n+1)}+2S_{15}^{(n+1)}+S_{16}^{(n+1)})-\frac{C_oh}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_{15}^{(n+1)}-S_{16}^{(n+1)})+\frac{C_oh}{3}\right) \\
0 & 0 & 0 & 0\ \cdots & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_{15}^{(n+1)}-S_{16}^{(n+1)})-\frac{C_oh}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(+S_{15}^{(n+1)}+S_{16}^{(n+1)})-\frac{C_oh}{3}\right)
\end{bmatrix}$$

$F_1\left(\phi^{(n)}\right)=$

$$\begin{bmatrix}
\frac{h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_1^{(n)}+S_2^{(n)})+\frac{C_oh}{6}\right) & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)}-S_2^{(n)})+\frac{C_oh}{3}\right) & 0 & 0\ \cdots & 0 & 0 \\
\frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)}-S_2^{(n)})-\frac{C_oh}{6}\right) & \frac{2h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_1^{(n)}+2S_2^{(n)}+S_3^{(n)})-\frac{C_oh}{6}\right) & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)}-S_2^{(n)})+\frac{C_oh}{3}\right) & 0\ \cdots & 0 & 0 \\
0 & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_2^{(n)}-S_3^{(n)})-\frac{C_oh}{6}\right) & \frac{2h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_2^{(n)}+2S_3^{(n)}+S_4^{(n)})-\frac{C_oh}{6}\right) & \cdots\ \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots\ \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0\ \cdots & \frac{2h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_{14}^{(n)}+2S_{15}^{(n)}+S_{16}^{(n)})-\frac{C_oh}{6}\right) & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_{15}^{(n)}-S_{16}^{(n)})+\frac{C_oh}{3}\right) \\
0 & 0 & 0 & 0\ \cdots & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_{15}^{(n)}-S_{16}^{(n)})-\frac{C_oh}{6}\right) & \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(S_{15}^{(n)}+S_{16}^{(n)})-\frac{C_oh}{3}\right)
\end{bmatrix}$$

$$\phi^{(n)} = \begin{bmatrix} S_1^{(n)} \\ S_2^{(n)} \\ S_3^{(n)} \\ \vdots \\ \vdots \\ S_{15}^{(n)} \\ S_{16}^{(n)} \end{bmatrix}$$

$$F(\phi^{(n)})=$$

$$\begin{bmatrix} \frac{h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_1^{(n)}+S_2^{(n)})+\frac{C_o h}{6}\right)S_1^{(n)}+\frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)}-S_2^{(n)})+\frac{C_o h}{3}\right)S_2^{(n)} \\ \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)}-S_2^{(n)})-\frac{C_o h}{6}\right)S_1^{(n)}+\frac{2h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_1^{(n)}+2S_2^{(n)}+S_3^{(n)})-\frac{C_o h}{6}\right)S_2^{(n)}+\frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_2^{(n)}-S_3^{(n)})+\frac{C_o h}{3}\right)S_3^{(n)} \\ \vdots \\ \vdots \\ \vdots \\ \frac{2h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_{14}^{(n)}+2S_{15}^{(n)}+S_{16}^{(n)})-\frac{C_o h}{6}\right)S_{15}^{(n)}+\frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_{15}^{(n)}-S_{16}^{(n)})+\frac{C_o h}{3}\right)S_{16}^{(n)} \\ \frac{h}{6}-(1-\delta)k\left(\frac{1}{2h}(-S_{15}^{(n)}-S_{16}^{(n)}-\frac{C_o h}{6}\right)S_{15}^{(n)}+\frac{h}{3}-(1-\delta)k\left(\frac{1}{2h}(S_{15}^{(n)}+S_{16}^{(n)})-\frac{C_o h}{3}\right)S_{16}^{(n)} \end{bmatrix}$$

where K is called global stiffness matrix and F is called global generalized force vector and N+1 is total number of global nodes .

**4.3.Imposing boundary conditions**: We now apply the boundary condition (14) to the global equation (25) of the problem and simplifying, we get,

$$[K(\phi^{(n+1)})]\phi^{(n+1)}=F \quad \ldots\ldots\ldots(26)$$

where

$$K(\phi^{(n+1)})=$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots\cdots & 0 & 0 \\ \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_1^{(n+1)}-S_2^{(n+1)})-\frac{C_o h}{6}\right) & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_1^{(n+1)}+2S_2^{(n+1)}+S_3^{(n+1)})-\frac{C_o h}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_2^{(n+1)}-S_3^{(n+1)})+\frac{C_o h}{3}\right) & 0\cdots & 0 & 0 \\ 0 & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_2^{(n+1)}-S_3^{(n+1)})-\frac{C_o h}{6}\right) & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_2^{(n+1)}+2S_3^{(n+1)}+S_4^{(n+1)})-\frac{C_o h}{6}\right) & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0\cdots & \frac{2h}{3}+\delta k\left(\frac{1}{2h}(S_{14}^{(n+1)}+2S_{15}^{(n+1)}+S_{16}^{(n+1)})-\frac{C_o h}{6}\right) & \frac{h}{6}+\delta k\left(\frac{1}{2h}(-S_{15}^{(n+1)}-S_{16}^{(n+1)})+\frac{C_o h}{3}\right) \\ 0 & 0 & 0 & 0\cdots & 0 & 1 \end{bmatrix}$$

$$F(\phi^{(n)}) = \begin{bmatrix} 0.445 \\ \frac{h}{6} - (1-\delta)k\left(\frac{1}{2h}(-S_1^{(n)} - S_2^{(n)}) - \frac{C_o h}{6}\right)S_1^{(n)} + \frac{2h}{3} - (1-\delta)k\left(\frac{1}{2h}(S_1^{(n)} + 2S_2^{(n)} + S_3^{(n)}) - \frac{C_o h}{6}\right)S_2^{(n)} + \frac{h}{6} - (1-\delta)k\left(\frac{1}{2h}(-S_2^{(n)} - S_3^{(n)}) + \frac{C_o h}{3}\right)S_3^{(n)} \\ \vdots \\ \vdots \\ \vdots \\ \frac{2h}{3} - (1-\delta)k\left(\frac{1}{2h}(S_{14}^{(n)} + 2S_{15}^{(n)} + S_{16}^{(n)}) - \frac{C_o h}{6}\right)S_{15}^{(n)} + \frac{h}{6} - (1-\delta)k\left(\frac{1}{2h}(-S_{15}^{(n)} - S_{16}^{(n)}) + \frac{C_o h}{3}\right)S_{16}^{(n)} \\ 1 \end{bmatrix}$$

Thus, equation (26) is the resulting system of nonlinear algebraic equation.

**4.4.Solution of nonlinear algebraic equation:** In the previous section, we obtained the assembled equation which is nonlinear. The assembled nonlinear equation after imposing boundary conditions is given by equation (26).We seek an approximate solution by the linearization which based on scheme

$$\left\lfloor K(\phi^{(n)}) \right\rfloor \phi^{(n+1)} = F \qquad \ldots \ldots \ldots (27)$$

where $\phi^{(n)}$ denotes the solution of the n iteration. Thus, the coefficient $K_{ij}$ are evaluated using the solution $\phi^{(n)}$ from the previous iteration and the solution at the $(n+1)^{th}$ iteration can be obtained by solving equation (26) using Gauss Elimination Method. At the beginning of the iteration (i.e. n=0), we assume the solution $\phi^{(0)}$ from initial condition (13) which requires to have $S_1^{(0)} = S_2^{(0)} = \ldots\ldots\ldots\ldots\ldots = S_{N+1}^{(0)} = 1$.

**5. Graphical representation:**A Matlab Code isprepared for 15 elements model and resulting equation (27) for N = 15 is solved by Gauss Elimination method.
Saturation of injected liquid at t = 0.1, 0.2, 0.3, 0.4 and 0.5 seconds are

| | | | | |
|---|---|---|---|---|
| 5.0000e-001 | 5.0000e-001 | 5.0000e-001 | 5.0000e-001 | 5.0000e-001 |
| 4.0544e-001 | 4.3033e-001 | 4.4043e-001 | 4.4503e-001 | 4.4718e-001 |
| 3.2910e-001 | 3.7082e-001 | 3.8824e-001 | 3.9625e-001 | 4.0002e-001 |
| 2.6604e-001 | 3.1908e-001 | 3.4170e-001 | 3.5218e-001 | 3.5711e-001 |
| 2.1355e-001 | 2.7365e-001 | 2.9974e-001 | 3.1186e-001 | 3.1758e-001 |
| 1.6987e-001 | 2.3353e-001 | 2.6160e-001 | 2.7465e-001 | 2.8080e-001 |
| 1.3368e-001 | 1.9797e-001 | 2.2670e-001 | 2.4005e-001 | 2.4634e-001 |
| 1.0319e-001 | 1.6636e-001 | 1.9459e-001 | 2.0769e-001 | 2.1385e-001 |
| 7.9602e-002 | 1.3816e-001 | 1.6489e-001 | 1.7725e-001 | 1.8305e-001 |
| 5.9986e-002 | 1.1291e-001 | 1.3726e-001 | 1.4848e-001 | 1.5373e-001 |
| 4.4219e-002 | 9.0167e-002 | 1.1140e-001 | 1.2115e-001 | 1.2570e-001 |
| 3.1680e-002 | 6.9504e-002 | 8.7050e-002 | 9.5066e-002 | 9.8794e-002 |
| 2.1579e-002 | 5.0529e-002 | 6.3951e-002 | 7.0053e-002 | 7.2884e-002 |
| 1.3383e-002 | 3.2858e-002 | 4.1878e-002 | 4.5960e-002 | 4.7878e-002 |
| 6.3668e-003 | 1.6129e-002 | 2.0624e-002 | 2.2649e-002 | 2.3584e-002 |
| 0 | 0 | 0 | 0 | 0 |

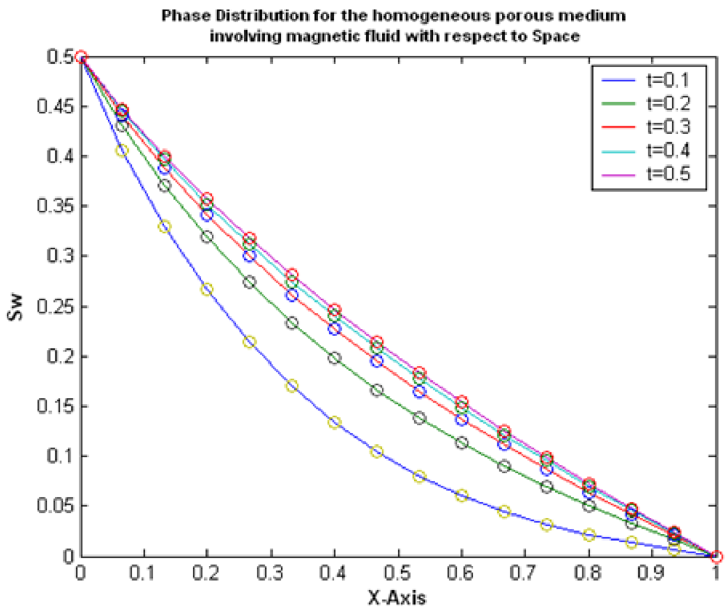


**figure 8.**

Solution is obtained with $\mathbf{C_o = 4.046 \times 10^{-11}}$ , **h =1/15** , **k = 0.002223** for **225** time levels.It is clear from graph that $S_w = S_0(=0.5)$ at layer x =0 and there is no saturation of injected liquid at other end (x =1) irrespective of time.It is clear from graph that, at particular time, saturation of injected liquid involving magnetic fluid decrease with increase in value of x(or as we move ahead) and at x = 1,saturation is decreased to zero and and at particular point x of observed region,saturation of injected fluid increases with increase in time but rate of increase of the saturation slows down at each point as time increases .

## References


[1]. Scheidegger, A. E. Physics of flow through porous media, (Revised Ed.). Univ., Toronto Press.,1974.
[2]. Jain M.K., Numerical solution of differential equations.2nd edition, Wiley Eastern, 1984.
[3]. Reddy J.N., An introduction to the finite element method, 3rd edition, McGraw-Hill, Inc., Newyork, 2006.
[4]. Brownscombe, E.R. and Dyes, A.B., Amer. Petrol Inst. Drill. Prod. Pract. , P. 383
[5]. Enright, R.J., Oil gas J, 53,954,104.
[6]. Graham, J.W. and Richardson, J.G., J. Petrol. Tech., 11,65.
[7]. Rijik, V.M., Izv. Akad. Nauk. SSSR, Utd. Tech. Nauk Mekhan Mashinostor, 2.
[8]. Fox, L., Numerical Solution of ordinary and partial differential equation, Macmillan (paraganon), NewYork, 1962.
[9]. Rijik, V.M., et al., (1962): Vliianie Svoistv jornih porodna advijenie vi nih jidkosti, Goatoplehizdt Moskva, p.202.
[10]. Boksermann, A.A., et al., (BZK, 1964): Sci, Phys. Doklady, (Eng. Trans.), 9, 285.
[11]. Verma, A.P., (1968): Rev. Roum. Sci. Tech. Ser. Mecan. Appli. 13, p. 177.
[12]. Verma, A.P., (1969): Can. J. Phys. 47,pp. 2519-24.
[13]. Muskat, M. Physical Principles of oil production, McGraw Hill, 1949.
[14]. Jones, D.P.J., (1946): Petrol. Prod. (Reinhold, Newyork).
[15]. Verma,A.P., Imbibition in a Cracked porous media, Can.J. Phys. 44, 22, 2519,1969.
[16]. Muskat, M., Physics of homogenous fluid, 5, 51, 1954.
[17]. Muskat, M. Flow of homogeneous fluids, Edward 1946, pp. 7, 400.
[18]. Mercer and Faust C.R., " The application of finite element techniques to immiscible flow in porous media, Proc. First Int. conf., Princeton University, U.S.A., June 1976 p. 121.
[19]. Zienkiewicz, o.c. and Taylor, R.C.(1989). The finite Element Method in Engineering Science, 2nd Edition, McGraw-Hill, London.
[20]. Whiteman, J., (1973): The mathematics of Finite Elements and Applications, Academic Press, NewYork.
[21]. Reddy, J.N., Energy Principles and Variational Methods in applied Mechanics, 2nd ed., John Wiley, NewYork, 2002.
[22]. Reddy, J.N., Applied Functional Analysis and Variational Methods in Engineering, McGraw-Hill, NewYork, 1986; Krieger, Melbourne,Fl, 1991.
[23]. Reddy, J.N.,An Introduction to Nonlinear Finite Element Analysis, Oxford University Press, Oxford, UK,2004.
[24]. Forsythe, G.E. and Wasow, W.R., Finite Difference Methods for Partial Differential Equations, Wiley, 1960.
[25]. MATLAB Appliction Program Interface Reference, Version 6, The MathWorks Inc., 2000.
[26]. Smith, G.D. Numerical Solution of Partial Differential Equations, Oxford University Press, 1966.
[27]. Zlamal,M. "Finite Element methods for parabolic equations ", Lecture notes in Math., 363, Springer-Verlag,1974.