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Numerical Evaluation of Two Dimensional Cpv Integrals

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Abstract: *Numerical methods have been formulated for the numerical approximation of real two dimensional integrals. The truncation error associated with the methods has been analyzed using the Taylors' series expansion. The methods have been verified by considering standard examples.*
Keywords: *Approximation rules, Degree of precision, Taylors' series expansion*.

I. Introduction

Cauchy Principal Value (CPV) integrals occur quite frequently in physics and applied mathematics. Quite a large number of methods have been devised for the numerical evaluation of one dimensional real CPV integral which is given by

$$J(g) = P\int_{x_0-h}^{x_0+h} \frac{g(x)}{x-x_0}\,dx \quad (1)$$

Some of these methods can be found in Davis and Rabinowitz [1], Monegato [2], etc.

Two dimensional CPV integral is specified in the following form

$$I(f) = P\iint_S \frac{f(x,y)}{(x-x_0)(y-y_0)}\,dx\,dy \quad (2)$$

where *S*is a square with vertices at $(x_0 \pm h, y_0 \pm h)$ and f is continuous on *S.* If $\varepsilon > 0$ is an arbitrarily small positive number, then the value of the integral $I(f)$is given by the following limit if it exists.

$$I(f) = \lim_{\varepsilon\to 0}\iint_{S-s} \frac{f(x,y)}{(x-x_0)(y-y_0)}\,dx\,dy \qquad (3)$$

where *s* is a square with vertices at $(x_0 \pm \varepsilon, y_0 \pm \varepsilon)$. The condition under which the limit in eqn. (3) exists is that the function *f* should satisfy Holder's inequality on the square *S*.

It is pertinent to note that substantial research work has not been conducted for the numerical approximation of the two dimensional CPV integral. However, Monegato [3],Nayak etal [4],Squire [5], Theocaris and Kazantzakis [6], Theocaris [7] have discussed the topic of numerical evaluation oftwo dimensional Cauchy principal value integral $I(f)$ given by equation (2).

The object of the present paper is to formulate some cubature rules for approximating numerically the CPV integral $I(f)$. To start with, construction of product rules is considered and subsequently a non-product interpolatory ruleis formulated for the numerical evaluation of the two dimensional CPV integral $I(f)$.

II. Generation of product rules

The easiest technique of generation of cubature rules meant for the integral $I(f)$ is by forming the Cartesian product of one dimensional rules meant for the CPV integral $J(g)$given by eqn(1). The most popular$n$ point Gauss type rule(n even)of degree of precision $2n$is the following rule

$$Q_n(f) = \sum_{j=1}^{n/2} \frac{w_j}{t_j}\{f(x_0 + h * t_j) - f(x_0 - h * t_j)\}.(4)$$

The quantities t_j's and w_j's in equation (4) are respectively the*n/2*positive nodesin (0,1) and coefficients of the Gauss-Legendre *n*-point (*n*even) one dimensional quadrature rulemeant for the weighted definite integral $\int_{-1}^{1} \omega(x)\theta(x)dx$ of the function $\theta(x)$ where $\omega(x) \equiv 1$.

So far as the two dimensionalCPV integral $I(f)$ is concerned,applying the rule $Q_n(f)$ for the numerical approximation with respect to the variable $x$first and then with respect to the variable y, then^2 –point rule is obtained in the followingmatrix form:

$$R_{n^2}(f) = (B \times T) \times B' \qquad (5)$$

where *B* is a $1 \times (n/2)$matrix with transpose B' and T is a $(n/2 \times n/2)$ matrix and

$$\left.\begin{array}{l} B = (b_i)_{1\times(n/2)},\ b_i = w_i/t_i, \\ T = \left(f_{ij}\right)_{(n/2\times n/2)}, \\ \quad f_{ij} = f\left(x_0 + ht_i, y_0 + ht_j\right) - f\left(x_0 - ht_i, y_0 + t_j\right) \\ \quad -f\left(x_0 + ht_i, y_0 - ht_j\right) + f\left(x_0 - ht_i, y_0 - ht_j\right). \end{array}\right\}(6)$$

Let the truncation error $E_{n^2}(f)$ associated with the product rule $R_{n^2}(f)$ i.e.

$$E_{n^2}(f) = I(f) - R_{n^2}(f).(7)$$

Let the function f be a regular function in the square $S$for whichthe Taylors' series expansion which is prescribed as

$$f(x,y) = \sum_{j=0}^{\infty}\frac{1}{j!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right)^j f(x_0,y_0)(8)$$

where all thepartial derivatives areevaluated at(x_0, y_0). Using eqn. (8) in eqn.(7) and simplifying, the leading term in $E_{n^2}(f)$ is given in the following form:

$$E_{n^2}(f) \approx 4\{\frac{1}{2n+1} - \sum_{j=1}^{n/2} w_j t_j^{2n}\}\frac{h^{2n+2}}{(2n+1)!}(f^{2n+1,1} + f^{1,2n+1}).(9)$$

The notation$f^{\gamma,\rho}$means $\frac{\partial^{\gamma+\rho}}{\partial x^{\gamma}\partial x^{\rho}} f(x_0, y_0)$ .Eqn. (9) signifies that the degree of precision of the product rule $R_{n^2}(f)$is $2n$. It is noteworthy that $E_{n^2}(f) = \mathrm{O}(h^{2(n+1)})$. Therefore the accuracy of the rule $R_{n^2}(f)$ is dependant upon the quantity h.

III. Generation of interpolatory rule

The product layout for Cartesian product rules involvesrelatively large number of nodes. So for reasonable accuracy with less function evaluations, interpolatory rules are preferable. Let the proposed interpolatory rule meant for the integral $I(f)$ involve the following set of seven nodes :

A=$\{(x_0, y_0),(x_0 \pm sh, y_0 \pm th),(x_0,y_0 \pm rh)\}$(10)

and the interpolatory rule be

$$R_7(f) = C_1 h^2 f_{xy}(x_0, y_0) + C_2\sum f\,(x_0 \pm sh, y_0 \pm th) + C_3 h \sum f_x(x_0, y_0 \pm rh) \quad (11)$$

where the coefficients C_j and the real parameters $r, s, t$are to be suitably determined.It is pertinent to note that the rule is exact i.e. $I(f) = R_7(f)$ whenever $f(x,y)$=$(x-x_0)^{\alpha} \times (y-y_0)^{\beta}$,$\alpha + \beta$ is odd or both are even. Therefore for the determination of seven unknowns in eqn.(11), it is sufficient to consider the cases (α, β)=(1,1),(3,1),(1,3),(5,1),(1,5),(3,3) and make the rule exact for monomials $(x-x_0)^{\alpha} \times (y-y_0)^{\beta}$.As a result of this the rule $R_7(f)$becomes exact for all monomials of degree ≤ 7. This leads to the following system of equations:

$$\left.\begin{array}{c} C_1 + 4C_2 st + 2C_3 r = 4, \\ C_2 s^3 t = 1/3, \\ 2C_2 st^3 + C_3 r^3 = 2/3, \\ C_2 s^5 t = 4/5, \\ 2C_2 st^5 + C_3 r^5 = 2/5, \\ C_2 s^3 t^3 = 1/9. \end{array}\right\}(12)$$

The solutions of the above system of equations are the following which yield the interpolatory cubature rule $R_7(f)$;

$$\left.\begin{array}{l} C_1 = 8/7, C_2 = 5\sqrt{5}/9, C_3 = 20\sqrt{15}/63\sqrt{14}, \\ s = \sqrt{3/5}, t = 1/\sqrt{3}, r = \sqrt{14/15}. \end{array}\right\} \qquad (13)$$

The truncation error associated with the rule $R_7(f)$ is given by

$$E_7(f) = I(f) - R_7(f). \qquad (14)$$

Proceeding in the same vein as in equations (7)-(9), the error in respect of the rule $R_7(f)$ is obtained in the following form:

$$E_7(f) \approx \frac{h^8}{225}\{\frac{-8}{6615}f^{1,7} + \frac{1}{245}f^{7,1} + \frac{1}{27}f^{3,5}\}. \qquad (15)$$

Equation (15) shows that the rule $R_7(f)$ has degree of precision seven has order of accuracy $\mathrm{O}(h^8)$.

IV. Numerical Experiments

For the numerical verification of the product rule as well as the interpolatory rule the following two double CPV integrals are considered:

$J_1 = P\int_{-h}^{h}\int_{-h}^{h}\frac{e^{x+y}}{xy}\,dxdy$=1.028182817310825 for h=0.5 , (16)

$J_2 = P\int_{-h}^{h}\int_{-h}^{h}\frac{\cos(x-y)}{xy}\,dxdy$= 0.972619702916399 for h=0.5. (17)

The third and fourth columns in Table-1 contain respectively the absolute error i.e. difference of exact value and the computed value of the integrals and the absolute values of the leading error terms ($|LET|$) for the rules . So far as the product rule is concerned the value of n is equal to 4 has been considered. It is noteworthy that the absolute error and the magnitude of the leading error term in case of the product rule coincide upto at least nine decimal places and those in respect of the interpolatory rule coincide upto six decimal places. This confirms that the leading terms in the truncation error of the rules almost account for the absolute error. It is pertinent to note that the product rule of degree 8 for n=4 involves sixteen nodes while the interpolatory rule of degree 7 involves only seven nodes. As a result of which the product rule possesses greater accuracy than the interpolatory rule.

Table 1

Integrals	Rules	$\|Error\|$	$\|LET\|$
J_1	$R_{4^2}(f)$	1.27×10^{-10}	1.25×10^{-10}
	$R_7(f)$	7.03×10^{-07}	6.92×10^{-07}
J_2	$R_{4^2}(f)$	1.22×10^{-10}	6.24×10^{-11}
	$R_7(f)$	6.82×10^{-07}	5.51×10^{-07}

References:

[1]. Davis,P.J. & Rabinowitz,P. Methods of numerical integration(2nd edn.),Academic Press,London,1984.

[2]. Monegato,G., The numerical evaluation of one dimensional Cauchy principal value integrals,computing,29,1982,337-354.

[3]. Monegato,G., convergence of product formulas for the numerical evaluation of cetain two dimensional CPV integral, Numer.Math., 43(2),1984,161-173.

[4]. Nayak,M.M.,Acharya,M. and Acharya,B.P., Numerical evaluation of two dimensional Cauchy principal value integrals,Appl.Math. and Comp.,225,2013,807-810.

[5]. Squire,W.,Numerical evaluation of a class of singular double integrals by symmetric pairing, Int.J.Numer.Methods Engg.,10(3),1976,703-708.

[6]. Theocaris,P.S. and Kazantzakis,J.V., On the numerical evaluation of two and three dimensional Cauchy principal value integrals,Acta Mech.,39,1981,105-115.

[7]. Theocaris,P.S.,Modified Gauss-Legendre,Lobatto and Radau cubature formulas for the numerical evaluation of 2-D singular integrals, Int.J. Math.Math. Sci,6(3), 1983, 567-587.