IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 1 Ver. I (Jan. - Feb. 2016), PP 71-75
www.iosrjournals.org

Extention of Mittag-Leffler function and its properties

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Abstract
The main aim of this paper is to present a new extension of Mittag-Leffler functions using extended Beta function. We derive several Integral representations and differentiation formulas for this new extended Mittag-Leffler function. Further, we establish Mellin transform of these functions in terms of Wright generalized hypergeometric functions.
Key words: *Mittag-Leffler function, extended Beta function, Mellin transform, Wright generalized hypergeometric functions.*

I. Introduction

The Mittag-Leffler functions has applications in many areas of science and engineering. The Mittag-Leffler functions arises naturally in the solution of fractional order integral equations or fractional order differential equations.
Mittag-Leffler [4] defined the function as :

$$\mathbf{E}_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}\ ;\ \alpha\in\mathbf{C},\ \operatorname{Re}(\alpha)>0\ ,\ z\in\mathbf{C} \qquad (1.1)$$

The general form of (1.1) was introduced by Winman [9] as follows:

$$\mathbf{E}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\ ;\ \alpha,\beta\in\mathbf{C}, \qquad (1.2)$$

$$\left(\operatorname{Re}(\alpha)>0\ ,\ \operatorname{Re}(\beta)>0,\ z\in\mathbf{C}\right)$$

Prabhakar [6] gave the generalized Mittag-Leffler function as

$$\mathbf{E}_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\cdot\frac{z^k}{k!} \qquad (1.3)$$

where α, β, $\gamma\in$ C, Re (α) > 0, Re (β) > 0 and $(\gamma)_k$ is Pochhammer symbol [7] defined by

$$(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}=\gamma(\gamma+1).....(\gamma+k-1),\ \gamma\neq 0,\ k\in\mathbf{N} \qquad (1.4)$$

and $(\gamma)_0=1$

In this paper, we present the new extended Mittag-Leffler function in terms of extended Beta function as follows :

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;\ p;\ m)=\sum_{k=0}^{\infty}\frac{\mathbf{B}_p(\gamma+k,\ c-\gamma;\ m)}{\mathbf{B}(\gamma,\ c-\gamma)}\cdot\frac{(c)_k}{\Gamma(\alpha k+\beta)}\cdot\frac{z^k}{k!} \qquad (1.5)$$

$$\left(p\geq 0,\ \operatorname{Re}(c)>\operatorname{Re}(\gamma)>0,\ m>0\right)$$

where $\mathbf{B}_p(x,y;m)$ is given by

$$\mathbf{B}_p(x,y\ ;\ m)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}dt \qquad (1.6)$$

$$\left(p\geq 0,\ m>0,\ \operatorname{Re}(x)>0, \operatorname{Re}(y)>0\right)$$

Here $\mathbf{B}_p(x,\ y\ ;m)$ is the extended Beta function due to Lee et.al [3] which is further extention of the extended Beta function introduced by Chaudhary et.al [1,2].

For m = 1, the new extended Mittag-Leffler function defined by (1.5) reduces to the function given by Ozarslan et.al [5].

II. Integral Representation of the Extended Mittag-Leffler Function

In this section, we present the various integral representations of the extended Mittag-Leffler function defined by (1.5).

Theorem 2.1. For the extended Mittag-Leffler function , we have the following integral representation:

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;\,p;\,m)=\frac{1}{\mathbf{B}(\gamma,\,c-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\exp\left(-\frac{p}{t^m(1-t)^m}\right)\mathbf{E}_{\alpha,\beta}^{c}(tz)\,dt, \quad (2.1)$$

where p ≥ 0, m > 0, Re (c) > Re (γ) > 0, Re (α) > 0, Re (β) > 0.

Proof : From the definition of extended Milttag-Leffler function given by (1.5) , we have

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p\,;m)=\sum_{k=0}^{\infty}\left[\int_0^1 t^{\gamma+k-1}(1-t)^{c-\gamma-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}\right.$$
$$\left.\times\frac{(c)_k}{\mathbf{B}(\gamma,c-\gamma)}\frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}\right]dt \quad (2.2)$$

On interchanging the order of integration and summation in (2.2), which is valid due to the conditions given in the statement of theorem, we get

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p\,;m)=\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}\sum_{k=0}^{\infty}\frac{(c)_k}{\mathbf{B}(\gamma,c-\gamma)}\frac{(tz)^k}{\Gamma(\alpha k+\beta)k!}\,dt$$
$$=\frac{1}{\mathbf{B}(\gamma,c-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}$$
$$\times\left[\sum_{k=0}^{\infty}\frac{(c)_k}{\Gamma(\alpha k+\beta)}.\frac{(tz)^k}{k!}\right]dt \quad (2.3)$$

By using (1.3) in (2.3) , we obtain the required result.

Corollary (i):

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)=\frac{1}{\mathbf{B}(\gamma,c-\gamma)}\int_0^{\infty}\frac{u^{\gamma-1}}{(1+u)^c}.\exp\left\{-p\left(2+u+\frac{1}{u}\right)^m\right\}.$$
$$\times\mathbf{E}_{\alpha,\beta}^{c}\left(\frac{uz}{1+u}\right)du \quad (2.4)$$

On taking $t=\frac{u}{1+u}$ in (2.1), we get the desired result (2.4)

Corollary (ii):

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)=\frac{1}{\mathbf{B}(\gamma,c-\gamma)}2\int_0^{\pi/2}\cos^{2\gamma-1}\theta\,\sin^{2c-2\gamma-1}\theta.$$
$$\times\exp\left\{-\frac{p}{\sin^{2m}\theta\cos^{2m}\theta}\right\}\mathbf{E}_{\alpha,\beta}^{c}\left(z\cos^2\theta\right)d\theta \quad (2.5)$$

Putting $t=\cos^2\theta$ in (2.1), we get (2.5).

Corollary (iii) : Taking $t=\sin^2\theta$ in (2.1), we have

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)=\frac{1}{\mathbf{B}(\gamma,c-\gamma)}2\int_0^{\pi/2}\sin^{2\gamma-1}\theta\cos^{2c-2\gamma-1}\theta$$

$$\times \exp\left\{-\frac{p}{\sin^{2m}\theta\cos^{2m}\theta}\right\}\mathbf{E}^{c}_{\alpha,\beta}\left(z\sin^{2}\theta\right)d\theta \qquad (2.6)$$

Corollary (iv) : On putting $t=\frac{1+u}{2}$ in (2.1), we get the following integral representation :

$$\mathbf{E}^{\gamma;c}_{\alpha,\beta}(z;p;m)=\frac{1}{\mathbf{B}(\gamma,c-\gamma)}2^{1-c}\int_{-1}^{1}(1+u)^{\gamma-1}(1-u)^{c-\gamma-1}$$

$$\times \exp\left\{-\frac{4^{m}p}{\left(1-u^{2}\right)^{m}}\right\}\mathbf{E}^{c}_{\alpha,\beta}\left(z\frac{(1+u)}{2}\right)du \qquad (2.7)$$

III. Differentiation of the extended Mittag-Leffler function

In this section, we give differentiation formula for the extended Mittag-Leffler function $\mathbf{E}^{\gamma;c}_{\alpha,\beta}(z;p;m)$.

Theorem 3.1. If Re(α) > 0, Re (β) > 0, Re (c) > Re (γ)> 0, p ≥ 0, m >0 and n $\in \mathbf{N}$ then the following formula holds true :

$$\frac{d^{n}}{dz^{n}}\left[\mathbf{E}^{\gamma;c}_{\alpha;\beta}(z;p;m)\right]=(c)_{n}\,\mathbf{E}^{\gamma+n;c+n}_{\alpha,\beta+n\alpha}(z;p;m) \qquad (3.1)$$

Proof : Differentiating (1.5) with respect to z, we have

$$\frac{d}{dz}\left[\mathbf{E}^{\gamma;c}_{\alpha;\beta}(z;p;m)\right]=c.\mathbf{E}^{\gamma+1;c+1}_{\alpha,\beta+\alpha}(z;p;m) \qquad (3.2)$$

Again, differentiating (3.2) with respect to z, we get

$$\frac{d^{2}}{dz^{2}}\left[\mathbf{E}^{\gamma;c}_{\alpha;\beta}(z;p;m)\right]=c(c+1).\mathbf{E}^{\gamma+2;c+2}_{\alpha,\beta+2\alpha}(z;p;m) \qquad (3.3)$$

On continuing the above process n times, we get the desired result.

IV. Mellin Transform of the extended Mittag-Leffler function

In this section, we establish the Mellin transform of the extended Mittag-Leffler function in terms of the Wright generalized hypergeometric function.

Definition : The Wright generalized hypergeometric function denoted by ${}_p\psi_q$, $\left(p,q\in\mathbf{N}_0\right)$ is defined as [8]

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots\ldots\ldots\ldots,(a_p,A_p)\\(b_1,B_1),\,(b_2,\,B_2),\ldots\ldots\ldots\ldots\ldots,(b_q,B_q)\end{matrix};z\right]$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_in)}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)}\cdot\frac{z^{n}}{n!}, \qquad (4.1)$$

where the coefficient $A_i\,(i=1,2,\ldots p)$ and $B_j\,(j=1,2,\ldots q)$ are positive real numbers such that $1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i\geq 0$.

Theorem 4.1. The Mellin Transform of the extended Mittag-Leffler function $\mathbf{E}^{\gamma,c}_{\alpha,\beta}(z;p;m)$ is given by

$$\mathbf{M}\left\{\mathbf{E}^{\gamma,c}_{\alpha,\beta}(z,\,p\,;m);s\right\}=\frac{\Gamma(s)\Gamma(c+sm-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\,{}_2\psi_2\left[\begin{matrix}(c,1),\,(\gamma+s,1)\\(\beta,\gamma),(c+2s,1)\end{matrix};z\right] \qquad (4.2)$$

where p ≥ 0, m >0, Re (c) > Re (γ) > 0, Re (α)> 0, Re (s) > 0, Re(β)>0 and ${}_2\psi_2$ is the Wright generalized hypergeometric
function.

Proof : On taking the Mellin transform of the extended Mittag Leffler function $\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;\ p\ ;m)$ and using (2.1), we have

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;\ p;m);\ s\right\} = \int_0^\infty p^{s-1}\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\,dp$$

$$= \frac{1}{\mathbf{B}(\gamma,c-\gamma)}\int_0^\infty p^{s-1}\left[\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}\right.$$

$$\left.\times\mathbf{E}_{\alpha,\beta}^{c}(t\ z)\ dt\right]dp\,, \tag{4.3}$$

Changing the order of integration in (4.3), which is valid due to conditions given in the theorem, we get

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\ ;\ s\right\} = \frac{1}{\mathbf{B}(\gamma,c-\gamma)}\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\,\mathbf{E}_{\alpha,\beta}^{c}(tz)$$

$$\times\left[\int_0^\infty p^{s-1}\exp\left\{-\frac{p}{t^m(1-t)^m}\right\}dp\right]dt \tag{4.4}$$

Using the definition of Gamma function in (4.4), we have

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\ ;s\right\}=\frac{\Gamma(s)}{\mathbf{B}(\gamma,c-\gamma)}\int_0^1 t^{\gamma+sm-1}(1-t)^{c+sm-\gamma-1}\ \mathbf{E}_{\alpha,\beta}^{c}(tz)\,dt \tag{4.5}$$

Substituting for $\mathbf{E}_{\alpha,\beta}^{c}(tz)$ using (1.3) in (4.5) then interchanging the order of integration and summation, we obtain

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\ ;s\right\}=\frac{\Gamma(s)}{\mathbf{B}(\gamma,c-\gamma)}\cdot\sum_{k=0}^{\infty}\frac{(c)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$

$$\times\int_0^1 t^{\gamma+k+sm-1}\ (1-t)^{c+sm-\gamma-1}\ dt\,, \tag{4.6}$$

On using Beta function in (4.6), we get

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\ ;s\right\}=\frac{\Gamma(s)}{\mathbf{B}(\gamma,c-\gamma)}\cdot\sum_{k=0}^{\infty}\frac{(c)_k}{\Gamma(\alpha k+\beta)}.$$

$$\times\frac{\Gamma(\gamma+sm+k)\Gamma(c+sm-\gamma)}{\Gamma(c+2sm+k)}\cdot\frac{z^k}{k!}$$

$$=\frac{\Gamma(s)\,\Gamma(c+sm-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\cdot\sum_{k=0}^{\infty}\frac{\Gamma(c+k)\Gamma(\gamma+sm+k)}{\Gamma(\beta+\alpha k)\Gamma(c+2sm+k)}.\frac{z^k}{k!}\,. \tag{4.7}$$

Using the definition of Wright generalized hypergeometric function (4.1) in (4.7), we get

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\ ;s\right\}=\frac{\Gamma(s)\Gamma(c+sm-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\ {}_2\psi_2\left[\begin{matrix}(c,1)\ ,\ (\gamma+sm,1)\\(\beta,\ \alpha)\ ,\ (c+2sm,\ 1)\end{matrix};\ z\right]$$

Corollary (i) : For s = 1, we have

$$\mathbf{M}\left\{\mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m),1\right\} = \int_0^\infty \mathbf{E}_{\alpha,\beta}^{\gamma;c}(z;p;m)\,dp$$

$$=\frac{\Gamma(c+m-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\ {}_2\psi_2\left[\begin{matrix}\Gamma(c,1)\ ,\ (\gamma+m,1)\\(\beta,\ \alpha)\ ,\ (c+2m,\ 1)\end{matrix};\ z\right]$$

Corollary (ii) :By taking the inverse Mellin transform of both sides of (4.2), we get

$$\mathbf{E}_{\alpha,\beta}^{\gamma;c}\left(z;p;m\right)=\frac{1}{2\pi i}\cdot\frac{1}{\Gamma\left(\gamma\right)\Gamma\left(c-\gamma\right)}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(s\right)\Gamma\left(c+sm-\gamma\right)$$
$$\times{}_2\psi_2\left[\begin{matrix}\left(c,1\right),\ \left(\gamma+sm,1\right)\\ \left(\beta,\alpha\right),\left(c+2sm,1\right)\end{matrix};z\right]p^{-s}\ ds,$$

where μ> 0.

Note : For m = 1, we get few results given by Ozarslan et.al. [5] as a special cases.

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