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Circulant Graphs without Cayley Isomorphism Property with m_j = 7

V. Vilfred[1*] and P. Wilson[2,]
Department of Mathematics
[1]*Central University of Kerala, Tejaswini Hills, Periye – 671 316, Kasaragod, Kerala, India.*
[2]*S.T. Hindu College, Nagercoil – 629 002, Kanyakumari District, Tamil Nadu, India.*

Abstract: *A circulant graph $C_n(R)$ is said to have the Cayley Isomorphism (CI) property if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which S = aR. In this paper, we prove that for $1 \le n$, $3 \le k$, $1 \le i \le 7$, $d_i = 7n(i-1)+1$ and $R_i = \{d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, . . . , 294n-d_i, 294n+d_i, 343n-d_i, 7p_1, 7p_2, . . . , 7p_{k-2}\}$, graphs $C_{343n}(R_i)$ are circulant without CI-property with $m_j = gcd(343n, r_j) = 7$, $r_j \in R_i$, $gcd(p_1,p_2,...,p_{k-2}) = 1$ and $n,p_1,p_2,...,p_{k-2} \in N$.*

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I. Introduction

In 1846 Catalan (cf. **[3]**) introduced circulant matrix. If a graph G is circulant, then its adjacency matrix $A(G)$ is circulant. It follows that if the first row of the adjacency matrix of a circulant graph is $[a_1,a_2,\ldots,a_n]$, then $a_1 = 0$ and $a_i = a_{n-i+2}$, $2 \le i \le n$ **[3], [8]**. Circulant graphs have been investigated by many authors **[1]-[15]**. An excellent account can be found in the book by Davis **[3]** and in **[6]**.

Cayley Isomorphism (CI) *problem* determines which graphs (or which groups) have the CI-property and its investigation started with the investigation of isomorphism of circulant graphs. An important achievement in this area is the complete classification of cyclic CI-groups by Muzychuk **[7], [9]**. But study on graphs without CI-property is not much done. Type-2 isomorphism, a new type of isomorphism of circulant graphs other than already known Adam's isomorphism, was defined and studied in **[10], [12]**. Type-2 isomorphic circulant graphs have the property that they are isomorphic circulant graphs without CI-property. Theorems 1.9, 1.10 and 1.11 give classes of isomorphic circulant graphs of Type 2 (and without CI-property) with m_j = 2, 3 or 5. In this paper, we obtain new families of circulant graphs without *CI*-property with m_j = 7 and prove that for $1 \le n$, $3 \le k$, $1 \le i \le 7$, $d_i = 7n(i-1)+1$ and $R_i = \{d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, . . . , 294n-d_i, 294n+d_i, 343n-d_i, 7p_1, 7p_2, . . . , 7p_{k-2}\}$, circulant graphs $C_{343n}(R_i)$ are graphs without *CI*-property with $m_j = gcd(343n, r_j) = 7$, $r_j \in R_i$, $gcd(p_1,p_2,\ldots,p_{k-2}) = 1$ and $n,p_1,p_2,\ldots,p_{k-2} \in N$.

Through-out this paper, for a set $R = \{r_1, r_2, \ldots, r_k\}$, $C_n(R)$ denotes circulant graph $C_n(r_1, r_2, \ldots, r_k)$ where $1 \le r_1 < r_2 < \cdots < r_k \le [n/2]$. We consider only connected circulant graphs of finite order, $V(C_n(R)) = \{v_0,v_1,v_2,\ldots,v_{n-1}\}$ with v_i adjacent to v_{i+r} for each $r \in R$, subscript addition taken modulo n and all cycles have length at least 3, unless otherwise specified, $0 \le i \le n-1$. However when $\frac{n}{2} \in R$, edge $v_i v_{i+\frac{n}{2}}$ is taken as a single edge for considering the degree of the vertex v_i or $v_{i+\frac{n}{2}}$ and as a double edge while counting the number of edges or cycles in $C_n(R)$, $0 \le i \le n-1$. We will often assume, with-out further comment, that the vertices of $C_n(R)$ are the corners of a regular n-gon, labeled clockwise. Circulant graph is also defined as a Cayley graph or digraph of a cyclic group.Isomorphic circulant graphs $C_{16}(1,2,7)$ and $C_{16}(2,3,5)$ are given in Figures 1 and 2 and isomorphic circulant graphs $C_{27}(1,3,8,10)$, $C_{27}(3,4,5,13)$ and $C_{27}(2,3,7,11)$ are shown in Figures 3, 4 and 5, respectively.

Theorem 1.1 [11] *If $C_n(R) \cong C_n(S)$, then there is a bijection f from R to S so that for all $r \in R$, gcd(n, r) = gcd(n,f(r)).*

Proof: The proof is by induction on the order of R. □

Definition 1.2 [7] A circulant graph $C_n(R)$ is said to have the *CI-property* if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which $S = aR$.

Lemma 1.3 [12] *Let S be a non-empty subset of Z_n and $x \in Z_n$. Define a mapping $\Phi_{n,x}: S \to Z_n$ such that $\Phi_{n,x}(s) = xs$ for every $s \in S$ under multiplication modulo n. Then $\Phi_{n,x}$ is bijective if and only if $S = Z_n$ and gcd(n,x) = 1.* □

Definition 1.4 [1] Circulant graphs, $C_n(R)$ and $C_n(S)$ for $R = \{r_1,r_2,\ldots,r_k\}$ and $S = \{s_1,s_2,\ldots,s_k\}$ are *Adam's isomorphic* or *Type*-1 *isomorphic* if there exists a positive integer x relatively prime to n with S =

$\{xr_1, xr_2, \ldots, xr_k\}_n^*$ where $< r_i >_n^*$, the *reflexive modular reduction* of a sequence $<r_i>$ is the sequence obtained by reducing each r_i modulo n to yield r_i' and then replacing all resulting terms r_i' which are larger than $\frac{n}{2}$ by n-r_i'.

Lemma 1.5 [12] *Let* $j,m,q,r,t,x \in Z_n$ *such that* $gcd(n, r) = m > 1$, $x = j+qm$, $0 \le j \le m$-1 *and* $0 \le q,t \le \frac{n}{m}$ -1. *Then the mapping* $\theta_{n,r,t}$: $Z_n \rightarrow Z_n$ *defined by* $\theta_{n,r,t}(x) = x+jtm$ *for every* $x \in Z_n$ *under arithmetic modulo n is bijective.*

Proof: From the definition of $\theta_{n,r,t}$ we get the following properties:

i) $\theta_{n,r,t}(km) = km$ for every $k \in Z_n$, $km \in Z_n$.

ii) For $0 \le i,j \le m$-1, $\theta_{n,r,t}(i) = \theta_{n,r,t}(j)$ if and only if $i = j$ if and only if $\theta_{n,r,t}(i+qm) = \theta_{n,r,t}(j+qm)$, $0 \le qm \le n$-1 and

iii) For $0 \le i \le m$-1 and $0 \le km,qm \le n$-1, $\theta_{n,r,t}(i+km) = \theta_{n,r,t}(i+qm)$ if and only if $k = q$.

From the above three properties, we get,

iv) For $0 \le i,j \le m$-1 and $0 \le km,qm \le n$-1, $\theta_{n,r,t}(i+km) = \theta_{n,r,t}(j+qm)$ if and only if $i = j$ and $k = q$. This implies that the mapping $\theta_{n,r,t}$ is bijective.
Hence the result follows. □

Theorem 1.6 [12] *Let* $V(C_n(R)) = \{v_0,v_1,v_2,\ldots,v_{n-1}\}$, $V(K_n) = \{u_0,u_1,u_2,\ldots,u_{n-1}\}$, $r \in R$ *and* $j,m,q,t,x \in Z_n$ *such that* $gcd(n, r) = m > 1$, $x = j+qm$, $0 \le j \le m$-1 *and* $0 \le q,t \le \frac{n}{m}$ -1. *Then the mapping* $\theta_{n,r,t}$: $V(C_n(R)) \rightarrow V(C_n(1,2,\ldots,n\text{-}1)) = V(K_n)$ *defined by* $\theta_{n,r,t}(v_x) = u_{x+jtm}$ *and* $\theta_{n,r,t}((v_x,v_{x+s})) = (\theta_{n,r,t}(v_x), \theta_{n,r,t}(v_{x+s}))$ *for every* $x \in Z_n$ *and* $s \in R$, *under subscript arithmetic modulo n, for a set* $R = \{r_1,r_2,\ldots,r_k, n\text{-}r_k,n\text{-}r_{k-1},\ldots,r_1\}$ *is one-to-one, preserves adjacency and* $\theta_{n,r,t}(C_n(R)) \cong C_n(R)$ *for* $t = 0,1,2,\ldots,\frac{n}{m}$ - 1. □

And for a particular value of t if $\theta_{n,r,t}(C_n(R)) = C_n(S)$ for some $S \subseteq [1, [n/2]]$ and $S \neq xR$ for all $x \in \Phi_n$ under reflexive modulo n, then $C_n(R)$ and $C_n(S)$ are called *Type-2isomorphiccirculant graphs w.r.t. r*, $0 \le q,t \le \frac{n}{m}$ -1.

Definition 1.7 [12] The *symmetric equidistance condition* with respect to v_i in $C_n(R)$ for a set $R = \{r_1,r_2,\ldots,r_k\}$ is that v_{i+j} is adjacent to v_i if and only if v_{n-j+i} is adjacent to v_i, using subscript arithmetic modulo n, $0 \le i,j \le n$-1.

Theorem 1.8 [12] *For a set* $R = \{r_1,r_2,\ldots,r_k\} \subseteq [1, n/2]$, $1 \le i \le k$ *and* $0 \le t \le \frac{n}{m}$ -1, $\theta_{n,r_i,t}(C_n(R)) = C_n(S)$ *for some* $S \subseteq [1, n/2]$ *if and only if* $\theta_{n,r_i,t}(C_n(R))$ *satisfies the symmetric equidistance condition w.r.t.* v_0. □

Theorem 1.9 [12] *For* $2 \le n$, $3 \le k$, $1 \le 2s$-1 $\le 2n$-1, $n \neq 2s$-1, $R = \{2s\text{-}1, 4n\text{-}2s\text{+}1, 2p_1,2p_2,\ldots,2p_{k-2}\}$ and $S = \{2n\text{-}2s\text{+}1, 2n\text{+}2s\text{-}1, 2p_1,2p_2,\ldots,2p_{k-2}\}$, *circulant graphs* $C_{8n}(R)$ *and* $C_{8n}(S)$ *are Type-2 isomorphic* (*and without CI-property*) *where* $gcd(p_1,p_2,\ldots,p_{k-2}) = 1$ and $n,s,p_1,p_2,\ldots,p_{k-2} \in N$. □

Theorem 1.10 [14] *For* $3 \le k$, $R = \{1, 9n\text{-}1, 9n\text{+}1, 3p_1,3p_2,\ldots,3p_{k-2}\}$, $S = \{3n\text{+}1, 6n\text{-}1, 12n\text{+}1, 3p_1,3p_2,\ldots,3p_{k-2}\}$ *and* $T = \{3n\text{-}1, 6n\text{+}1, 12n\text{-}1, 3p_1,3p_2,\ldots,3p_{k-2}\}$, $C_{27n}(R)$, $C_{27n}(S)$ *and* $C_{27n}(T)$ *are Type-2 isomorphic* (*and without CI-property*) *where* $gcd(p_1,p_2,\ldots,p_{k-2}) = 1$ and $n,p_1,p_2,\ldots,p_{k-2} \in N$. □

Theorem 1.11 [15] *For* i = 1 to 5, $d_i = 5n(i\text{-}1)+1$, $3 \le k$ *and* $R_i = \{d_i, 25n\text{-}d_i, 25n\text{+}d_i, 50n\text{-}d_i, 50n\text{+}d_i, 5p_1,5p_2,\ldots,5p_{k-2}\}$, *circulant graphs* $C_{125n}(R_i)$ *are Type*-2 *isomorphic* (*and without CI-property*) *where* $gcd(p_1,p_2,\ldots,p_{k-2}) = 1$ *and* $n,p_1,p_2,\ldots,p_{k-2} \in N$. □

Theorem 1.12 [12] *For* $R = \{2, 2s\text{-}1, 2s'\text{-}1\}$, $1 \le t \le [\frac{n}{2}]$, $1 \le 2s\text{-}1 < 2s'\text{-}1 \le [\frac{n}{2}]$ *and* $n,s,s',t \in N$, *if* $C_n(R)$ *and* $\theta_{n,2,t}(C_n(R))$ *are Type-2 isomorphic circulant graphs for some t, then* $n \equiv 0$ (*mod* 8), $2s\text{-}1+2s'\text{-}1 = \frac{n}{2}$, $t = \frac{n}{8}$ *or* $\frac{3n}{8}$, $2s'\text{-}1 \neq \frac{n}{8}$, $1 \le 2s\text{-}1 \le \frac{n}{4}$ *and* $16 \le n$. □

Theorem 1.13 [12] *Let* $x \in Z_n$. *Define mapping* $\Phi_{n,x}$: $V(C_n(R)) \rightarrow V(K_n)$ *for a set* $R = \{r_1,r_2,\ldots,r_k, n\text{-}r_k,n\text{-}r_{k-1}, \ldots, n\text{-}r_1\}$ *such that* $\Phi_{n,x}(v_i) = u_{xi}$ *and* $\Phi_{n,x}((v_i, v_{i+s})) = (\Phi_{n,x}(v_i), \Phi_{n,x}(v_{i+s}))$ *for every* $s \in R$ *and* $i \in Z_n$ *under subscript arithmetic modulo n where* $V(C_n(R)) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $V(K_n) = \{u_0,u_1,\ldots,u_{n-1}\}$. *Then* $\Phi_{n,x}(C_n(R)) = C_n(xR)$ *and the mapping* $\Phi_{n,x}$ *is one-to-one if and only if* $gcd(n, x) = 1$. □

Definition 1.14 [12] Let $Ad_n(C_n(R)) = T1_n(C_n(R)) = \{\Phi_{n,x}(C_n(R)): x \in \Phi_n\} = \{C_n(xR): x \in \Phi_n\}$ for a set $R = \{r_1,r_2,\ldots,r_k, n\text{-}r_k,n\text{-}r_{k-1},\ldots,n\text{-}r_1\}$. Define 'o' in $Ad_n(C_n(R))$ such that $\Phi_{n,x}(C_n(R))$ o $\Phi_{n,y}(C_n(R)) = \Phi_{n,xy}(C_n(R))$ and $C_n(xR)$ o $C_n(yR) = C_n((xy)R)$ for every $x,y \in \Phi_n$, under arithmetic modulo n. Clearly, $Ad_n(C_n(R))$ is the set of all circulant graphs which are Adam's isomorphic to $C_n(R)$ and $(Ad_n(C_n(R))$, o) = $(T1_n(C_n(R))$, o) is an abelian group called *the Adam's group* or *theType*-1 *group on* $C_n(R)$ under 'o'.

Definition 1.15 [12] Let $V(C_n(R)) = \{v_0,v_1,v_2,\ldots,v_{n-1}\}$, $V(K_n) = \{u_0,u_1,u_2,\ldots,u_{n-1}\}$, $r \in R$, $m,q,t,t',x \in Z_n$ such that $gcd(n, r) = m > 1$, $x = j+qm$, $0 \le j \le m$-1 and $0 \le q,t,t' \le \frac{n}{m}$ -1. Define $\theta_{n,r,t}$: $Z_n \rightarrow Z_n$ and $\theta_{n,r,t}$: $V(C_n(R)) \rightarrow V(C_n(1,2,\ldots,n\text{-}1)) = V(K_n)$ such that $\theta_{n,r,t}(x) = x+jtm$, $\theta_{n,r,t}(v_x) = u_{x+jtm}$ and $\theta_{n,r,t}((v_x,v_{x+s})) = (\theta_{n,r,t}(v_x), \theta_{n,r,t}(v_{x+s}))$ for every $x \in Z_n$ and $s \in R$, under arithmetic modulo n. Let $s \in Z_n$, $V_{n,r} = \{\theta_{n,r,t}: t = 0,1,\ldots,\frac{n}{m} -1\}$,

$V_{n,r}(s)$ = {$\theta_{n,r,t}(s)$: t = 0,1,…,$\frac{n}{m}$ -1} and $V_{n,r}(C_n(R))$ = {$\theta_{n,r,t}(C_n(R))$: t = 0,1,…,$\frac{n}{m}$ -1}. Define 'o' in $V_{n,r}$ such that $\theta_{n,r,t}$ o $\theta_{n,r,t'}$ = $\theta_{n,r,t+t'}$, ($\theta_{n,r,t} o \theta_{n,r,t'}$)($x$) (= $\theta_{n,r,t}(\theta_{n,r,t'}(x))$ = $\theta_{n,r,t}$(x+$jt'm$) = (x+$jt'm$)+jtm = x+j(t+t')m) = $\theta_{n,r,t+t'}(x)$ and $\theta_{n,r,t}(C_n(R))$ o $\theta_{n,r,t'}(C_n(R))$ = $\theta_{n,r,t+t'}(C_n(R))$ for every $\theta_{n,r,t},\theta_{n,r,t'} \in V_{n,r}$ where t+t' is calculated under addition modulo $\frac{n}{m}$. Clearly, for every $s \in Z_n$, ($V_{n,r}(s)$, o) and ($V_{n,r}(C_n(R))$, o) are *abelian groups*.

$V_{n,r}(C_n(R))$ contains all isomorphic circulant graphs of Type 2 of $C_n(R)$ w.r.t. r, if exist. Let $T2_{n,r}(C_n(R))$ = {$C_n(R)$} ∪ {$C_n(S)$: $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. r}. Thus, $T2_{n,r}(C_n(R))$ = {$C_n(R)$} ∪ {$\theta_{n,r,t}(C_n(R))$: $\theta_{n,r,t}(C_n(R))$ = $C_n(S)$ and $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. r, 0 ≤ t ≤ $\frac{n}{m}$ -1} ⊆ $V_{n,r}(C_n(R))$ and ($T2_{n,r}(C_n(R))$, o) is a subgroup of ($V_{n,r}(C_n(R))$, o). Clearly, $T1_n(C_n(R)) \cap T2_{n,r}(C_n(R))$ = {$C_n(R)$}. $C_n(R)$ has Type-2 isomorphic circulant graph w.r.t. r iff $T2_{n,r}(C_n(R)) \neq$ {$C_n(R)$} iff $T2_{n,r}(C_n(R)) \cap$ {$C_n(R)$} ≠Φ iff |$T2_{n,r}(C_n(R))$| > 1 **[14]**.

Definition 1.16 [14] For any circulant graph $C_n(R)$, if $T2_{n,r}(C_n(R)) \neq$ {$C_n(R)$}, then ($T2_{n,r}(C_n(R))$, o) is called *the Type-2 group of* $C_n(R)$ *w.r.t. r* under 'o'.

Effort to obtain new families of circulant graphs without CI-property is the motivation for this work. For all basic ideas in graph theory, we follow **[5]**.

II. Main result

Theorem 2.1 *For i = 1 to 7, n∈N,* d_i = 7n(i-1)+1 *and* R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, *circulant graphs* $C_{343n}(R_i)$ *are isomorphic.*

Proof: We prove that for i = 1 to 7, d_i = 7n(i-1)+1 and R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, $\theta_{343n,7,in}(C_{343n}(R_1))$ = $C_{343n}(R_{i+1})$ where i+1 is calculated under addition modulo 7.

To simplify our calculation let us consider R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, … , 294n-d_i, 294n+d_i, 343n-d_i, 343n-7}, d_i = 7n(i-1)+1 and i = 1 to 7. In particular,

R_1 = {1, 7, 49n-1, 49n+1, 98n-1, 98n+1, 147n-1, 147n+1, 196n-1, 196n+1, 245n-1, 245n+1, 294n-1, 294n+1, 343n-7, 343n-1},

R_2 = {7, 7n+1, 42n-1, 56n+1, 91n-1, 105n+1, 140n-1, 154n+1, 189n-1, 203n+1, 238n-1, 252n+1, 287n-1, 301n+1, 336n-1, 343n-7},

R_3 = {7, 14n+1, 35n-1, 63n+1, 84n-1, 112n+1, 133n-1, 161n+1, 182n-1, 210n+1, 231n-1, 259n+1, 280n-1, 308n+1, 329n-1, 343n-7},

R_4= {7, 21n+1, 28n-1, 70n+1, 77n-1, 119n+1, 126n-1, 168n+1, 175n-1, 217n+1, 224n-1, 266n+1, 273n-1, 315n+1, 322n-1, 343n-7},

R_5 = {7, 21n-1, 28n+1, 70n-1, 77n+1, 119n-1, 126n+1, 168n-1, 175n+1, 217n-1, 224n+1, 266n-1, 273n+1, 315n-1, 322n+1, 343n-7},

R_6 = {7, 14n-1, 35n+1, 63n-1, 84n+1, 112n-1, 133n+1, 161n-1, 182n+1, 210n-1, 231n+1, 259n-1, 280n+1, 308n-1, 329n+1, 343n-7},

R_7 = {7, 7n-1, 42n+1, 56n-1, 91n+1, 105n-1, 140n+1, 154n-1, 189n+1, 203n-1, 238n+1, 252n-1, 287n+1, 301n-1, 336n+1, 343n-7}.

For 1 ≤ i,j ≤ 7, using the definition of $\theta_{n,r,t}$, we get the following:

$\theta_{343n,7,n}(R_1)$ = $\theta_{343n,7,n}$({1, 7, 49n-1, 49n+1, 98n-1, 98n+1, 147n-1, 147n+1, 196n-1, 196n+1, 245n-1, 245n+1, 294n-1, 294n+1, 343n-7, 343n-1}) = $\theta_{343n,7,n}$({7, 343n-7}) ∪ $\theta_{343n,7,n}$({1, 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1}) ∪ $\theta_{343n,7,n}$({49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1}) = {7, 343n-7} ∪ (7n+({1, 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1})) ∪ (42n+({49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1})) = {7, 343n-7} ∪ {7n+1, 56n+1, 105n+1, 154n+1, 203n+1, 252n+1, 301n+1} ∪ {91n-1, 140n-1, 189n-1, 238n-1, 287n-1, 336n-1, 42n-1} = R_2;

$\theta_{343n,7,in}(R_1)$ = $\theta_{343n,7,in}$({7, 343n-7}) ∪ $\theta_{343n,7,in}$({1, 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1}) ∪ $\theta_{343n,7,in}$({49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1}) = {7, 343n-7} ∪ (7in+({1, 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1})) ∪ (42in+({49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1})) = {7, 343n-7} ∪ {7in+1, 49n+7in+1, 98n+7in+1, 147n+7in+1, 196n+7in+1, 245n+7in+1, 294n+7in+1} ∪ {49n+42in-1 = (49+49i)n-(7in+1), 98n+42in-1 = (2x49+49i)n-(7in+1), 147n+42in-1 = (3x49+49i)n-(7in+1), 196n+42in-1 = (4x49+49i)n-(7in+1), 245n+42in-1 = (5x49+49i)n-(7in+1), 294n+42in-1 = (6x49+49i)n-(7in+1), 343n+42in-1 = (7x49+49i)n-(7in+1) = (0x49+49i)n-(7in+1)} = R_{i+1} where d_{i+1} = 7in+1.

In a similar way we can prove that for 1 ≤ i,j ≤ 7, $\theta_{343n,7,jn}(R_i)$ = R_{i+j} where i+j is calculated under addition modulo 7. This implies that for 1 ≤ i,j ≤ 7, $\theta_{343n,7,jn}(C_{343n}(R_i))$ = $C_{343n}(R_{i+j})$ where i+j is calculated under addition modulo 7.

Hence the result follows since the mapping $\theta_{n,r,t}$ is one-to-one and preserves adjacency on circulant graph $C_n(R)$. □

Theorem 2.2 *For* i = 1 to 7, $n \in N$, $d_i = 7n(i\text{-}1)+1$ *and* R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, $\theta_{343n,7,jn}(C_{343n}(R_i)) = C_{343n}(R_{i+j})$ *where i+j is calculated under addition modulo* 7 *and* $C_{343n}(R_i)$ *are Type-*2 *isomorphic circulant graphs.*

Proof: To prove that for i = 1,2,…,7, circulant graphs $C_{343n}(R_i)$ are of Type-2 isomorphic, it is enough to prove that every pair of the circulant graphs are different (not the same), isomorphic and not of Adam's isomorphic.

When R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, d_i = 7n(i-1)+1, 1 ≤ i,j ≤ 7 and $n \in N$, R_i = R_j iff i = j. Thus for different i, the set of jump sizes of the seven circulant graphs $C_{343n}(R_i)$ are different and thereby the seven circulant graphs are also different.

In the proof of Theorem 2.1, we have seen that when R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, d_i = 7n(i-1)+1, 1 ≤ i,j ≤ 7and $n \in N$, $\theta_{343n,7,in}(C_{343n}(R_j)) = C_{343n}(R_{i+j})$ where i+j is calculated under addition modulo 7. This implies that for i = 1 to 7 all the seven circulant graphs $C_{343n}(R_i)$ are isomorphic since the mapping $\theta_{n,r,t}$ is one-to-one and preserves adjacency on circulant graph $C_n(R)$.

To complete the proof we are left with establishing their isomorphism is of Type-2. Now it is enough to prove that each pair of isomorphic circulant graphs $C_{343n}(R_i)$ and $C_{343n}(R_j)$ for i≠j are not of Type-1, 1 ≤ i,j ≤ 7. At first let us prove the result for the circulant graph $C_{343n}(R_1)$.

Claim: $C_{343n}(R_1)$ and $C_{343n}(R_i)$ are Type-2 isomorphic for every i, 2 ≤ i ≤ 7.

If not, they are of Adam's isomorphic. This implies, there exists $s \in N$ such that $C_{343n}(sR_1) = C_{343n}(R_i)$ where 2 ≤ i ≤ 7, s = 7x-j, $x \in N$, j = 1 to 6, 1 ≤ 7x-j ≤ 343n-1 and gcd(343n, s) = 1. In particular, now choose s such that s = 7x-1, gcd(343n, 7x-1) = 1, $C_{343n}((7x-1)R_1) = C_{343n}(R_i)$, 2 ≤ i ≤ 7 and $x \in N$. This implies, (7x-1){1,7,49n-1, 49n+1, 98n-1, 98n+1, 147n-1, 147n+1, 196n-1, 196n+1, 245n-1, 245n+1, 294n-1, 294n+1, 343n-7, 343n-1} = {7x-1,7(7x-1),(7x-1)(49n-1), (7x-1)(49n+1), (7x-1)(98n-1), (7x-1)(98n+1), (7x-1)(147n-1), (7x-1)(147n+1), (7x-1)(196n-1), (7x-1)(196n+1), (7x-1)(245n-1), (7x-1)(245n+1), (7x-1)(294n-1), (7x-1)(294n+1), (7x-1)(343n-7), (7x-1)(343n-1)} under arithmetic modulo 343n. This implies, 7(7x-1), (7x-1)(343n-7), 7+343np_1 and 343n-7+343np_2 are the only numbers, each is a multiple of 7, in the two sets for some p_1,$p_2 \in N_0$. Here the following two cases arise.

Case i 7(7x-1) = 7+343np_1, $p_1 \in N_0$, $x \in N$, 1 ≤ 7x-1 ≤ 343n-1.

In this case, p_1 = 0,1,…,5 or 6 since 1 ≤ 7x-1≤ 343n-1 and n,$x \in N$. When p_1 = 0, 7x-1 = 1; p_1 = 1, 7x-1 = 49n+1; p_1 = 2, 7x-1 = 98n+1; p_1 = 3, 7x-1 = 147n+1; p_1 = 4, 7x-1 = 196n+1; p_1 = 5, 7x-1 = 245n+1; p_1 = 6, 7x-1 = 294n+1. Now let us calculate $(7x-1)R_1$ for 7x-1 = 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1 under arithmetic modulo 343n.

When 7x-1 = 49n+1, under arithmetic modulo 343n,

$(7x-1)R_1 = (49n+1)R_1$ = (49n+1){1, 7, 49n-1, 49n+1, 98n-1, 98n+1, 147n-1, 147n+1, 196n-1, 196n+1, 245n-1, 245n+1, 294n-1, 294n+1, 343n-7, 343n-1}
= {49n+1, 7, 343n-1, 98n+1, 49n-1, 147n+1, 98n-1, 196n+1, 147n-1, 245n+1, 196n-1, 294n+1, 245n-1, 1, 343n-7, 294n-1}= R_1.

Similarly, we can prove that $(7x-1)R_1 = R_1$ when 7x-1 = 98n+1, 147n+1, 196n+1, 245n+1 or 294n+1 under arithmetic modulo 343n. This implies, $C_{343n}((7x-1)R_1) = C_{343n}(R_1)$ when 7x-1 = 49n+1, 98n+1,147n+1, 196n+1, 245n+1 or 294n+1. Similarly, we can prove that for j = 2,3,4,5,6, $(7x-j)R_1 = R_1$under arithmetic modulo 343n when 7x-j = 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1. This implies, $C_{343n}((7x-j)R_1) = C_{343n}(R_1)$ for j = 1,2,…,6 and 7x-j = 49n+1, 98n+1, 147n+1, 196n+1, 245n+1, 294n+1.

Case ii 7(7x-1) = 343n-7+343np_2, $p_2 \in N_0$, $x \in N$, 1 ≤ 7x-1 ≤ 343n-1.

In this case, p_2 = 0,1,2,3,4,5 or 6 since 1 ≤ 7x-1 ≤ 343n-1 and n,$x \in N$. When p_2 = 0, 7x-1 = 49n-1; p_2 = 1, 7x-1 = 98n-1; p_2 = 2, 7x-1 = 147n-1; p_2 = 3, 7x-1 = 196n-1; p_2 = 4, 7x-1 = 245n-1; p_2 = 5, 7x-1 = 294n-1; p_2 = 6, 7x-1 = 343n-1. Now let us calculate $(7x-1)R_1$ for 7x-1 = 49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1 under arithmetic modulo 343n.

When $(7x-1)$ = 49n-1, under arithmetic modulo 343n,

$(7x-1)R_1 = (49n-1)R_1$ = (49n-1){1, 7, 49n-1, 49n+1, 98n-1, 98n+1, 147n-1, 147n+1, 196n-1, 196n+1, 245n-1, 245n+1, 294n-1, 294n+1, 343n-7, 343n-1}
= {49n-1, 343n-7, 245n+1, 343n-1, 196n+1, 294n-1, 147n+1, 245n-1,98n+1, 196n-1, 49n+1, 147n-1, 1, 98n-1, 7, 294n+1} = R_1.

Similarly, we can prove that $(7x-1)R_1 = R_1$ when 7x-1 = 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1 under arithmetic modulo 343n. This implies, $C_{343n}((7x-1)R_1) = C_{343n}(R_1)$ when 7x-1 = 49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1. Similarly, we can prove that $(7x-j)R_1 = R_1$, under arithmetic modulo 343n, when 7x-j = 49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1for j = 2,3,4,5,6. This implies,

$C_{343n}((7x-j)R_1) = C_{343n}(R_1)$ when $7x$-j = 49n-1, 98n-1, 147n-1, 196n-1, 245n-1, 294n-1, 343n-1for j = 1,2,3,4,5,6.

This implies, $C_{343n}(R_1)$ is not Adam's isomorphic to all the other six isomorphic circulant graphs. Similarly, we can prove that $C_{343n}(R_i)$ is not Adam's isomorphic to all the other six circulant graphs, 1≤ i ≤7. This implies, all the seven isomorphic circulant graphs $C_{343n}(R_i)$ are Type 2 isomorphic circulant graphs only, 1 ≤ i ≤ 7. ☐

Theorem 2.3 *For* i = 1 to 7, d_i = 7n(i-1)+1, 3 ≤ k *and* R_i = {d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i, 7p_1,7p_2,…,7p_{k-2}}, *circulant graphs* $C_{343n}(R_i)$ *are Type*-2 *isomorphic* (*and without CI-property*) *where gcd*(p_1,p_2,…,p_{k-2}) = 1 *and n*,p_1,p_2,…,p_{k-2}∈N.

Proof: For i = 1 to 7, d_i = 7n(i-1)+1, 3 ≤ k and R_i = {7, d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i}, circulant graphs $C_{343n}(R_i)$ are Type-2 isomorphic, using Theorem 2.2, n∈N. Lemma 1.5 helps us while searching for possible value(s) of t such that the transformed graph $\theta_{n,r,t}(C_n(R))$ is circulant of the form $C_n(S)$ for some $S \subseteq$ [1, n/2], the calculation on r_j which are integer multiples of m = gcd(n, r) need not be done as there is no change in these r_j under the transformation $\theta_{n,r,t}$. Therefore, for i = 1 to 7, d_i = 7n(i-1)+1 and R_i = {d_i, 49n-d_i, 49n+d_i, 98n-d_i, 98n+d_i, 147n-d_i, 147n+d_i, 7p_1,7p_2,…,7p_{k-2}}, circulant graphs $C_{343n}(R_i)$ are Type-2 isomorphic circulant graphs where 3 ≤ k, gcd(p_1,p_2,…,p_{k-2}) = 1 and n,p_1,p_2,…,p_{k-2}∈N. Type-2 isomorphic circulant graphs are graphs *without CI-property*. Hence the result follows. ☐

For n = 1, let

C_{343}(1, 7, 48, 50, 97, 99, 146, 148, 195, 197, 244, 246,293,295,336,342) = $C_{343}(R_1)$,
C_{343}(7, 8, 41, 57, 90, 106, 139, 155, 188, 204, 237, 253, 286, 302, 335, 336)= $C_{343}(R_2)$,
C_{343}(7, 15, 34, 64, 83, 113, 132, 162, 181, 211, 230, 260, 279, 309, 328, 336)= $C_{343}(R_3)$,
C_{343}(7, 22, 27, 71, 76, 120, 125, 169, 174, 218, 223, 267, 272, 316, 321, 336)= $C_{343}(R_4)$,
C_{343}(7, 20, 29, 69, 78, 118, 127, 167, 176, 216, 225, 265, 274, 314, 323, 336)= $C_{343}(R_5)$,
C_{343}(7, 13, 36, 62, 85, 111, 134, 160, 183, 199, 232, 258, 281, 307, 330, 336)= $C_{343}(R_6)$,
C_{343}(7, 6, 43, 55, 92, 104, 141, 153, 190, 192, 239, 251, 288, 300, 337, 336)= $C_{343}(R_7)$.

Then, circulant graphs $C_{343}(R_i)$ are Type 2 isomorphic, 1 ≤ i ≤ 7.

Theorem 2.4 *For i* = 1 to 7, d_i = 7n(i-1)+1, 3 ≤ k *and* R_i = {d_i,49n-d_i,49n+d_i,98n-d_i,98n+d_i,147n-d_i,147n+d_i, 7p_1,7p_2,…,7p_{k-2}}, ($V_{343n,5}(C_{343n}(R_i))$, o) *is an abelian group where gcd*(p_1,p_2,…,p_{k-2}) = 1, n,p_1,p_2,…,p_{k-2}∈N.

Proof: The result follows from Theorem 2.3 and from the definitions of $\theta_{n,r,t}$ and $V_{n,r}$. □

For n = 1 and R_is as given just above Theorem 2.4, ($T2_{343,7}(C_{343}(R_i))$, o) is the required Type 2 group of $C_{343}(R_i)$ w.r.t. r = 7 where $T2_{343,7}(C_{343}(R_i))$ = {$\theta_{343,7,j}(C_{343}(R_i))$: j = 0,1,2,3,4,5,6} = {$C_{343}(R_j)$: j = 1,2,3,4,5,6,7} since $\theta_{343,7,j}(C_{343}(R_i))$ = $C_{343n}(R_{i+j})$ where i+j is calculated under addition modulo 7, 1 ≤ i ≤ 7.

III. Conclusion

In this paper and in **[12]**, **[14], [15]** we obtained families of isomorphic circulant graphs of Type-2 (and without CI-property), each with m_i = gcd(n, r_i) = 2, 3, 5 or 7. One can go for general result with m_i, an odd number greater than 7.

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<u>Figures</u>

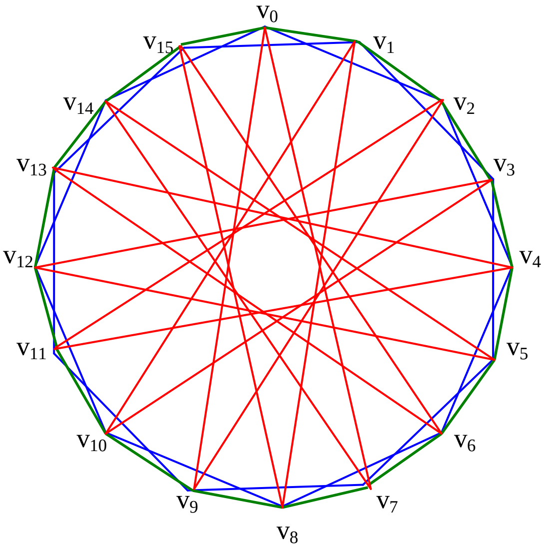

Fig. 1.C_{16}(1,2,7)

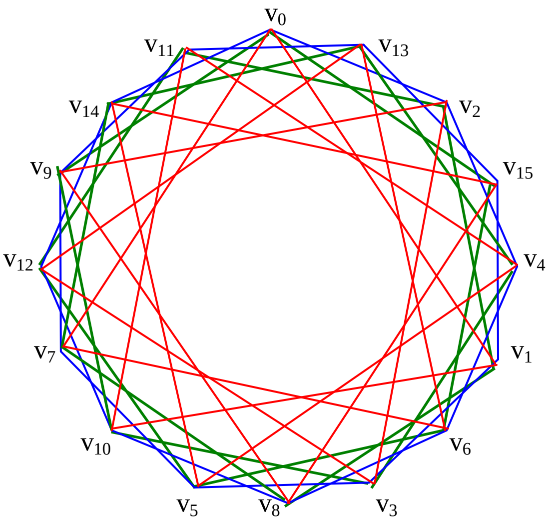

Fig. 2. C_{16}(2,3,5)

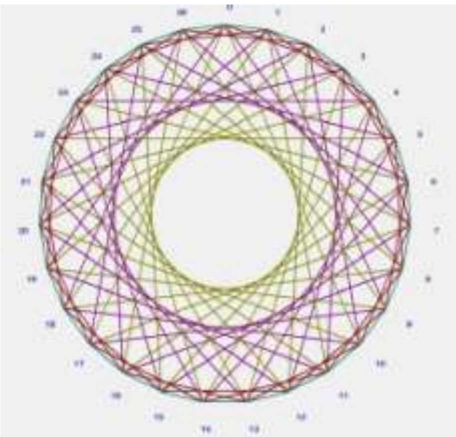

Fig. 3 C_{27}(1,3,8,10)

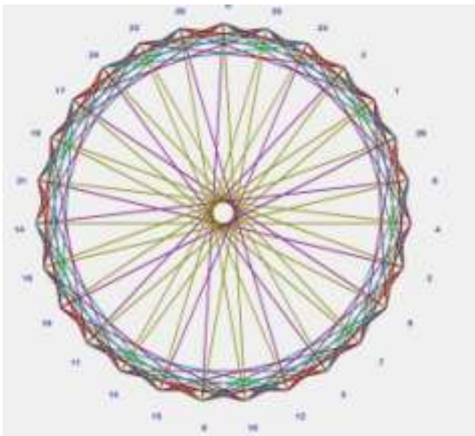

Fig. 4 C_{27}(3,4,5,13)

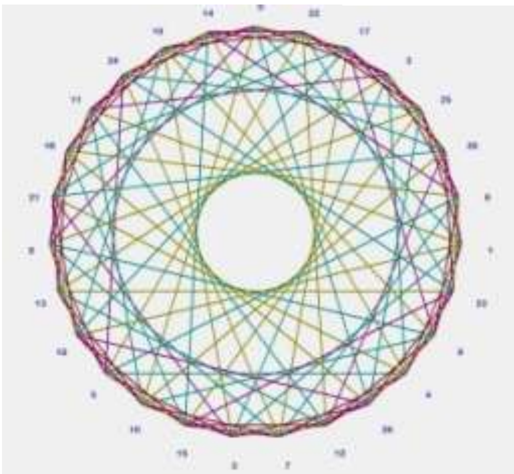

Fig. 5 C_{27}(2,3,7,11)