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$I-Ces\acute{a}$ro Statistical Core Of Double Sequences

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Abstract: *The concept of statistical limit inferior and superior for single sequences was introduced by Fridy and Orhan (1997) where some inequalities and Knopp type core theorem were obtained. Cakan and Altay extended these results to double sequences, similar results were obtained for $C_{1.1}(St_2)$ −sequences by Siddiqui et al. (2012). Demirci (2001), Lahiri and Das (2003) used the same concept to define the I −limit inferior, I −limit superior and obtained some I −analogue of the properties of limit inferior and superior for single sequences. Kumar (2007) further extended this concept to double sequences. In this paper we define $C,^{I}_{1.1}(St_2)$ −limit inferior, superior and $C,^{I}_{1.1}(St_2)$ −Core which are I −analogues of $C_{1.1}(St_2)$ −limit inferior, superior and $C_{1.1}(St_2)$ −Core for double sequences respectively.*
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I. Introduction

Pringsheim (1900) introduced the notion of convergence for double sequences. Using this definition, Robison (1926) and Hamilton [(1936),(1938a),(1938b),(1939)], defined and extensively studied the four dimensional matrix transformation $(Ax)_{mn} = \sum_{ij} a^{mn}_{ij} x_{ij}$. Siddiqui et al. (2012) defined and studied Ces$\acute{a}$ro statistical Core of double sequences $(C_{1.1}(St_2) - Core)$. Using these concepts and the notion of I −Core defined and studied by Kumar (2007), we analogously, define $C^{I}_{1.1}(St_2)$ −limit inferior and superior, $C^{I}_{1.1}(St_2)$ −Core which are I −analogues of the $C_{1.1}(St_2)$ −limit inferior and superior, $C_{1.1}(St_2)$ −Core of double sequences respectively. Inequalities relating $C_{1.1}(St_2)$ −Core and $C^{I}_{1.1}(St_2)$ −Core were also presented.

II. Background and Preliminaries

Throughout the paper $\mathbb{N}, \mathbb{R}$ will denote respectively the sets of positive integers and real numbers where as $\mathbb{N}^2$ will denote the usual product set $\mathbb{N} \times \mathbb{N}$. For any set $X, P(X)$ stands for the power set of X and A^c will denote the complement of the set A.

Definition 2.1: If X is a non-empty set then a family of set $I \subset P(X)$ is called an ideal in X if and only if (i) $\Phi \in I$; (ii) For each $A, B \in I$ we have $A \cup B \in I$; (iii) For each $A \in I$ and $B \subset A$ we have $B \in I$.

Definition 2.2: Let X is a non-empty set. A non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if (i) $\Phi \notin F$; (ii) For each $A, B \in F$ we have $A \cap B \in F$; (iii) For each $A \in F$ and $B \supset A$ we have $B \in F$.

An ideal I is called non-trivial if $I \neq \Phi$ and $X \notin I$.

Definition 2.3: A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\}: x \in X\}$.

For further study we shall take $X = \mathbb{N}^2$ and I will denote an ideal of subsets of $\mathbb{N}^2$. The following proposition express a relation between the notions of an ideal and a filter.

Proposition 2.1: Let $I \subset P(\mathbb{N}^2)$ be a non- trivial ideal. Then the class $F = F(1) = \{M \subset \mathbb{N}^2: M = \mathbb{N}^2 - A, for\ some\ A \in I\}$ is a filter on $\mathbb{N}^2$ (we shall call $F = F(1)\ the\ filter\ associated\ with\ I$).

Definition 2.4: Let $I \subset P(\mathbb{N}^2)$ be a non-trivial ideal in $\mathbb{N}^2$. A double sequence $x = (x_{ij})$ of real numbers is said to be I −convergent to a number L if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{(i,j) \in \mathbb{N}^2: |x_{ij} - L| \geq \varepsilon\}$ belongs to I.The number L is called the I −limit of the sequence (x_{ij}) and we write $I - \lim_{ij} x_{ij} = L$.

Remark 2.1: If we take $I = \{E \subset \mathbb{N}^2: E\ is\ contained\ (\mathbb{N} \times A) \cup (A \times \mathbb{N}) where\ A\ is\ a\ finite\ subset\ of\ N\}$. Then $I -$ convergent is equivalent to the usual Pringsheim's convergence.

Definition 2.5: A double sequence $x = (x_{ij})$ is said to be convergent to L in the Pringsheim's sense (1900) if for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j \geq m$. The number L is called the Pringsheim limit of the sequence x.

Definition 2.6: A double sequence $x = (x_{ij})$ is said to be bounded if ther exists a real number $M > 0$ such that $|x_{ij}| < M$ for each i and$j$, i.e., if $\|x\|_{(\infty,2)} = \sup_{ij} |x_{ij}| < \infty$.

We shall denote the set of al bounded double sequence by ℓ^2_∞. Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded. By $c^\infty_2, we\ mean\ the\ space\ of\ all$ P−convergent and bounded double sequences whereas I_2 denotes the space of

I −convergent double sequences. Let $A=[a_{ij}^{mn}]_{j,k=0}^{\infty}$ be a four dimensional infinite matrix of real numbers for all $m,n=0,1,2,\ldots$ The sums $y_{mn}=\sum_{i,j}^{\infty,\infty}a_{ij}^{mn}x_{ij}$ are called the A −transformation of the sequence (x_{ij}). A double sequence $x=(x_{ij})$ is said to be A −summable to the number L if A −transformed sequence (y_{mn}) of the sequence x converges to L (in the Pringsheim's sense) as $m,n\to\infty$. A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into convergent sequence with the same limit. In 1926, Robison presented a 4- dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness: A four dimensional matrix $A=[a_{ij}^{mn}]_{i,j=0}^{\infty}$ is said to be RH −regular if and only if it maps every bounded P −convergent sequence into a P −convergent sequence with the same P −limit. Let X and Y be two sequence spaces. We denote by (X,Y) the class of all matrices A which map X into Y, and by $(X,Y)_{reg}$ we mean $A\in(X,Y)$ such that the limit preserved. A matrix $A=[a_{ij}^{mn}]_{i,j=0}^{\infty}$ is said to be RH −regular if and only if $A\in(c_2^{\infty},c_2^{\infty})$ reg [see Hamilton (1936) and Robison (1926)]. In 1936 Hamilton proved the following theorem for the regularity of any four dimensional infinite matrix.

Theorem 2.1: A four dimensional matrix $A=[a_{ij}^{mn}]_{i,j=0}^{\infty}$ is RH −regular if and only if (i) $P-\lim_{m,n\to\infty}a_{ij}^{mn}=0$ for each *i,j, (ii) P*−lim*m,n→∞i,jaijmn=1, iii P*−lim*m,n→∞iaijmn=0, ivP*−lim*m,n→∞i,ja/mn=0, (v) P*−lim*m,n→∞i,jaijmn*exists; (vi) *A*=sup*m,ni,jaijmn<∞.*

The idea of P −limit inferior, P −limit superior and P −Core for double sequences was presented by Patterson (1999). We state here the two important theorems of Patterson (1999) related to the Core of double sequence.

Theorem 2.2: If A is a non-negative RH −regular summability matrix, then the P −Core$\{Ax\}\subseteq P$ −Core$\{x\}$, for any bounded sequence $x=(x_{ij})$ for which (Ax) exists.

Theorem 2.3: If a four dimensional matrix, then the following are equivalent (i) for all real double sequences $(x_{ij}),P-limsup\,Ax\le p-limsup\,x$; (ii) A is RH −regular summability matrix with $P-\lim_{m,n\to\infty}\sum_{i,j}\left|a_{ij}^{mn}\right|=1$.

Mursaleen and Osama (2003), Morciz (2003) introduced the two dimensional analogue of natural density as follows; Let $K\subset\mathbb{N}^2$ and $K(m,n)$ denotes the number of (i,j) in K such that $i\le m$ and $j\le n$. Then the lower asymptotic density of K is defined by $\underline{\delta_2}(K)=\lim_{m,n\to\infty}inf_{m,n\to\infty}\frac{K(m,n)}{mn}$. In case the sequence $\left(\frac{K(m,n)}{mn}\right)$ has a limit in Pringsheim's sense then we say that K has double natural density and is defined by $\lim_{m,n\to\infty}\frac{K(m,n)}{mn}=\delta_2(K)$. Cakan and Altay (2006) defined the statistical limit superior and inferior for a double sequence as follow.

Let $B_x=\{b\in\mathbb{R}:\delta_2(\{(i,j):x_{ij}>b\})\ne0\}$, and $A_x=\{a\in\mathbb{R}:\delta_2(\{(i,j):x_{ij}<a\})\ne0\}$. where $\delta_2(E)\ne0$, means that either $\delta_2(E)>0$ or does not have double natural density.

Definition 2.7: If $x=(x_{ij})$ be a double sequence. Then statistical-limit superior of x is defined by

$$St_2-limsupx=\begin{cases}supB_x, & if\ B_x\ne\Phi\\ -\infty & B_x=\Phi\end{cases}$$

Also the statistical-limit inferior of x is defined by

$$St_2-liminfx=\begin{cases}infA_x, & if\ A_x\ne\Phi\\ +\infty & A_x=\Phi\end{cases}$$

Remark 2.2: Define the first means $\sigma_{m,n}$ of a double sequence (x_{ij}) by setting

$$\sigma_{m,n}^x=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}$$

Definition 2.5.1[Siddiqui et al., (2012)]: We say a double sequence $x=(x_{ij})$ is statistically summable $(C,1.1)$ to L, if the sequence $\sigma=(\sigma_{mn})$ is statistically convergent to L in Pringsheim's sense, that is, $St_2-\lim_{m,n}\sigma_{mn}=L$. We denote by $C_{1.1}(St_2)$, the set of all double sequences which are statistically summable (C,1.1).

Definition 2.6.1 [Siddiqui et al., (2012)]: (i) A double sequence $x=(x_{ij})$ is said to be lower $C_{1.1}$ −staistically bounded if there exists a constant M such that $\delta\{(i,j):\sigma_{ij}^x<M\}=0$, or equivalently, we write $\delta_{C_{11}}\{(i,j):x_{ij}<$ *M=0.*

(ii) A double sequence $x=x_{ij}$ is said to be upper $C_{1.1}$ −statistically bounded if there exists a constant N such that $\delta\{(i,j):\sigma_{ij}^x>N\}=0$, or equivalently, we write

$$\delta_{C_{11}}\{(i,j):x_{ij}>N\}=0.$$

(iii) If $x=x_{ij}$ is both lower and upper $C_{1.1}$ −statistically bounded, we say that $x=(x_{ij})$ is C_{11} −statistically bounded, equivalently written x is $C_{1.1}(st)$ −bdd.

We denote the set of all $C_{11}(st)$ −bdd sequences by $C_{1.1}(St_{2\infty})$ whereas $C_{,1.1}^I(St_2)$ denotes the space of I −convergent $C_{,1.1}$ −statistically convergent double sequences.

Definition 2.7.1[Siddiqui et al., (2012)]: For $M,N\in\mathbb{R}$, let

$$K_x=\{M:\delta(\{(i,j):\sigma_{ij}^x<M\})\},$$
$$L_x=\{N:\delta(\{(i,j):\sigma_{ij}^x>N\})\}.$$

Then

$$C_{\mathbf{1.1}}(St_{\mathbf{2}})\text{ −superior of } x=\inf L_x,$$
$$C_{\mathbf{1.1}}(St_{\mathbf{2}})\text{ −inferior of } x=\sup K_x.$$

Remark 2.3: Note that every bounded double sequence is Pringsheim bounded and every Pringsheim bounded double sequence is $C_{1.1}$ −statistically bounded but not conversely, in general.

The following is an example of $x=(x_{ij})$ which is neither bounded above nor bounded below, but the Pringsheim limit superior and inferior are both finite numbers:

$$x_{ij}=\begin{cases} i & if\ j=0\\ -j & if\ i=0\\ (-1)^i & i=j\\ 0, & otherwise.\end{cases}$$

Thus $P-liminf[x]$ and $P-limsup[x]=1$.

III. I −Limit Superior and Inferior

The following concept and results are by Kumar (2007).

Definition 3.1 [Kumar (2007)]: A real double sequence $x=(x_{ij})$ is said to be I −bounded below if there exists a real number M such that $\{(i,j):x_{ij}<M\}\in I. x=(x_{ij})$ is said to be I −bounded above if there exists a real number M such that $\{(i,j):x_{ij}>M\}\in I$. A sequence which is both I −bounded below as well as I −bounded above is called I −bounded.

Remark 3.1[Kumar (2007)]**:** One can observe easily that any bounded double sequence is I −bounded. Let ℓ_∞^2 denote the space of all I −bounded double sequences. Let $I\subset P(\mathbb{N}^2)$ be an admissible ideal. For a real double sequence $x=(x_{ij})$, let

$B_x=\{b\in\mathbb{R}:\{(i,j):x_{ij}>b\}\notin I\}$ and $A_x=\{a\in\mathbb{R}:\{(i,j):x_{ij}<a\}\notin I\}$.

Definition 3.2 [Kumar (2007)]: Let $I\subset P(\mathbb{N}^2)$ be an admissible ideal. If $x=(x_{ij})$ be a real double sequence. Then I −limit ssuperior of x is defined by

$$I-limsupx=\begin{cases} supB_x, & if\ B_x\neq\Phi\\ -\infty & if\ B_x=\Phi.\end{cases}$$

Also the I −limit inferior of x is defined by

$$I-liminfx=\begin{cases} supA_x, & if\ A_x\neq\Phi\\ +\infty & if\ A_x=\Phi.\end{cases}$$

Proposition 3.1: If $\beta=I-limsupx$ is finite, then for each $\varepsilon>0$, the sets

$\{(i,j):x_{ij}>\beta-\varepsilon\}\notin I$ and $\{(i,j):x_{ij}>\beta+\varepsilon\}\in I$ (1)

Conversely if (1) holds for each $\varepsilon>0$, then $\beta=I-limsupx$.

Proposition 3.2: If $\alpha=I-liminfx$ is finite, then for each $\varepsilon>0$, the sets

$\{(i,j):x_{ij}>\alpha-\varepsilon\}\notin I$ and $\{(i,j):x_{ij}>\alpha+\varepsilon\}\in I$ (2)

Conversely if (2) holds for each $\varepsilon>0$, then $\alpha=I-liminfx$.

Theorem 3.1 For every real double sequence $x=(x_{ij}),\ I-liminfx\le I-limsupx$.

Remark 3.2: For any real double sequence $x=(x_{ij})$ $P-liminfx\le I-liminfx\le I-limsupx\le P-limsupx$.

Remark 3.3: I-boundedness of a sequence $x=(x_{ij})$ implies that $I-liminfx$ and $I-limsupx$ are finite, so (1) and (2) hold.

Theorem 3.2: If $x=(x_{ij}),y=(y_{ij})$ are two I −bounded sequences, then we have (i) $I-limsup(x+y)\le I-limsupx+I-limsupy.(ii)I-liminf(x+y)\ge I-liminfx+I-liminfy$.

IV. I −Core of double sequences

The following notions and results are by Kumar (2007)

Definition 4.1: For any I −bounded real sequence (x_{ij}), the I −Core of x is defined as the closed interval $[I-liminf$ and $I-limsupx]$. In case, x is not I −bounded, I −Core of the sequence x is defined by either $(-\infty,\infty)$. I −Core$\{x\}$ will denote I −Core of the sequence $x=(x_{ij})$.

It is clear from remark 3.2 that I −Core$\{x\}\subseteq P$ −Core$\{x\}$, for any real double sequence x.

Lemma 4.1: Let $c_2^{\infty,0}$ be the space of all sequences which are bounded and P −convergent to zero. Then $A\in(\ell_\infty^2,c_2^{\infty,0})$, if and only if,

(i) $\|A\| = \sup_{m,n} \sum_{i,j} |a_{ij}^{mn}| < \infty$, $(ii)\ P - \lim_{m,n\to\infty} a_{ij}^{mn} = 0$ for each $i,j \in \mathbb{N}$, (iii) $P - \lim_{m,n\to\infty} \sum_j a_{ij}^{mn}$ for each $i \in \mathbb{N}$, $(iv) P - \lim_{m,n\to\infty} \sum_i a_{ij}^{mn} = 0$ for each $j \in \mathbb{N}$, $(v)\ P - \lim_{m,n\to\infty} \sum_{i,j} |a_{ij}^{mn}| = 0$.

Let us assume that I be an admissible ideal of such that I contains all sets of the form $H \times \mathbb{N}, \mathbb{N} \times H$ where H is a finite subset of $\mathbb{N}$.

Lemma 4.2: If $A = [a_{ij}^{mn}]$ be a four dimensional matrix. Then,

$A \in \left(I_2 \cap \ell_\infty^2, c_2^\infty\right)_{reg}$, if and only if, (3)

A is RH −regular and $P - \lim_{m,n\to\infty} \sum_{i,j\in E} |a_{ij}^{mn}| = 0$ for every $E \subset \mathbb{N}^2$ in I.

Theorem 4.1: Let $\|A\| < \infty$ and $x = (x_{ij}) \in \ell_\infty^2$. Then, $P - limsupAx \leq I - limsupx$ if and only if

$A \in \left(I_2 \cap \ell_\infty^2, c_2^\infty\right)_{reg}$, and $P - \lim_{m,n\to\infty} \sum_{i,j} |a_{ij}^{mn}| = 1$ (5)

V. $C_{1.1}^I(St_2)$ −Limit Superior and Inferior

In this section we shall in analogy to Kumar (2007), define the concepts of $C_{,1.1}(St_2)$ −superior and inferior for double sequences and prove some fundamental properties of $C_{,1.1}^I(St_2)$ −limit superior and inferior.

Definition 5.1: A real $C_{,1.1}$ −statistically convergent double sequences $[C_{1.1}(St_2)]\ x = (x_{ij})$ is said to be $C_{1.1}^I(St_2)$ −bounded below if there exists a real number M such that $\{(i,j): \sigma_{ij} < M\} \in I$. $x = (x_{ij})$ is said to be $C_{1.1}^I(St_2)$ −bounded above if there exists a real number M such that $\{(i,j): \sigma_{ij} > M\} \in I$. A sequence which is $C_{1.1}^I(St_2)$ −bounded below and also $C_{1.1}^I(St_2)$ −bounded above is called $C_{1.1}^I(St_2)$ −bounded.

Remark 5.1: It can be easily seen that all I −bounded double sequences are $C_{1.1}^I(St_2)$ −bounded. Let $C_{,1.1}^I(St_{2\infty})$ denote the space of all I −bounded−Cesáro $C_{1.1}$ −statistically convergent double sequences.

Following the introduction of the concept of I −limit superior and inferior [see Kumar (2007)] we introduce $C_{,1.1}^I(St_2)$ −limit superior and inferior as follows:

Let $I \subset P(\mathbb{N}^2)$ be an admissible ideal. For a real double sequence $x = (x_{ij})$, let,

$A_x = \{a \in \mathbb{R}: \{(i,j): \sigma_{ij} < a\} \in I\}$ and $B_x = \{b \in \mathbb{R}: \{(i,j): \sigma_{ij} > b\} \in I\}$.

Definition 5.2: Let $I \subset P(\mathbb{N}^2)$ be an admissible ideal. Let $x = (x_{ij})$ be a real double sequence. Then $C_{,1.1}^I(St_2)$ −limit superior of x is defined by

$$C_{,1.1}^I(St_2) - limsupx = \begin{cases} supB_x, & if\ B_x \neq \Phi \\ -\infty & if\ \ B_x = \Phi \end{cases}$$

Also

$$C_{,1.1}^I(St_2) - liminfx = \begin{cases} supA_x, & if\ A_x \neq \Phi \\ -\infty & if\ \ A_x = \Phi \end{cases}$$

Proposition 5.1 If $\beta = C_{,1.1}^I(St_2) - limsupx$ is finite, then for each $\varepsilon > 0$, the sets $\{(i,j): \sigma_{ij} > \beta - \varepsilon\} \in I$ and $\{(i,j): \sigma_{ij} > \beta + \varepsilon\} \in I$ (1.1)

Conversely, if (1.1) holds for each $\varepsilon > 0$, then $\beta = C_{,1.1}^I(St_2) - limsupx$.

Proposition 5.2 If $\alpha = C_{,1.1}^I(St_2) - liminfx$ is finite, then the sets

$\{(i,j): \sigma_{ij} < \alpha + \varepsilon\} \notin I$ and $\{(i,j): \sigma_{ij} < \alpha - \varepsilon\} \in I$ (1.2)

Conversely, if (1.2) holds, for each $\varepsilon > 0$, then $\alpha = C_{1.1}^I(St_2) - liminfx$.

Theorem 5.1: For any real double sequence $\sigma = (\sigma_{ij})$, $C_{1.1}^I(St_2) - liminfx \leq C_{1.1}^I(St_2) - limsupx$

Proof: Case (i): If $C_{1.1}^I(St_2) - limsupx = -\infty$, then we have $B_x = \emptyset$ and therefore for each $b \in \mathbb{R}$, $\{(i,j): \sigma_{ij}^x > b\in I$. This implies that for each *a∈ℝ, i,j:σijx∈FI.* Hence *C1.1ISt2−limsupx=−∞.* Case (ii): $C_{1.1}^I(St_2) - limsupx = \infty$, then we have nothing to prove. Case (iii): Suppose that $\beta = C_{1.1}^I(St_2) - limsupx$ is finite then we have $B_x \neq \emptyset$ and $\beta = supB_x$. Let $\alpha = C_{1.1}^I(St_2) - liminfx$. To prove the result it is sufficient to prove that given $\varepsilon > 0, \beta + \varepsilon \in A_x$, so that $\alpha \leq \beta + \varepsilon$. Let $\varepsilon > 0$. By proposition 5.1, $\left\{(i,j): \sigma_{ij}^x > \beta + \frac{\varepsilon}{2}\right\} \in I$ and therefore the set $\left\{(i,j): \sigma_{ij}^x \leq \beta + \frac{\varepsilon}{2}\right\} \in F(I)$. Since $\left\{(i,j): \sigma_{ij}^x \leq \beta + \frac{\varepsilon}{2}\right\} \subseteq \left\{(i,j): \sigma_{ij}^x < \beta + \frac{\varepsilon}{2}\right\}$ and $F(I)$ is a filter on $\mathbb{N}^2$ therefore $\{(i,j): \sigma_{ij}^x < \beta + \varepsilon\} \in F(I)$. This implies that the set $\{(i,j): \sigma_{ij}^x < \beta + \varepsilon\} \notin I$ and therefore by definition of $A_x, \beta + \varepsilon \in A_x$. As $\alpha = infA_x$ so we have, $\alpha \leq \beta + \varepsilon$. Since ε is arbitrary this proves that $\alpha \leq \beta$.

Remark 5.2: For any real double sequence $x = (x_{ij}), I - liminfx \leq C_{1.1}^I(St_2) - liminfx \leq C_{1.1}^I(St_2) - limsupx \leq I - limsupx$.

Remark 5.3: $C_{1.1}^I(St_2)$ −boundedness of a sequence $x = (x_{ij})$ implies $C_{1.1}^I(St_2) - liminfx$ and $C_{1.1}^I(St_2) - limsupx$ are finite, as such (1.1) and (1.2) hold.

Theorem 5.2: If $x = (x_{ij}), y = (y_{ij})$ are two $C_{1.1}^I(St_2)$ −bounded sequences, then we have

(i) $\boldsymbol{C_{1.1}^I}(St_2) - limsup(x + y) \leq C_{1.1}^I(St_2) - limsupx + C_{1.1}^I(St_2) - limsupy$

(ii) $C^I_{1.1}(St_2) - liminf(x+y) \geq C^I_{1.1}(St_2) - liminfx + C^I_{1.1}(St_2) - liminfy$

Proof: Since x and y are $C^I_{1.1}(St_2)$ −bounded sequences therefore by remark 5.3, $C^I_{1.1}(St_2) - limsupx$ and $C^I_{1.1}(St_2) - limsupy$ are both finite. We may also assume that B_{x+y} is non-void. Let $\alpha = C^I_{1.1}(St_2) - limsupx, \beta = C^I_{1.1}(St_2) - limsupy$ and $\gamma = C^I_{1.1}(St_2) - limsup(x+y)$. Take $\varepsilon > 0$. Then by proposition 5.1 we have the sets $A = \left\{(i,j): \sigma^x_{ij} > \alpha + \frac{\varepsilon}{2}\right\}$ and $B = \left\{(i,j): \sigma^y_{ij} > \beta + \frac{\varepsilon}{2}\right\}$ belong to I. Now it is clear that the set $C = \left\{(i,j): \sigma^x_{ij} + \sigma^y_{ij} > \alpha + \beta + \varepsilon\right\} \in A \cup B$. Since the set on the right side belongs to I therefore $C \in I$. Next we shall prove that for every $b \in B_{x+y}, b \leq \alpha + \beta + \varepsilon$.

Let $b \in B_{x+y}$, then by definition $\left\{(i,j): \sigma^x_{ij} + \sigma^y_{ij} > b\right\} \notin I$. If $b > \alpha + \beta + \varepsilon$, then $\left\{(i,j): \sigma^x_{ij} + \sigma^y_{ij} > b\right\} \subset C$ and therefore $\left\{(i,j): \sigma^x_{ij} + \sigma^y_{ij} > b\right\}$ belongs to I as $C \in I$. In this way we obtained a contradiction as $\{(i,j): \sigma^x_{ij} +$ *σijy>b∉I.* Hence *b≤α+β+ε* for every *b∈Bx+y.*This shows that *C1.1ISt2−limsupx+y=supBx+y≤α+β+ε.* Since ε>0 is arbitrary so $\gamma \leq \alpha + \beta$. We can prove (ii) analogously. This completes the proof.

Theorem 5.3: A real double $C^I_{1.1}(St_2)$ −bounded sequence $x = (x_{ij})$ is $C^I_{1.1}(St_2)$ −convergent if and only if

$$C^I_{1.1}(St_2) - liminfx = C^I_{1.1}(St_2) - limsupx$$

Proof: Since $x = (x_{ij})$ is $C^I_{1.1}(St_2)$ −bounded therefore by remark 5.3 $C^I_{1.1}(St_2) - liminfx$ and $C^I_{1.1}(St_2) - limsupx$ are both finite. Let, $\alpha = C^I_{1.1}(St_2) - liminfx$ and $\beta = C^I_{1.1}(St_2) - limsupx$. Suppose that $x = (x_{ij})$ is convergent with $C^I_{1.1}(St_2) - \lim_{i,j\to\infty} x_{ij} = L$. Let $\varepsilon > 0$ be given. Then we have the set $\left\{(i,j): \left|\sigma^x_{ij} - L\right| \geq \varepsilon\right\} \in I$. Thus for each $b > L + \varepsilon$, we have $\left\{(i,j): \sigma^x_{ij} > b\right\} \subset \left\{(i,j): \sigma^x_{ij} > L + \varepsilon\right\}$. Since I is an ideal and the set on the right side belongs to I, therefore the set $\left\{(i,j): \sigma^x_{ij} > b\right\} \in I$. Thus for each $b > L + \varepsilon$, the set $\left\{(i,j): \sigma^x_{ij} > b\right\} \in I$, which implies that $\beta \leq L + \varepsilon$. As ε is arbitrary, we have $\beta \leq L$. We also have by proposition 5.2, $\{(i,j): \sigma^x_{ij} <$ *L−ε∈I,* which yields that *L≤α.* Therefore we have *β≤α.* Combining this with theorem 5.1, we conclude that $\alpha = \beta$. Conversely, suppose that $C^I_{1.1}(St_2) - liminfx = C^I_{1.1}(St_2) - limsupx = L$ i.e., both are finite then by proposition 5.1 and 5.2, we have the set $A(\varepsilon) = \left\{(i,j): \left|\sigma^x_{ij} - L\right| \geq \varepsilon\right\} \in I$. Hence $C^I_{1.1} - \lim_{ij\to\infty} x_{ij} = L$.

VI. $C^I_{1.1}(St_2)$ −Core of Double Sequences

Analogous to I −Core [see Kumar (2007)] and $C_{1.1}(St_2)$ −Core [see Siddiqui (2012)], we define $C^I_{1.1}(St_2)$ −Core as follows:

Definition 6.1: For any $C^I_{1.1}(St_2)$ −bounded sequence $x = (x_{ij})$, then $C^I_{1.1}(St_2)$ −Core of x is defined as the closed interval $[C^I_{1.1}(St_2) - liminfx, C^I_{1.1}(St_2) - limsupx$. In case x is not $C^I_{1.1}(St_2)$ −bounded, $C^I_{1.1}(St_2)$ −Core is given by either $(-\infty, C^I_{1.1}(St_2) - limsupx]$ $(C^I_{1.1}(St_2) - liminfx]$ or $(-\infty,\infty)$. We shall denote by $C^I_{1.1}(St_2)$ −Core$\{x\}$, the $C^I_{1.1}(St_2)$ −Core of the sequence $x = (x_{ij})$.

It can be seen from remark 5.2 that $C^I_{1.1}(St_2)$ −Core$\{x\} \subseteq I$ −Core$\{x\}$, for any double sequence x.

Lemma 6.1: Let $C^I_{1.1}(St^0_{2\infty})$, be the space of all double sequences which are bounded and I −statistically convergent to zero.Then $A \in \left(\ell^2_\infty, C^I_{1.1}(St^0_{2\infty})\right)$ if and only if

(i) $\|A\| = sup \sum_{ij} \left|a^{mn}_{ij}\right| < \infty$

(ii) $I - \lim_{mn\to\infty} \sum_{i,j} a^{mn}_{ij} = 0$ for each $i,j \in \mathbb{N}$

(iii) $I - \lim_{mn\to\infty} \sum_{j} a^{mn}_{ij} = 0$ for each $i \in \mathbb{N}$

(iv) $I - \lim_{mn\to\infty} \sum_{i} a^{mn}_{ij} = 0$ for each $j \in \mathbb{N}$

(v) $I - \lim_{mn\to\infty} \sum_{i,j} \left|a^{mn}_{ij}\right| = 0$

Assuming that I is an admissible ideal such that I contains all sets of the form
$H \times \mathbb{N}, \mathbb{N} \times H$ where H is a finite subset of $\mathbb{N}$.

Lemma 6.2: If $A = [a^{mn}_{ij}]$ is a four dimensional matrix. Then,

$$A \in [C^I_{1.1}(St_{2\infty}, C_{1.1}(St_2)]_{reg} \text{ If and only if} \quad (6.1)$$

A is RH −regular and $C_{1.1}(St_2) - \lim_{mn\to\infty} \sum_{ij} \left|a^{mn}_{ij}\right| = 0$ for every $E \subset \mathbb{N}^2$ in I.

Proof: Suppose that $A \in (C^I_{1.1}(St_{2\infty}), C_{1.1}(St_2))_{reg}$.Since I is an admissible ideal which contain all sets of the form $H \times \mathbb{N}, \mathbb{N} \times H$ wher H is a subset of $\mathbb{N}$. Therefore $C_{1.1}(St_2)$ −convergene for any bounded double sequence (x_{ij}).This implies that $C_{1.1}(St_2) \subset C^I_{1.1}(St_{2\infty})$ and the RH −regular of A as follows. Let $E \subset \mathbb{N}^2$ be any set belongs to the ideal I and $x = (x_{ij}) \in \ell^2_\infty$. Define the sequence $\sigma = (\sigma_{ij})$ as follows:

$$\sigma_{ij} = \begin{cases} \sigma^x_{ij}, & if\ (i,j) \in E \\ 0 & otherwise. \end{cases}$$

Then it is obvious that $\sigma \in C_{1.1}^{I}(St_{2,\infty})$ with $C_{1.1}(St_2) - \lim_{ij\to\infty} z_{ij} = 0$. By (6.1) we have $A\sigma \in C_{1.1}(St_{2,\infty}^{0})$. Also we have $A\sigma = \frac{1}{mn}\sum_{ij}^{mn} a_{ij}^{mn} x_{ij} = \frac{1}{mn}\sum_{i,j\in E} a_{ij}^{mn} x_{ij}$. Define the matrix $B = [b_{ij}^{mn}]$ as follows: For each $m,n \in \mathbb{N}$

$$b_{ij}^{mn} = \begin{cases} a_{ij}^{mn}, & if\ (i,j) \in E \\ 0 & otherwise. \end{cases}$$

It is obvious that the matrix B defined above is in the class $\left(C_{1.1}^{I}(St_{2,\infty})\right)$, and therefore by (lemma 6.1), we have, $C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j}|b_{ij}| = 0$. This implies that

$$C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j\in E}|a_{ij}^{mn}| = 0.$$

Conversely, suppose that A is RH −regular and $C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j\in E}|a_{ij}^{mn}| = 0$ for every$E \subset \mathbb{N}^2$ in . Let $x \in C_{1.1}^{I}(St_{2,\infty})$ and suppose that $C_{1.1}^{I}(St_2) - \lim_{i,j} x_{ij} = L$. Then for each $\varepsilon > 0$, the set

$\{(i,j): |\sigma_{ij}^{x} - L| \geq \varepsilon\} \in I$ and $\{(i,j): |\sigma_{ij}^{x} - L| < \varepsilon\} \in F(I)$. (6.2)

Now we can write $\sum_{ij} a_{ij}^{mn} x_{ij} = \sum_{i,j} a_{ij}^{mn}(x_{ij} - L) + L\sum_{ij} a_{ij}^{mn}$.This implies that $C_{1.1}(St_2) - \lim_{m,n\to\infty} A\sigma^{x} = C_{1.1}(St_2) -$
$\lim_{m,n\to\infty}\sum_{i,j} a_{ij}^{mn} x_{ij} = C_{1.1}(St_2) - \lim_{m,n\to\infty}(\sum_{i,j} a_{ij}^{mn}(x_{ij} - L)) + C_{1.1}(St_2) - \lim_{m,n\to\infty}(L\sum_{i,j} a_{ij}^{mn})$. Since A is RH −regular matrix therefore by lemma(6.1), we have $C_{1.1}(St_2) - \lim_{m,n\to\infty} A\sigma^{x} = C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j} a_{ij}^{mn}(x_{ij} - L) + L$. Also we have $|\sum_{i,j} a_{ij}^{mn}(x_{ij} - L)| = |\sum_{i,j\in E} a_{ij}^{mn}(x_{ij} - L)| + \sum_{i,j\notin E} a_{ij}^{mn}(x_{ij} -$ *L)≤xij−Li,jaijmn+εA* by (6.2). It follows that *C1.1St2−*lim*m,n→∞i,jaijmn(xij−L)=0.* Since$C_{1.1}(St_2) - \lim_{m,n\to\infty} A\sigma^{x} = C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j} a_{ij}^{mn}(x_{ij} - L) + L$ we have, therefore $C_{1.1}(St_2) - \lim_{m,n\to\infty} A\sigma^{x} = L$. Hence $A \in [C_{1.1}^{I}(St_{2,\infty}), C_{1.1}(St_2)]_{reg}$.

Theorem 6.1: Let $\|A\| < \infty$ and $x = (x_{ij}) \in \ell_{\infty}^{2}$. Then, $C_{1.1}(St_2) - limsup A\sigma^{x} \leq C_{1.1}^{I}(St_2) - limsup\sigma^{x}$ if and only if

$A \in \left(C_{1.1}^{I}(St_{2,\infty}), C_{1.1}(St_2)\right)_{reg}$, and $C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{ij}|a_{ij}^{mn}| = 1$ (6.3)

Proof First suppose that $C_{1.1}(St_2) - limsup A\sigma \leq C_{1.1}^{I}(St_2) - limsup\sigma$ holds. Then, by an easy argument, one has that

$$C_{1.1}^{I}(St_2) - liminf\sigma \leq c_{1.1}(St_2) - liminf A\sigma \leq C_{1.1}(St_2) - limsup A\sigma \leq C_{1.1}^{I}(St_2) - limsup\sigma \qquad (6.4)$$

Let $x = (x_{ij}) \in C_{1.1}^{I}(St_{2,\infty})$. As x is $C_{1.1}^{I}(St_2)$ −convergent we take $C_{1.1}^{I}(St_2) - \lim_{ij\to\infty} x_{ij} = L$. Since $C_{1.1}^{I}(St_{2,\infty})$ is a subspace of ℓ_{∞}^{2}, so $x \in \ell_{\infty}^{2}$ and therefore by assumption we have

$C_{1.1}(St_2) - \lim\sup A\sigma \leq C_{1.1}^{I}(St_2) - \lim\sup\sigma$. As (x_{ij}) is $C_{1.1}^{I}(St_2)$ −convergent to L, so by theorem 5.2 we have $C_{1.1}^{I}(St_2) - \lim\inf\sigma = C_{1.1}^{I}(St_2) - \lim\sup\sigma = L$ by (6.4) we can observe that $C_{1.1}(St_2) - \lim A\sigma = L$. This implies that $A\sigma^{x} \in \left(C_{1.1}^{I}(St_{2,\infty}), C_{1.1}(St_2)\right)_{reg}$. Also by remark 5.2

We have $C_{1.1}^{I}(St_2) - limsup\sigma \leq C_{1.1}(St_2) - limsup\sigma$. On combining the above inequalities we get, $C_{1.1}(St_2) - limsup A\sigma \leq C_{1.1}(St_2) - limsup\sigma$. But then by theorem (4.1), We have A is RH −regular summability matrix with $C_{1.1}(St_2) - \lim_{m,n\to\infty}\sum_{i,j}|a_{ij}^{mn}| = 1$.

Conversely suppose that (6.3) holds. Let $x = (x_{ij})$ be any bounded double sequence, then we have $A\sigma$ is bounded and x is $C_{1.1}^{I}(St_2)$ −bounded sequences. Therefore by remark 5.3, $C_{1.1}^{I}(St_2) - limsup\sigma$ is finite say L.So by proposition 5.1, for each $\varepsilon > 0$ the set $E = \{(i,j): \sigma_{ij}^{x} > L + \varepsilon\} \in I$. Also we have $\sigma_{ij}^{x} \leq L + \varepsilon$ whenever $i,j \notin E$. Now we can write

$$\begin{aligned} A\sigma = \sum_{ij} a_{ij}^{mn} x_{ij} &\leq \left|\sum_{ij}\frac{|a_{ij}^{mn} x_{ij}| + a_{ij}^{mn} x_{ij}}{2} + \sum_{ij}\frac{|a_{ij}^{mn} x_{ij}| - a_{ij}^{mn} x_{ij}}{2} + \right| \\ &\leq \left|\sum_{ij} a_{ij}^{mn} x_{ij}\right| \\ &+ \sum_{ij}(|a_{ij}^{mn}| - a_{ij}^{mn})|x_{ij}| \\ &\leq \left|\sum_{i,j\in E} a_{ij}^{mn} x_{ij} + \sum_{i,j\notin E} a_{ij}^{mn} x_{ij}\right| \\ &+ \|x\|\sum_{ij}(|a_{ij}^{mn}| - a_{ij}^{mn}) \leq \|x\|\sum_{ij\in E}|a_{ij}^{mn}| + (L+\varepsilon)\sum_{i,j\notin E}|a_{ij}^{mn}| + \|x\|(|a_{ij}^{mn}| - a_{ij}^{mn}). \end{aligned}$$

Since $A \in \left(C_{1.1}^{I}\left(St_{2,\infty}\right), C_{1.1}(St_2)\right)_{reg}$, therefore by lemma (5.2) A is RH −regular and $C_{1.1}(St_2) - \lim_{m,n\to} \sum_{i,j\in E} \left|a_{ij}^{mn}\right| = 0$ for every $E \subset \mathbb{N}^2$ in I. Since A is RH −regular and $C_{1.1}(St_2) - \lim_{m,n\to} \sum_{i,j} \left|a_{ij}^{mn}\right| = 1$, therefore theorem (2.2) and theorem (2.3) implies that $C_{1.1}(St_2) - limsupA\sigma \leq (L + \varepsilon)$. This completes the proof as ε was arbitrary selected.

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