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Falling Factorials, Compositions and the Ordered Integers of Curious Patterns: Some Arithmetic Rhythms

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Abstract: *Product of consecutive integers is expressible by the products of falling factorials which have connection with compositions of a positive integer. There exists a recurrence relation involving falling factorials which has close connection with the ordered compositions and the ordered integers of amazing properties. The ordered compositions and integers hold some arithmetic rhythms.*
Keyword: *Sequence; Recurrence; Falling factorial; Composition; Summand.*

I. Introduction

First we represent the product of consecutive integers as the products of falling factorials which involve with the compositions of a positive integer. Then we show the existence of a recurrence relation involving falling factorials which has connection with the ordered compositions and the ordered integers of curious properties. In fact the ordered compositions and integers have some arithmetic rhythms.

II. Product of Falling Factorials Involving Compositions

Product of n consecutive positive integers with the greatest integer: g is expressible by the falling factorial notation: $(g)_n$ such that $(g)_n = g\,(g-1)\ldots(g-n+1)$. The preliminary condition is: $(g)_1 = g$, that is, any positive integer is expressible by the falling factorial notation. Let $i, j, k \in \mathbb{N}$ and $i + j + k = n$. Then we can write: $(g)_n = (g)_i\,(g-i)_j\,(g-i-j)_k$. This implies that $(g)_n$ can be expressed as a product of some falling factorials such that the sum of the bottom indices in the expression is equal to n. Let the expression be named as "falling factorial expression" for the product: $g\,(g-1)\ldots(g-n+1)$. If $i_1, i_2, \ldots, i_m$ are m bottom indices in such an expression then $i_1 + i_2 + \ldots + i_m$ can be any composition of n. Since the number of the compositions of n is 2^{n-1}, it follows that including $(g)_n$, the product has 2^{n-1} falling factorial expressions where the integers in the successive parentheses of an expression occur in descending order. In general,

$$g\,(g-1)\ldots(g-n+1) = (g)_{i_1}(g-i_1)_{i_2}(g-i_1-i_2)_{i_3}\ldots(g-i_1-\ldots-i_{m-1})_{i_m},\ \text{for } i_1+i_2+\cdots+i_m = n.$$

Product of the first n natural number is usually expressed by the factorial notation: $n!$. This is the fundamental case of the product: $g\,(g-1)\ldots(g-n+1)$ for $g = n$. Evidently $n!$ has 2^{n-1} falling factorial expressions. For instance, 4! has 2^3 or 8 expressions such that:

$$4! = (4)_4 = (4)_3\,\mathbf{(1)_1} = \mathbf{(4)_2\,(2)_2} = (4)_1\,\mathbf{(3)_3} = (4)_2\,(2)_1\,(1)_1 = (4)_1(3)_2(1)_1 = (4)_1\,(3)_1\,(2)_2 = (4)_1(3)_1(2)_1(1)_1.$$

The general form of a falling factorial expression for $n!$ is:

$$(n)_{i_1}\,(i_2+\ldots+i_m)_{i_2}\,(i_3+\ldots+i_m)_{i_3}\ \ldots\ (i_{m-1}+i_m)_{i_{m-1}}\,(i_m)_{i_m},\ \text{where } i_1+i_2+\ldots+i_m = n.$$

III. A Recurrence Relation, 2^{n-1} Falling Factorial Expressions for $n!$, and Ordered Compositions

(a) A Recurrence Relation

Surprisingly there exists a recurrence relation which by the process of recursive substitution, can generate a summation series of 2^{n-1} terms that are exactly 2^{n-1} falling factorial expressions for $n!$. The sets of 2^{n-1} bottom indices with respect to 2^{n-1} terms of the series occur in a definite order of the compositions of n. First we establish Theorem 1 which is indeed the solution of the recurrence function in the recurrence relation involving falling factorials.

Theorem 1: *If the* $(n+1)$*-th order recurrence function*: R_{n+1} *is defined by Recurrence relation* 1:
$R_{n+1} = (n+r)_1\,R_n + (n+r)_2\,R_{n-1} + \ldots + (n+r)_n\,R_1 + (n+r)_{n+1}$, *where* $R_1 = r$ and $(n+r)_m = (n+r)\,(n+r-1)\ldots(n+r-m+1)$ *then* $R_{n+1} = (n+r)_{n+1} \cdot 2^n$.

Proof: The proof is short and simple by induction on *n*. We have: $R_2 = (1+r)_2$ and $R_3 = (2+r)_3 \cdot 2$. Assume that the theorem is true for the first *n* natural numbers for a given value *n*. Then we deduce that

$R_{n+2} = (n+r+1)_1 R_{n+1} + (n+r+1)_2 R_n + \ldots + (n+r+1)_{n+1} R_1 + (n+r+1)_{n+2}$

$= (n+r+1)_1 (n+r)_{n+1} 2^n + (n+r+1)_2 (n+r-1)_n 2^{n-1} + \ldots + (n+r+1)_{n+1} r$

$+ (n+r+1)_{n+2}$

$= (n+r+1)_{n+2} (2^n + 2^{n-1} + \ldots + 2 + 1 + 1)$.

$= (n+r+1)_{n+2}\ 2^{n+1}$.

The theorem follows. ▌

Theorem 1.1 below is the reduced version of Theorem 1.

Theorem 1.1: *If the (n + 1)-th order recurrence function*: R_{n+1} *is defined by Recurrence relation* 1.1*:* $R_{n+1} = (n+1)_1 R_n + (n+1)_2 R_{n-1} + \ldots + (n+1)_n R_1 + (n+1)_{n+1}$ *where* $R_1 = 1$ *and* $(n+1)_m =$ (*n* + 1) *n* … (*n* – *m* + 2)*, then* $R_{n+1} = (n+1)!\,2^n$.

(b) 2^{n-1} Falling Factorial Expressions for *n*! from Recurrence Relation 1.1

(i) We have:

$$R_1 = 1 = (1)_1 . \tag{1.1}$$

Then we get the following successive results using Recurrence relation 1.1.

(ii) From (1.1) and Recurrence relation 1.1 for *n* = 1, we get:

$$R_2 = (2)_1(1)_1 + (2)_2 . \tag{1.2}$$

2 terms of (1.2) are 2 falling factorial expressions for 2!; and 2 sets of bottom indices in 2 successive terms of (1.2) are (1,1) and 2 such that 1 + 1 = 2.

(iii) Substituting 2 terms of (1.2) and 1 term of (1.1) for R_2 and R_1 respectively in Recurrence relation 1.1 for *n* = 2, we get (1.3) of 4 terms as:

$$R_3 = (3)_1(2)_1(1)_1 + (3)_1(2)_2 + (3)_2(2)_1 + (3)_3 . \tag{1.3}$$

4 terms of (1.3) are 4 falling factorial expressions for 3!. 4 sets of bottom indices in 4 successive terms of (1.3) are (1,1,1), (1,2), (2,1) and 3 such that 1 + 1 + 1 = 1 + 2 = 2 + 1 = 3.

(iv) Substituting 4 terms of (1.3), 2 terms of (1.2) and 1 term of (1.1) for R_3 , R_3 and R_1 respectively in Recurrence relation 1.1 for *n* = 3, we get (1.4) of 8 terms as:

$$R_4 = (4)_1(3)_1(2)_1(1)_1 + (4)_1(3)_1(2)_2 + (4)_1(3)_2(1)_1 + (4)_1(3)_3 + (4)_2(2)_1(1)_1 + (4)_2(2)_2 + (4)_3(1)_1 + (4)_4 \tag{1.4}$$

8 terms of (1.4) are 8 falling factorial expressions for 4!. 8 sets of bottom indices in 8 successive terms of (1.4) are : (1,1,1,1), (1,1,2), (1,2,1), (1,3), (2,1,1), (2,2), (3,1) and (4) such that 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 1 + 3 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4.

(v) Substituting 8 terms of (1.4), 4 terms of (1.3), 2 terms of (1.2) and 1 term of (1.1) for R_4, R_3, R_3 and R_1 respectively in Recurrence relation 1.1 for *n* = 4, we get (1.5) of 16 terms as:

$$R_5 = (5)_1(4)_1(3)_1(2)_1(1)_1 + (5)_1(4)_1(3)_1(2)_2 + (5)_1(4)_1(3)_2(1)_1 + (5)_1(4)_1(3)_3 + (5)_1(4)_2(2)_1(1)_1 + (5)_1(4)_2(2)_2 + (5)_1(4)_3(1)_1 + (5)_1(4)_4 + (5)_2(3)_1(2)_1(1)_1 + (5)_2(3)_1(2)_2 + (5)_2(3)_2(2)_1 + (5)_2(3)_3 + (5)_3(2)_1(1)_1 + (5)_3(2)_2 + (5)_4(1)_1 + (5)_5 \tag{1.5}$$

16 terms of (1.5) are 16 falling factorial expressions for 5!. 16 sets of bottom indices in 16 successive terms of (1.5) are : (1,1,1,1,1), (1,1,1,2), (1,1,2,1), (1,1,3), (1,2,1,1), (1,2,2), (1,3,1), (1,4), (2,1,1,1), (2,1,2), (2,2,1), (2,3), (3, 1,1) , (3, 2), (4, 1) and 5 such that: 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 2 = 1 + 1 + 2 + 1 = 1 + 1 + 3 = 1 +

2 + 1 + 1 = 1 + 1 + 2 + 2 = 1 + 3 + 1 = 1 + 4 = 2 + 1 + 1 + 1 = 2 + 1 + 2 = 2 + 2 + 1 = 2 + 3 = 3 + 2 = 4 + 1 = 5.
…. …

$$R_n = (n)_1(n-1)_1 \dots (2)_1(1)_1 + (n)_1(n-1)_1 \dots (2)_2 + \dots + \dots + (n)_n . \qquad (1.n)$$

For the subsequent demonstration, we use the following notations.

1. Compositions of n = $C(n)$.
2. All $C(n)$ = $\{C(n)\}$. Only for the mathematical representations, we use the notation: $\{C(n)\}$ to mean all $C(n)$. Otherwise we use an adjective to specify $C(n)$.
3. Number of all $C(n)$ = $NC(n)$. We know: $NC(n) = 2^{n-1}$.
4. $x_1 + \dots + x_{r-1} + \{\mathrm{C}(x_r)\}$ for $x_1 + \dots + x_r = n$ denotes some $C(n)$ which start with the common summands: $x_1, \dots, x_{r-1}$ in succession.

We use the symbol of equivalence (≡) between $\{C(n)\}$ and its implication; and similarly between $x_1 + \dots + x_{r-1} + \{\mathrm{C}(x_r)\}$ and its implication.

Examples:

(i) $\{C(3)\} \equiv 1 + 1 + 1,\ 1 + 2,\ 2 + 1,\ 3$.
(ii) $NC(3) = 4$.
(iii) $2 + 5 + \{C(3)\}$ denotes some $C(10)$ such that:
$2 + 5 + \{C(3)\} \equiv 2 + 5 + 1 + 1 + 1,\ 2 + 5 + 1 + 2,\ 2 + 5 + 2 + 1,\ 2 + 5 + 3$

Remark 1: Alternate Way of Counting *NC*(*n*) :
$NC(n) = 2^{n-1}$. This is a familiar result. It is easy to obtain the result also from the simple notations used here.
From the notations, we get***:***
For $n \geq 2$, $\{C(n)\} \equiv 1 + \{C(n-1)\},\ 2 + \{C(n-2)\},\ \dots,\ (n-1) + \{C(1)\},\ n$
Consequently For $n \geq 2$, $NC(n) = NC(n-1) + NC(n-2) + \dots + NC(1) + 1$
Then we have the successive results as shown.
$\{C(1)\} \equiv 1$. Hence $NC(1) = 1$.
$\{C(2)\} \equiv 1 + \{C(1)\},\ 2$. Hence $NC(2) = 2$.
$\{C(3)\} \equiv 1 + \{C(2)\},\ 2 + \{C(1)\},\ 3$.
Hence, $NC(3) = NC(2) + NC(1) + 1 = 2 + 1 + 1 = 2^2$.
Similarly, $NC(4) = NC(3) + NC(2) + NC(1) + 1 = 2^2 + 2 + 1 + 1 = 2^3$.
Proceeding thus, we get: For $n \geq 2$, $NC(n) = 2^{n-1}$.
Furthermore $NC(1) = 1 = 2^0$. Hence for $n \geq 1$, $NC(n) = 2^{n-1}$.
In the example (iii) above, $2 + 5 + \{C(3)\}$ denotes 2^{3-1} or 4 particular compositions of $C(10)$, that start with two common summands: 2 and 5 in succession.

(c) Ordered Compositions of *n*

The preliminary condition is (1.1). Then we find that (1.2), (1.3), (1.4), … involve with the significant ordered compositions of 2, 3, 4, … in succession. In general, (1.n) involves with the 'significant ordered compositions of *n' or*, in brief *'SOC*(*n*)' for $n \geq 2$. *SOC*(*n*) is demonstrated in the paper: Bera soumendra, 'Relationships between Ordered Compositions and Fibonacci Numbers', Journal of Mathematics Research (Canadian Center of Science and Education) Vol.7, No.3, 2015. We state below the rule for *SOC*(*n*).

Rule for *SOC*(*n*): *Under SOC(n), the summands of the 1st C(n) are all 1; the last C(n) is n itself; and for* $n \geq r \geq 2$*, if any k-th C(n) is* $x_1 + \dots + x_r$ *then (k + 1)-th C(n) is* $x_1 + \dots + x_{r-2} + (x_{r-1} + 1)$ *+ the sum of* $(x_r - 1)$ *summands which are all 1 such that if* $r \geq 3$ *then the first r – 2 summands of k-th C(n) appear also in (k +1)-th C(n) in the same order and if r = 2 then no such common part of k-th and (k +1)-th C(n) exists.*

Example: We use the symbol of equivalence (≡) between *SOC*(*n*) and its implication.
SOC(5) ≡ 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 1 + 3, 1 + 2 + 1 + 1, 1 + 2 + 2, 1 + 3 + 1, 1 + 4, 2 + 1 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1, 2 + 3, 3 + 1 + 1, 3 + 2, 4 + 1, 5. Obviously starting with any *C*(5), the successive *C*(5) under *SOC*(5) can be written by the help of the above rule.
How *SOC*(*n*) occurs in (1.n) is shown in Topic V.

IV. A Special Integer-Sequence with respect to *SOC*(*n*) for *n* ≥ 2

We introduce a concept of 'ancillary integer set' for a *C*(*n*) as follows.

Ancillary Integer Set for a *C*(*n*): We define a set of *r* ordered integers: $y_1, \ldots, y_r$ as the 'ancillary integer set' for a *C*(*n*): $x_1 + \ldots + x_r$ such that $y_1 = x_1 + \ldots + x_r$, $y_2 = x_2 + \ldots + x_r$, ..., $y_{r-1} = x_{r-1} + x_r$ and $y_r = x_r$. We use the notation: *A*(*n*) for this integer-sequence.

Example: One *C*(11) is 4 + 3 + 2 + 2. Then *A*(11) for this *C*(11) is (11, 7, 4, 2).

From the definition of *A*(*n*) for a *C*(*n*), we find some basic properties of the integers of the ancillary integer sets. When (y_1, ..., y_r) is *A*(*n*) for a *C*(*n*), then this particular *C*(*n*) is the sum of *r* successive summands: $(y_1 - y_2)$, $(y_2 - y_3)$, …, $(y_{r-1} - y_r)$, y_r. In an *A*(*n*), *r* integers appear in descending order: $y_1 > y_2 > \ldots > y_r$. The number of summands of a *C*(*n*) is equal to the number of integers of its *A*(*n*). An integer appears once only in *A*(*n*) for a *C*(*n*). In other words, there is no repetition of an integer in *A*(*n*) for a *C*(*n*). The integer *n* appears first in each of all 2^{n-1}*A*(*n*). The last integer of *A*(*n*) for a *C*(*n*) = the last summand of that *C*(*n*).

Corresponding to *SOC*(*n*), we denote the **'significant order of ancillary integer sets'** by '***SOA*(*n*)**'.

Examples: We use the symbol of equivalence (≡) between *SOA*(*n*) and its implication. For convenience, both *SOC*(*n*) and *SOA*(*n*) for some successive values of *n* are shown below.

(i) *SOC*(2) ≡ 1 + 1, 2.
SOA(2) ≡ (2, 1), (2).

(ii) *SOC*(3) ≡ 1 + 1 + 1, 1 + 2, 2 + 1, 3.
SOA(3) ≡ (3, 2, 1), (3, 2), (3, 1), (3).

(iii) *SOC*(4) ≡ 1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 1 + 3, 2 + 1 + 1, 2 + 2, 3 + 1, 4.
SOA(4) ≡ (4, 3, 2, 1), (4, 3, 2), (4, 3, 1), (4, 3), (4, 2, 1), (4, 2), (4, 1), (4).

(iv) *SOC*(5) ≡ 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 1 + 2 + 1, 1 + 1 + 3, 1 + 2 + 1 + 1, 1 + 2 + 2, 1 + 3 + 1, 1 + 4, 2 + 1 + 1 + 1, 2 + 1 + 2, 2 + 2 + 1, 2 + 3, 3 + 1 + 1, 3 + 2, 4 + 1, 5.
SOA(5) ≡ (5, 4, 3, 2, 1), (5, 4, 3, 2), (5, 4, 3, 1), (5, 4, 3), (5, 4, 2, 1), (5, 4, 2), (5, 4, 1), (5, 4), (5, 3, 2, 1), (5, 3, 2), (5, 3, 1), (5, 3), (5, 2, 1), (5, 2), (5,1), (5).… …

In the context, we define the initial conditions: *SOC*(1) ≡ 1 and *SOA*(1) ≡ (1), where *C*(1) = *A*(1) = 1.

We notice that two sets of parentheses in two terms of (1.2), four sets of parentheses in four terms of (1.3), eight sets of parentheses in eight terms of (1.4), and sixteen sets of parentheses in sixteen terms of (1.5) involve with *SOA*(2), *SOA*(3), *SOA*(4) and *SOA*(5) respectively. In general 2^{n-1} sets of parentheses in 2^{n-1} terms of (1.n) involve with *SOA*(*n*) for *n* ≥ 2. An explanation of this involvement is given in Topic V.

We find the following interesting properties of *SOA*(*n*) from its successive values.

Property 1: Under *SOA*(2), the number of occurrences of 1 in all two *A*(2) = 1.
Under *SOA*(3), the number of occurrences of each of 1 and 2 in all four *A*(3) = 2.
Under *SOA*(4), the number of occurrences of each of 1, 2 and 3 in all eight *A*(4) = 4.
Under *SOA*(5), the number of occurrences f each of 1, 2, 3and 4 in all sixteen *A*(5) = 8

…. …

The successive results show the existence of the mathematical rule which is stated in Theorem 2.

Theorem 2: *Each of 1, …, (n – 1) occurs* 2^{n-2} *times in all A(n) for n ≥ 2.*

Proof: Let $x_1, \ldots, x_r$, *h* and *k* are all positive integers such that $x_1 + \ldots + x_r + h = k + 1$ and *h* ∈ (1,…, *k*). $x_1 + \ldots + x_r + \{\mathrm{C}(h)\}$ represents some C(*k* + 1) which contain the first *r* common summands : $x_1, \ldots, x_r$ in succession; and number of these *C*(*k* + 1) is : *NC*(*h*) $= 2^{h-1}$. Consequently *r* + 1 distinct integers**:** $x_1 + \ldots + x_r + h$, $x_2 + \ldots + x_r + h$, …, $x_r + h$, and *h* in descending order occur as the first *r* + 1 common integers in 2^{h-1} *A*(*k* + 1) for these 2^{h-1} C(*k* + 1). An integer occurs once only in *A*(*k* + 1) for a *C*(*k* + 1). Hence the integer *h* occurs 2^{h-1} times in these 2^{h-1} *A*(*k* + 1). $x_1 + \ldots + x_r = k - h + 1$ such that $x_1 + \ldots + x_r$ is one *C*(*k* – *h* + 1) among all $2^{k-\mathrm{h}}$ *C*(*k* – *h* + 1). Hence *h* for *k* ≥ *h* ≥ 1 occurs $2^{k-\mathrm{h}} \cdot 2^{h-1}$ or 2^{k-1} times in all 2^k *A*(*k* + 1). Replacing *k* by *n* for *n* ≥ 2, we get: *h* for *n* – 1 ≥ *h* ≥ 1 occurs 2^{n-2} times in all 2^{n-1} *A*(*n*). This completes the proof. ▌

Corollary 1: *Number of summands in all C*(*n*) *for n* ≥ 1 = $(n+1)\ 2^{n-2}$.

Proof: Number of summands in all 2^{n-1} *C*(*n*) for *n* ≥ 2
= Number of integers in all 2^{n-1} *A*(*n*) for *n* ≥ 2 .
= [Number of occurrences of *n* + This of 1, … , (*n* – 1))] in all 2^{n-1} *A*(*n*) .
= $2^{n-1} + (n-1)\, 2^{n-2}$.
= $(n+1)\ 2^{n-2}$.
Furthermore *NC*(1) = 1. Hence we have the corollary.▌

Property 2: Under *SOA*(2), 1 occurs once in the 1st *A*(2).
Under *SOA*(3), 1 occurs in the 1st and 3rd *A*(3);

2 occurs in 1^{st} two $A(3)$.

Under $SOA(4)$, 1 occurs in the 1^{st}, 3^{rd}, 5^{th} and 7^{th} $A(4)$;
2 occurs in 1^{st} and 3^{rd} two $A(4)$;
3 occurs in 1^{st} four $A(4)$.

Under $SOA(5)$, 1 occurs the 1^{st}, 3^{rd}, 5^{th}, 7^{th}, 9^{th}, 11^{th}, 13^{th} and 15^{th} $A(4)$;
2 occurs in 1^{st}, 3^{rd}, 5^{th} and 7^{th} two $A(4)$;
3 occurs in 1^{st} and 3^{rd} four $A(4)$.
4 occurs in 1^{st} eight $A(4)$.

… …

The general form of the successive results is Theorem 3 below.

Theorem 3: *In SOA(n) for $n \geq 2$, every integer k for $n - 1 \geq k \geq 1$ occurs in each of the alternate 2^{k-1} ordered A(n) only starting with the first 2^{k-1} ordered A(n).*

Proof: Let $m + \{C(n - m)\}$ be a set of C(n). Evidently it represents 2^{n-m-1} C(n). Then each of $2^{n-m-1}A(n)$ for these C(n) contains the first two integers: n and $n - m$ in succession such that no integer d for $n > d > n - m$ occurs in these $A(n)$. The sets of $C(n)$, which occur next to $m + \{C(n - m)\}$ are: $(m + 1) + \{C(n - m - 1)\}$, $(m + 2) + \{C(n - m - 2)\}$, …, $(n - 2) + \{C(2)\}$, $(n - 1) + 1$, n in succession such that all these successive sets of C(n) altogether represent $(2^{n-m-2} + \ldots + 2 + 1 + 1)$ or 2^{n-m-1}C(n). It follows that 2^{n-m-1} $A(n)$ for the last 2^{n-m-1} C(n) cannot contain the integer: $n - m$ where $n > n - m > k$ and $n - m - 1 \geq k \geq 1$.

This is the basic law to find how an integer i for $n - 1 \geq i \geq 1$ occurs in $SOA(n)$. We can write:

$\{C(n)\} \equiv \underline{1 + \{C(n-1)\}},\ 2 + \{C(n-2)\},\ 3 + \{C(n-3)\},\ 4 + \{C(n-4)\},\ \ldots,\ (n-2) + \{C(2)\},\ (n-1) + 1,\ n.$

$\equiv \underline{1 + 1 + \{C(n-2)\}},\ 1 + 2 + \{C(n-3)\},\ 1 + 3 + \{C(n-4)\},\ \ldots, 1 + (n-3) + \{C(2)\},\ 1 + (n-2) + 1,$

$1 + (n-1),\ \underline{2 + \{C(n-2)\}},\ 3 + \{C(n-3)\},\ 4 + \{C(n-4)\},\ \ldots,\ (n-2) + \{C(2)\},\ (n-1) + 1,\ n.$

$\equiv \underline{1 + 1 + 1 + \{C(n-3)\}},\ 1 + 1 + 2 + \{C(n-4)\},\ \ldots, 1 + 1 + (n-4) + \{C(2)\},$

$1 + 1 + (n-3) + 1, 1 + 1 + (n-2),\ \underline{1 + 2 + \{C(n-3)\}},\ 1 + 3 + \{C(n-4)\},\ \ldots, 1 + (n-3) + \{C(2)\},$

$1 + (n-2) + 1,\ 1 + (n-1),\ \underline{2 + 1 + \{C(n-3)\}},\ 2 + 2 + \{C(n-4)\},\ \ldots,\ 2 + (n-4) + \{C(2)\},$

$2 + (n-3) + 1,\ 2 + (n-2),\ \underline{3 + \{C(n-3)\}},\ 4 + \{C(n-4)\},\ \ldots,\ (n-2) + \{C(2)\},\ (n-1) + 1,\ n.$

$\equiv \underline{1 + 1 + 1 + 1 + \{C(n-4)\}},\ 1 + 1 + 1 + 2 + \{C(n-5)\},\ \ldots,\ 1 + 1 + 1 + (n-5) + \{C(2)\},$

$1 + 1 + 1 + (n-4) + 1, 1 + 1 + 1 + (n-3),\ \underline{1 + 1 + 2 + \{C(n-4)\}},\ 1 + 1 + 3 + \{C(n-5)\}, \ldots,$

$1 + 1 + (n-4) + \{C(2)\},\ 1 + 1 + (n-3) + 1,\ 1 + 1 + (n-2),\ \underline{1 + 2 + 1 + \{C(n-4)\}},\ 1 + 2 + 2 + \{C(n-5)\},$

$\ldots, 1 + 2 + (n-5) + \{C(2)\},\ 1 + 2 + (n-4) + 1,\ 1 + 2 + (n-3),\ \underline{1 + 3 + \{C(n-4)\}},\ 1 + 4 + \{C(n-5)\}, \ldots,$

$1 + (n-3) + \{C(2)\},\ 1 + (n-2) + 1,\ 1 + (n-1),\ \underline{2 + 1 + 1 + \{C(n-4)\}},\ 2 + 1 + 2 + \{C(n-5)\},\ \ldots,$

$2 + 1 + (n-5) + \{C(2)\},\ 2 + 1 + (n-4) + 1,\ 2 + 1 + (n-3),\ \underline{2 + 2 + \{C(n-4)\}},\ 2 + 3 + \{C(n-5)\},\ \ldots,$

$2 + (n-4) + \{C(2)\},\ 2 + (n-3) + 1,\ 2 + (n-2), \underline{3 + 1 + \{C(n-4)\}},\ 3 + 2 + \{C(n-5)\},\ \ldots,$

$3 + (n-5) + \{C(2)\},\ 3 + (n-4) + 1,\ 3 + (n-3),\ \underline{4 + \{C(n-4)\}},\ 5 + \{C(n-5)\},\ \ldots,\ (n-2) + \{C(2)\},$

$(n-1) + 1,\ n$

... …

The under-lined part of the 1^{st} step is expanded only so that the 2^{nd} step is obtained. The under-lined 2 parts of the 2^{nd} step are expanded only so that the 3^{rd} step is obtained. The under-lined 2^2 or 4 parts of the 3^{rd} step are expanded only so that the 4^{th} step is obtained; and so on. Then we find the following results.

(i) Each of 1^{st} 2^{n-2}ordered $A(n)$ for the 1^{st} 2^{n-2} ordered $C(n)$: $1 + \{C(n - 1)$ contains the integer: $n - 1$. But none of 2^{nd} or last 2^{n-2}ordered $A(n)$ for the last $(2^{n-3} + \ldots + 2 + 1 + 1)$ or 2^{n-2}ordered $C(n)$: $2 + \{C(n - 2)\}$, …, $(n - 2) + \{C(2)$, $(n - 1) + 1$ and n contains the integer $n - 1$.

(ii) Each of the 1st and 3rd 2^{n-3}ordered $A(n)$ for the 1st and 3rd 2^{n-3}ordered $C(n)$: 1 + 1 + {$C(n – 2)$} and 2 + {$C(n – 2)$} respectively contain $n – 2$. We also notice that the first part of 1 + 1 + {$C(n – 2)$} is 1 + 1 and this of 2 + {$C(n – 2)$} is 2; and these two parts are the 1st and 2nd $C(2)$ respectively in $SOC(2)$. On the contrary, none of the 2nd2^{n-3} ordered $A(n)$ for the (2^{n-4} + … + 2 + 1 + 1) or 2^{n-3}ordered $C(n)$: 1 + 2 + {$C(n – 3)$}, …, 1 + (n – 3) + {C(2)}, 1 + (n – 2) + 1, 1 + (n – 1); and the 4th or last 2^{n-3}ordered $A(n)$ for the (2^{n-4} + … + 2 + 1 + 1) or 2^{n-3}ordered $C(n)$: 3 + {C(n – 3)}, …, (n – 2) + {$C(2)$}, (n – 1) + 1 and n contains n – 2.
(iii) Each of the 1st, 3rd, 5th and 7th 2^{n-4}ordered $A(n)$ for the 1st, 3rd, 5th and 7th 2^{n-4}ordered $C(n)$: 1 + 1 + 1 + {C(n – 3)}, 1 + 2 + {C(n – 3)}, 2 + 1 + {C(n – 3)} and 3 + {C(n – 3)} contains n – 3. We also notice that the first parts of the four sets of ordered $C(n)$ are: 1 + 1 + 1, 1 + 2, 2 + 1 and 3, which are the 1st, 2nd, 3rd and 4th $C(3)$ respectively in $SOC(3)$. On the contrary, none of the 2nd 2^{n-4}ordered $A(n)$ for the 2nd 2^{n-4} ordered $C(n)$: 1 + 1 + 2 + {C(n – 4)}, …, 1 + 1 + (n – 4) + {C(2)}, 1 + 1 + (n – 3) + 1 and 1 + 1 + (n – 2); the 4th 2^{n-4}ordered $A(n)$ for the 4th 2^{n-4} ordered $C(n)$: 1 + 3 + {$C(n – 4)$}, …, 1 + (n – 3) + {C(2)}, 1 + (n – 2) + 1 and 1 + (n – 1); the 6th 2^{n-4} ordered $A(n)$ for the 6th 2^{n-4}ordered $C(n)$: 2 + 2 + {C(n – 4)}, …, 2 + (n – 4) + {C(2)}, 2 + (n – 3) + 1 and 2 + (n – 2); and the 8th or last 2^{n-4}ordered $A(n)$ for the 8th or last 2^{n-4}ordered $C(n)$: 4 + {C(n – 4)}, …, (n – 2) + {C(2)}, (n – 1) + 1 and n contains n – 3.
… …

In general, an integer k for $n – 1 \geq k \geq 1$ occurs in each of the 1st, 3rd, … , (2^{n-k} – 1)th 2^{k-1} ordered $A(n)$ for the 1st, 3rd, … , (2^{n-k} – 1)th 2^{k-1} ordered $C(n)$; but k cannot appear in the other ordered $A(n)$ for the remaining sets of ordered $C(n)$. Thus the theorem is established.▌

V. Simultaneous Existence of Ordered Compositions and Ordered Ancillary Sets in (1.n)

In this topic, we give an explanation of simultaneous occurrences of $SOC(n)$ and $SOA(n)$ in the special expansion in (1.n).

(i) *SOC*(*n*) : The preliminary condition is: $SOC(1) \equiv 1$ Then we find in succession:

$$\begin{aligned}
SOC(2) &\equiv 1 + SOC(1),\ 2\\
&\equiv 1 + 1,\ 2.\\
SOC(3) &\equiv 1 + SOC(2),\ \ 2 + SOC(1),\ 3\\
&\equiv 1 + 1 + 1,\ 1 + 2,\ 2 + 1,\ 3.\\
SOC(4) &\equiv 1 + SOC(3),\ 2 + SOC(2),\ 3 + SOC(1),\ 4.\\
&\equiv 1 + 1 + 1 + 1,\ 1 + 1 + 2,\ 1 + 2 + 1,\ 1 + 3,\ 2 + 1 + 1,\ 2 + 2,\ 3 + 1,\ 4.
\end{aligned}$$

In this way,

$$SOC(n) \text{ for } n \geq 2 \ \equiv\ 1 + SOC(n-1),\ 2 + SOC(n-2),\ \ldots,\ (n-1) + SOC(1),\ n. \tag{2}$$

The general form of the sets of $C(n)$ on the right of (2) is $a + SOC(b)$ for $a + b = n$ such that:

$$a + SOC(b) \equiv a + 1^{\text{st}}\ C(b) \text{ in } SOC(b),\ a + 2^{\text{nd}}\ C(b) \text{ in } SOC(b),\ \ldots,\ a + 2^{b-1}\text{-th } C(b) \text{ in } SOC(b). \tag{3}$$

(ii) *SOA*(*n*) : The preliminary condition is: $SOA(1) \equiv 1$. We can find $SOA(2)$, $SOA(3)$, … in succession by using the notation : [k, $SOA(n)$] such that:

$$[k, SOA(n)\ ] \equiv (k,\ 1^{\text{st}}A(n) \text{ in } SOA(n)),\ (k,\ 2^{\text{nd}}A(n) \text{ in } SOA(n)),\ \ldots,\ (k,\ 2^{n-1}\text{-th}\,A(n) \text{ in } SOA(n)). \tag{4}$$

Then,

$$\begin{aligned}
SOA(2) &\equiv [2, SOA(1)],\ 2\\
&\equiv (2, 1),\ 2.\\
SOA(3) &\equiv [3, SOA(2)],\ \ [3, SOA(1)],\ 3\\
&\equiv (3, 2, 1),\ (3, 2),\ (3,1),\ 3.\\
SOA(4) &\equiv [4, SOA(3)],\ [4, SOA(2)],\ [4, SOA(1)],\ 4\\
&\equiv (4, 3, 2, 1),\ (4, 3, 2),\ (4, 3, 1),\ (4, 3),\ (4, 2, 1),\ (4, 2),\ (4, 1),\ 4.
\end{aligned}$$

Thus,

$$SOA(n) \text{ for } n \geq 2 \ \equiv\ [n, SOA(n-1)],\ [n, SOA(n-2)],\ldots,[n, SOA(1)],\ n \tag{5}$$

We obtain $SOC(2)$, $SOC(3)$, $SOC(4)$, … successively following the rule that each sequence is the immediate consequence of all the preceding sequences. We follow the same rule in the process of obtaining $SOA(2)$, $SOA(3)$, $SOA(4)$, … in succession. In fact any higher sequence under $SOC(n)$ for $n \geq 2$ occur by the method of recursive substitution which is followed in 2^{n-1} sets of bottom indices of the falling factorial expressions in (1.n); and similarly any higher sequence under $SOA(n)$ for $n \geq 2$ occur by the same method which is followed in 2^{n-1} sets of integers of 2^{n-1} sets of parentheses of the falling factorial expressions in (1.n).

VI. Ordered Integers with respect to *SOC*(*n*) and *SOA*(*n*) for $2 \le n \le 12$

(a) Ordered Integers with respect to *SOC*(*n*) for $2 \le n \le 12$

SOC(*n*) involves nicely with the ascending order of 2^{n-1} integers if $2 \le n \le 12$. Let us take the example of *SOC*(4). Omitting + signs from all 8 *C*(4) under *SOC*(4), we find 8 integers as 1111, 112, 121, 13, 211, 22, 31 and 4 in succession. The first one of these is of maximum 4 digits; and then 0s are put on the rights of next 7 integers to make them all as the integers of 4 digits. Thus the first one and new 7 integers in succession are 1111, 1120, 1210, 1300, 2110, 2200, 3100, 4000; which obviously occur in ascending order. Now the question is, "Is this property of *SOC*(*n*) is true for the compositions of every positive integer *n*?" To find the solution, in general we can consider an inequality involving the successive summands of *k*-th and (*k* + 1)-th compositions as shown.

$$10^{n-1}x_1 + \dots + 10^{n-r}x_r < 10^{n-1}x_1 + \dots + 10^{n-r+2}x_{r-2} + 10^{n-r+1}(x_{r-1}+1) + 10^{n-r} + \dots + 10^{n-r-x_r+2} \quad (6)$$

[Last $x_r - 1$ terms on the right of the inequality occur when $x_r \ge 2$]

$$\Rightarrow 10^{n-r}x_r < 10^{n-r+1} + 10^{n-r} + \dots + 10^{n-r-x_r+2}.$$

$$\Rightarrow x_r \in (1, \dots, 11).$$

The smallest number of summands of *k*-th composition is 2. The smallest and greatest values of a summand of *k*-th compositions are 1 and $n - 1$ respectively. When $n = 2$ then $x_r = 1$ and when $n = 12$ then the condition of x_r is: $1 \le x_r \le 11$; but both $1 \le x_r \le 11$ and $x_r \ge 12$ are possible when $n \ge 13$. It follows that the inequality (6) is true for $2 \le n \le 12$ and always not true for $n \ge 13$. Hence corresponding to *SOC*(*n*) for $2 \le n \le 12$, we can find 2^{n-1} integers in ascending order; but we cannot find the ascending order of all 2^{n-1} integers corresponding to *SOC*(*n*) if $n \ge 13$.

(b) Ordered Integers with respect to *SOA*(*n*) for $2 \le n \le 12$

It has been shown in the above topic that *SOC*(*n*) for $2 \le n \le 12$ has close connection with 2^{n-1} integers in ascending order. In like manner, we can find that *SOA*(*n*) for $2 \le n \le 12$ has close connection with 2^{n-1} integers in descending order. If *k*-th *A*(*n*) in *SOA*(*n*) is $y_1, \dots, y_r$ then (*k* + 1)-th *A*(*n*) is $y_1, \dots, y_{r-1}, y_r - 1, y_r - 2, \dots, 1$. We find the following inequality with respect to *k*-th and (*k* + 1)-th *A*(*n*): For $2 \le n \le 12$,

$$10^{n-1}y_1 + \dots + 10^{n-r}y_r > 10^{n-1}y_1 + \dots + 10^{n-r+1}y_{r-1} + 10^{n-r}(y_r - 1) + 10^{n-r-1}(y_r - 2) + 10^{n-r-2}(y_r - 3) + \dots + 10^{n-r-y_r+2} \quad (7)$$

That is, we find all 2^{n-1} integers in descending order corresponding to *SOA*(*n*) for $2 \le n \le 12$. For example, regarding *SOA*(4): (4, 3, 2, 1), (4, 3, 2), (4, 3, 1), (4, 3), (4, 2, 1), (4, 2), (4, 1), (4); we find 8 integers in descending order: 4321 > 4320 > 4310 > 4300 > 4210 > 4200 > 4100 > 4000. But we cannot find the ascending order of all 2^{n-1} integers corresponding to *SOA*(*n*) for $n \ge 13$.

Remark 2: A Math Rule for the last integers in 2^{n-1} Sets of Integers Corresponding to *SOA*(*n*):

The rule is guessed from the successive forms of *SOA*(2), *SOA*(3), *SOA*(4), *SOA*(5), … and is stated in Conjecture 1.

Conjecture 1: *The last integer in any k-th set of integers for* $1 \le k \le 2^{n-1}$ *under SOA*(*n*) is $z + 1$ *if the lowest power in the expression of k in binary scale:* $k = 2^{h_1} + 2^{h_2} + \dots$, $h_1 > h_2 > \dots$ *is z.*

Example: The expressions of the first 8 natural numbers in binary scale are: $1 = 2^0$; $2 = 2^1$; $3 = 2^1 + 2^0$; $4 = 2^2$; $5 = 2^2 + 2^0$; $6 = 2^2 + 2^1$; $7 = 2^2 + 2^1 + 2^0$; and $8 = 2^3$. The lowest powers with respect to the 8 expressions are: 0, 1, 0, 2, 0, 1, 0 and 3 respectively. Then the last integers in the 8 sets of integers under *SOA*(4) are: (0 + 1, 1 + 1, 0 + 1, 2 + 1, 0 + 1, 1 + 1, 0 + 1, 3 + 1) or, (1, 2, 1, 3, 1, 2, 1, 4) in succession.

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