A Study on Soft α -open Sets

D. Sivaraj and V.E. Sasikala*

Meenakshi Academy of Higher Education and Research, Meenakshi University, Chennai, Tamil Nadu, India.

Abstract: The aim of this paper is to define the concepts of soft α -open sets and soft α -closed sets in soft topological spaces by studying the basics of soft set theory formulated by D. Molodstov. **Keywords:** soft set, soft topology, soft open sets and soft closed sets.

I. Introduction

After the introduction of Soft set theory by Molodtsov [4] in 1999 as a generic mathematical application in dealing with the vagueness of not well defined objects, many researchers followed him. Since the well established mathematical applications are very crisp and clear while computing on formal modeling, reasoning and computing, some of the engineering sciences, life sciences, social sciences and ecological sciences are not very clear always. Shabir and Naz [10] explained soft topological spaces and its basic notations in detail.

Maji [7] defined equality of two soft sets, subset and superset of Soft set, complement of a soft set, null soft set and absolute soft set with examples. With the latest advanced techniques researcher have now applied them in operations research, Riemans integration, Game theory, theory of probability and researched several basic notations of soft set theory. Naim et.al., [8] presented the foundations of the theory of soft topological spaces to open the door for experiments for soft mathematical concepts and structures that are based on soft set-theoretic operations. Recently Soft topology has been studied in depth in the papers [1, 2, 3, 5, 6, 9, 11, 12, 13, 14]. In this paper, we make a theoretical study of the new set called soft α -open set and soft α -closed sets in soft topological space.

II. Preliminaries

The following definitions are essential for the development of the paper.

Definition 2.1. [4] Let U be an initial universe and E be a set of parameters. Let P(U) denote the power set of U. The pair (F, E) or simply F_E , is called a *soft set* over U, where F is a mapping given by $F : E \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in U$, F(e) may be considered as the set of e - approximate elements of the soft set F. The collection of all soft sets over U and E is denoted by S(U). If $A \subseteq E$, then the pair (F, A) or simply F_A , is called a *soft set* over U, where F is a mapping $F : A \to P(U)$. Note that for $e \notin A$, $F(e) = \phi$.

Definition 2.2. [7] The union of two soft sets of F_B and G_C over the common universe U is the soft set H_D , where B and C are subsets of the parameter set E, $D = B \cup C$ and for all $e \in D$, H(e) = F(e) if $e \in B - C$, H(e) = G(e) if $e \in C - B$ and $H(e) = F(e) \cup G(e)$ if $e \in B \cap C$, we write $F_B \widetilde{U} G_C = H_D$.

Definition 2.3. [7] The *intersection of two soft sets* of F_B and G_C over the common universe U is the soft set H_D , where $D = B \cap C$ and for all $e \in D$, $H(e) = F(e) \cap G(e)$ if $D = B \cap C$. We write $F_B \cap G_C = H_D$.

Definition 2.4. [7] Let F_B and G_C be soft sets over a common universe set U and B, $C \subseteq E$. Then F_B is a *soft subset* of G_C , denoted by $F_B \cong G_C$, if (i) $B \subseteq C$ and (ii) for all $e \in B$, F(e) = G(e). Also, G_C , is called the soft super set of F_B and is denoted by $F_B \cong G_C$.

Definition 2.5. [7] The soft sets F_B and G_C over a common universe set U are said to be *soft equal*, if $F_B \cong G_C$, and $F_B \cong G_C$. Then we write $F_B = G_C$.

Definition 2.6. [7] A soft set F_B over U is called a *null soft set* denoted by F_{ϕ} , if for all $e \in B$, $F(e) = \phi$.

Definition 2.7. [8] The *relative complement* of a soft set F_A , denoted by F_A^c , is defined by the approximate function $f_{A^c}(e) = f_A^c(e)$, where $f_A^c(e)$ is the complement of the set $f_A(e)$, that is $f_A^c(e) = U - f_A(e)$ for all $e \in E$. It is easy to see that $(F_A^c)^c = F_A, F_{\Phi}^c = F_E$ and $F_E^c = F_{\Phi}$.

Definition 2.8. [7] Let U be an initial universe and E be a set of parameters. If $B \subseteq E$, the soft set F_B over U is called an *absolute soft set*, if for all $e \in B$, F(e) = U.

The following is the definition of soft topology used by various authors.

Definition 2.9. [1] Let U be an initial universe and E be a set of parameters. Let τ be a sub collection of S(U), the collection of soft sets defined on U. Then τ is a *soft topology* if it satisfies the following conditions.

(i) F_{Φ} , $F_E \in \tau$.

(ii) The union of any number of soft sets in τ belongs to τ .

(iii) The intersection of any two soft sets in τ belongs to τ .

If this definition is considered for further development, then the results are similar to that of results in topological spaces. Therefore, throughout the paper the following definition of soft topology is used.

Definition 2.10. [8] Let U be an initial universe and E be a set of parameters. Let $F_A \in S(U)$. A *soft topology* on F_A , denoted by $\tilde{\tau}$, is a collection of soft subsets of F_A having the following properties: 1. F_{Φ} , $F_A \in \tilde{\tau}$.

2. $\{F_{A_i} \cong F_A : i \in I\} \cong \tilde{\tau} \Rightarrow \tilde{U}_{i \in I} F_{A_i} \in \tilde{\tau}$.

3. $\{F_{A_i} \cong F_A : 1 \le i \le n, n \in N\} \cong \tilde{\tau} \Rightarrow \widetilde{\cap}_{i=1}^n F_{A_i} \in \tilde{\tau}$. The pair $(F_A, \tilde{\tau})$ is called a soft topological spaces.

Definition 2.11. [10] Let $(F_A, \tilde{\tau})$ be a soft topological space in F_A . Elements of $\tilde{\tau}$ are called *soft open* sets. A soft set F_B in F_A is said to be a *soft closed set* in F_A , if its relative complement F_B^C belongs to $\tilde{\tau}$.

In [13] it is established that under this definition of *soft closed sets*, Example 2.1 of [13] shows that no soft closed set exist in a soft topological space and moreover F_{ϕ} and F_A are not soft closed sets. Therefore, we follow the following definition of soft closed sets.

Definition 2.12. [13] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \cong F_A$. Then F_B is said to be *soft closed* set in F_A , if the soft set $F_{B/A}^c$, where *soft open* in F_A , where $F_{B/A}^c = F_B^c \cap F_A$.

Definition 2.13. [14] Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B be a soft set in F_A . (i) The *soft interior* of F_B is the soft set $int(F_B) = \tilde{U} \{F_C : F_C \text{ is soft open set and } F_C \cong F_B \}$. (ii) The *soft closure* of F_B is the soft set $cl(F_B) = \tilde{\cap} \{F_C : F_C \text{ is soft closed set and } F_C \cong F_B \}$.

Lemma 2.14. [13] Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B and F_C be a soft set in F_A . Then (i) $\operatorname{int}(\operatorname{int}(F_B)) = \operatorname{int}(F_B)$. (ii) $F_B \cong F_C$ implies $\operatorname{int}(F_B) \cong \operatorname{int}(F_C)$. (iii) $\operatorname{int}(F_B) \widetilde{\cap} \operatorname{int}(F_C) = \operatorname{int}(F_B \widetilde{\cap} F_C)$. (iv) $\operatorname{int}(F_B) \widetilde{\cup} \operatorname{int}(F_C) \cong \operatorname{int}(F_B \widetilde{\cup} F_C)$.

Lemma 2.15. [13] Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B and F_C be a soft set in F_A . Then the following hold.

(i) $\operatorname{cl}(\operatorname{cl}(F_B)) = \operatorname{cl}(F_B)$. (ii) $F_B \cong F_C$ implies $\operatorname{cl}(F_B) \cong \operatorname{cl}(F_C)$. (iii) $(\operatorname{cl}(F_B))_{/A}^c = \operatorname{int}(F_{B/A}^c)$ and $(\operatorname{int}(F_B))_{/A}^c = \operatorname{cl}(F_{B/A}^c)$ (iv) $\operatorname{cl}(F_B) \cap \operatorname{cl}(F_C) \cong \operatorname{cl}(F_B \cap F_C)$. (v) $\operatorname{cl}(F_B) \widetilde{\operatorname{Ucl}}(F_C) = \operatorname{cl}(F_B \widetilde{\operatorname{UF}}_C)$.

III. Soft α -open Set and Soft α -closed Set

This section is devoted to the study of soft α -open sets and Soft α -closed sets where we use the Definition 2.11 of closed sets and discuss its properties.

Definition 3.1. Let F_B be a soft subset of a soft topological space $(F_A, \tilde{\tau})$. F_B is said to be a *soft* α -open set, if $F_B \cong int(cl(int(F_B)))$.

Example 3.2. Let $U = \{a, b, c\}$, $E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\} \subseteq E$. $F_A = \{(e_1, \{a, b\}), (e_2, \{b, c\})\}, F_1 = \{(e_1, \{a\})\}, F_2 = \{(e_1, \{b\})\}, F_3 = \{(e_1, \{a, b\})\}, F_4 = \{(e_2, \{b\})\}, F_5 = \{(e_2, \{c\})\}, F_6 = \{(e_2, \{b, c\})\}, F_7 = \{(e_1, \{a\}), (e_2, \{b\})\}, F_8 = \{(e_1, \{a\}), (e_2, \{c\})\}, F_9 = \{(e_1, \{a\}), (e_2, \{b, c\})\}, F_{10} = \{(e_1, \{b\}), (e_2, \{b\})\}, F_{11} = \{(e_1, \{a\}), (e_2, \{b, c\})\}, F_{12} = \{(e_1, \{b\}), (e_2, \{b, c\})\}, F_{13} = \{(e_1, \{a, b\}), (e_2, \{b\})\}, F_{14} = \{(e_1, \{a, b\}), (e_2, \{b\})\}, F_{15} = F_A, F_{16} = F_{\phi}.$ Let $\tilde{\tau} = \{F_{\phi}, F_A, F_2, F_3, F_{11}, F_{12}, F_{14}\}$. Then $(F_A, \tilde{\tau})$ is a soft topological space. The family of all soft closed sets is $\{F_A, F_{\phi}, F_9, F_6, F_7, F_1, F_4\}$. The family of soft α -open sets is $\{F_A, F_{\phi}, F_2, F_3, F_{10}, F_{11}, F_{12}, F_{14}\}$.

Theorem 3.3. Every soft open set in a soft topological space $(F_A, \tilde{\tau})$ is a soft α -open set. **Proof.** The proof follows from the Definition 3.1 \Box The following Example 3.4 shows that the reverse implication of Theorem 3.3 is not true.

Example 3.4. Consider the soft topological space of Example 3.2. Here F_{10} and F_{13} are soft α -open sets but not soft open sets.

Theorem 3.5. Let $(F_A, \tilde{\tau})$ be a soft topological space and $\{F_B : B \in I\}$ be a family of soft α -open sets in $(F_A, \tilde{\tau})$. Then $\bigcup_{B \in I} F_B$ is also a soft α -open set.

Proof. Let $\{F_B : B \in I\}$ be a family of soft α -open sets in $(F_A, \tilde{\tau})$. Then for each F_B ,

 $F_B \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(F_B)))$ and so $F_B \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(F_B))) \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(\cup F_B)))$, since soft interior and closure are monotonic functions. This implies that $\bigcup F_B \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(\cup F_B)))$ and $\bigcup_{B \in I} F_B$ is a soft α -open set. \Box

The following Lemma 3.6 is essential to prove that the intersection of two soft α -open sets is a soft α -open set. If U is the universal set, E is the parameter set and $A \subseteq E$, then $P = (e, \{u\})$ where $e \in A$ and $u \in U$, is a soft point in F_A if $u \in F_A(e)$. In Theorem 2.10 of [13], it is established that a point $P \in cl(F_B)$ if and only if every soft open set F_C containing P soft intersects F_B .

Lemma 3.6. Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B and F_C are any two soft sets in $(F_A, \tilde{\tau})$. If F_C is a soft open set, then $\operatorname{cl}(F_B) \cap F_C \cong \operatorname{cl}(F_B \cap F_C)$. **Proof**. Let $\operatorname{P} \in \operatorname{cl}(F_B) \cap F_C$. Then $\operatorname{P} \in \operatorname{cl}(F_B)$ and $\operatorname{P} \in F_C$. If F_D is soft open set containing $P \in F_A$, then $F_C \cap F_D$ is a soft open set containing P. Therefore, $P \in \operatorname{cl}(F_B) \Rightarrow (F_C \cap F_D) \cap F_B \neq F_{\phi}$ $\Rightarrow F_C \cap (F_D \cap F_B) \neq F_{\phi} \Longrightarrow P \in \operatorname{cl}(F_B \cap F_C)$. Thus $P \in \operatorname{cl}(F_B) \cap F_C \Rightarrow P \in \operatorname{cl}(F_B \cap F_C)$. Therefore $\operatorname{cl}(F_B) \cap F_C \subseteq \operatorname{cl}(F_B \cap F_C)$. \Box

Theorem 3.7. Let $(F_A, \tilde{\tau})$ be a soft topological space. If F_B and F_C are any two soft α -open sets in $(F_A, \tilde{\tau})$, then $F_B \cap F_C$ is a soft α -open set.

Proof. If F_B and F_C are any two soft α -open sets in $(F_A, \tilde{\tau})$, then $F_B \cong int(cl(int(F_B)))$ and $F_C \cong int(cl(int(F_C)))$. Now, $F_B \cap F_C \cong int(cl(int(F_B))) \cap int(cl(int(F_C))))$ $= int[cl(int(F_B))) \cap int(cl(int(F_C)))]$ $\cong int[cl[int(F_B) \cap int(cl(int(F_C)))]]$, by Lemma 3.6 $\cong int[cl[cl[int(F_B) \cap int(F_C)]]$, by Lemma 3.6 $= int(cl(int(F_B \cap F_C)))$. Therefore, $F_B \cap F_C$ is a soft α -open set. \Box

The following Theorem follows from Theorem 3.5 and Theorem 3.7.

Theorem 3.8. Let $(F_A, \tilde{\tau})$ be a soft topological space. If $\alpha(\tilde{\tau})$ is the family of all soft α -open sets, then $\alpha(\tilde{\tau})$ is a soft topology.

The following Theorem 3.9 gives a characterization of soft α -open sets.

Theorem 3.9. Let F_B be a subset of a soft topological space $(F_A, \tilde{\tau})$. Then F_B is a soft α -open set if and

only if there exists a soft open set F_C such that $F_C \cong F_B \cong int(cl(F_C))$.

Proof. Suppose that there exists a soft open set F_C such that $F_C \cong F_B \cong \operatorname{int}(\operatorname{cl}(F_C))$. Since $F_B \cong \operatorname{int}(\operatorname{cl}(F_C)) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(F_C))) \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(F_B)))$ by hypothesis and so by Lemma 2.14 (ii), F_B is soft α -open set. On the other hand, let F_B be soft α -open set. Then $F_B \cong \operatorname{int}(\operatorname{cl}(\operatorname{int}(F_B)))$. Let $\operatorname{int}(F_B) = F_C$. Since $\operatorname{int}(F_B) \cong F_B$, $F_C \cong F_B$ and also $F_B \cong \operatorname{int}(\operatorname{cl}(F_C))$. Hence there exists a soft open set F_C such that $F_C \cong F_B \subseteq \operatorname{int}(\operatorname{cl}(F_C))$. \Box

Definition 3.10. Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B be a soft subset of F_A . F_B is said to be *soft a-closed set*, if its soft complement is a soft *a*-open set.

One can easily prove the following Theorem 3.11.

Theorem 3.11. Let $(F_A, \tilde{\tau})$ be a soft topological space. Then the following hold. (i) Arbitrary intersection of soft α -closed sets is a soft α -closed sets. (ii) Finite union of soft α -closed sets is a soft α -closed sets. (iii) F_A and F_{ϕ} are soft α -closed set.

Theorem 3.12. Let F_B be a subset of a soft topological space $(F_A, \tilde{\tau})$. Then F_B is a soft α -closed set if and only if there exists a soft closed set F_C such that $int(cl(F_C)) \cong F_B \cong F_C$.

Proof. Let F_B be soft α -closed set. Then $cl(int(cl(F_B))) \cong F_B$. Let $cl(F_B) = F_C$. Then F_C is soft closed set. Since $F_B \cong cl(F_B)$, $F_B \cong F_C$ and $cl(int(cl(F_B))) \cong F_B \Rightarrow cl(int(F_C)) \cong F_B$. Thus there exists a soft closed set F_C such that $cl(int(F_C)) \cong F_B \cong F_C$. On the other hand, suppose there exists a soft closed set F_C such that $cl(int(F_C)) \cong F_B \cong F_C$. Since F_C is soft closed set, $cl(F_C)=F_C$, by hypothesis, $cl(int(F_C)) \cong F_B \Rightarrow cl(int(cl(F_C))) \cong cl(int(cl(F_C))) \cong F_B \Rightarrow cl(int(cl(F_C))) \cong cl(int(cl(F_C))) \subseteq cl(F_C) \Rightarrow cl(F_C)$

Theorem 3.13. Let F_B be a subset of a soft topological space $(F_A, \tilde{\tau})$. Then F_B is a soft α -closed set if and only if $cl(int(cl(F_B))) \cong F_B$.

Proof. Let F_B be a soft α -closed set. So F_B^c is a soft α -open set. Therefore, by Lemma 2.15(iii), $F_B^c \subseteq int(cl(int(F_B^c))) = int(cl(cl(F_B)^c)) = int[int(cl(F_B))]^c = [cl(int(cl(F_B)))]^c \Rightarrow cl(int(cl(F_B))) \subseteq F_B$. The converse part is similar. \Box

Definition 3.14. Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B be a soft set in F_A .

(i) The soft α -interior of F_B is the soft set \widetilde{U} { F_C : F_C is soft α -open set and $F_C \cong F_B$ } and is denoted by α -int(F_B).

(ii) The soft α -closure of F_B is the soft set $\widetilde{\cap} \{F_C : F_C \text{ is soft } \alpha\text{-closed set and } F_C \cong F_B\}$ and is denoted by $\alpha\text{-cl}(F_B)$.

Clearly, α -cl(F_B) is the smallest soft α -closed set containing F_B and α - int(F_B) is the largest soft α -open set contained in F_B .

Remarks 3.15. Let $(F_A, \tilde{\tau})$ be a soft topological space and let F_B be a soft set in F_A . Then $(F_A, \tilde{\tau})$ is also a soft topological space. Hence proof of the following Theorems 3.16, 3.17 and 3.18 follow from the similar already established results.

Theorem 3.16. Let $(F_A, \tilde{\tau})$ be a soft topological space and let F_B be a soft set in F_A . Then

(i) F_B belongs to soft α -closed set in $(F_A, \tilde{\tau})$ if and only if $F_B = \alpha$ -cl (F_B) .

(ii) F_B belongs to soft α -open set in $(F_A, \tilde{\tau})$ if and only if $F_B = \alpha$ -int (F_B) .

- (iii) α -cl(F_{ϕ}) = F_{ϕ} and α -cl(F_A) = F_A .
- (iv) α -int $(F_{\phi}) = F_{\phi}$ and α -int $(F_A) = F_A$.
- (v) α -cl(α -cl(F_B)) = α -cl(F_B).
- (vi) α -int(α -int(F_B)) = α -int(F_B).
- (vii) $(\alpha$ -cl $(F_B))^C = \alpha$ -int (F_B^C) .
- (viii) $(\alpha \operatorname{int}(F_B))^C = \alpha \operatorname{cl}(F_B^C).$

Theorem 3.17. Let $(F_A, \tilde{\tau})$ be a soft topological space and let F_B and F_C be two soft sets in F_A , then (i) $F_B \cong F_C \Rightarrow \alpha \operatorname{-int}(F_B) \cong \alpha \operatorname{-int}(F_C)$.

(ii) $F_B \cong F_C \Rightarrow \alpha \operatorname{-cl}(F_B) \cong \alpha \operatorname{-cl}(F_C)$. (iii) $\alpha \operatorname{-cl}(F_B \tilde{U}F_C) = \alpha \operatorname{-cl}(F_B)\tilde{U}\alpha \operatorname{-cl}(F_C)$. (iv) $\alpha \operatorname{-int}(F_B \tilde{\Omega} F_C) = \alpha \operatorname{-int}(F_B) \tilde{\Omega}\alpha \operatorname{-int}(F_C)$.

DOI: 10.9790/5728-1205067074

www.iosrjournals.org

(v) α -cl($F_B \cap F_C$) $\cong \alpha$ -cl(F_B) $\cap \alpha$ -cl(F_C).

(vi) α -int($F_B \tilde{U} F_C$) $\cong \alpha$ -int(F_B) $\tilde{U} \alpha$ -int(F_C).

The proof of the following Theorem follows from Lemma 3.6.

Theorem 3.18. Let $(F_A, \tilde{\tau})$ be a soft topological space and F_B and F_C are any two soft sets in $(F_A, \tilde{\tau})$. If F_C is a soft α -open set, then $F_C \cap \alpha$ -cl $(F_B) \cong \alpha$ -cl $(F_B \cap F_C)$.

References

- I. Arockiarani and A. Arokia Lancy, "Generalized soft $g\beta$ closed sets and soft $gs\beta$ closed sets in soft topological spaces," [1]. International Journal of Mathematical Archive 4(2), 2013, 17-23.
- [2]. M.I. Ali, F.Feng, X.Liu, W.K.Min, M.Shabir, "On some new operations in soft set theory," Computers and Mathematics with Applications, 57, 2009, 1547-1553. J. Krishnaveni and C. Sekar, "Soft semi connected and soft locally semi connected properties in soft topological spaces,"
- [3]. International Journal of Mathematics and soft Computing, 3(3), 2013, 85-91.
- [4]. D. Molodtsov, "Soft set theory-first results," Computers and Mathematics with Applications, 37(4-5), 1999, 19-31.
- D.Molodtsov, V.Y. Leonov, D.V.Kovkov, "Soft sets technique and its application," Nechetkie Sistemy i Myagkie Vychisleniya [5]. 1(1), 2006, 8-39.
- [6]. J. Mahanta, and P.K. Das, "On soft topological space via semiopen and semiclosed soft sets," arXiv:1203.4133v1[math.GN] 2012.
- P.K. Maji, R.Biswas, and A.R. Roy, "Soft set theory," Computers and Mathematics with Applications, 45(4-5), (2003), 555-[7]. 562.
- Naim Çağman, Serkan Karataş, and Serdar Enginoglu, "Soft topology," Computers and Mathematics with Applications, [8]. 62, 351-358, doi: 10.1016/j.camwa.2011.05.016 , 2011.
- Naim Çağman, Serdar Enginoglu, "Soft set theory and uni-int decision making," European Journal of Operational Research [9]. 207, 2010, 848-855.
- [10]. M. Shabir and M. Naz, "On soft topological spaces," Computers and Mathematics with Applications, 61(7), 2011, 1786-1799. M. Shabir and M. Naz, "Some properties of soft topological spaces," Computers and Mathematics with Applications, 62, [11]. 2011, 4058-4067.
- T. Som, "On the theory of soft sets soft relation and fuzzy soft relation," Proc. of the National Conference on Uncertainty: [12]. A Mathematical Approach, UAMA-06, Burdwan, 1-9, 2006.
- V.Renukadevi, and S.D. Shanthi, "Note on soft topological spaces," Journal of Advanced Research in Pure Mathematics, [13]. 7(12), 2015, 1-15.
- [14]. I. Zorlutuna, M. Akdag, W.K. Min, and S. Atmaca, "Remarks on soft topological spaces," Annals of Fuzzy Mathematics and Informatics, 3(2), (2012), 171-185.