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On Some Coefficient Estimates For Certain Subclass of Analytic And Multivalent Functions

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Abstract: *In this paper, motivated by the works of Jenkins [11], Leung [12] and Panigrahi and Murugusundaramoorthy [16] we defined a subclass of p - valent analytic functions using a generalized differential operator and compute coefficient differences. We also point out, as particular cases, the results obtained earlier by various authors.*

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I. Introduction and Definition

Let A_p denote the class of analytic functions in the open unit disk $\mathrm{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad \left(p \in \mathbb{N} = \{1,2,3,\ldots\}\right) \tag{1.1}$$

and let $\mathrm{A} = \mathrm{A}_1$.

Let S denote the subclass of A_p consisting of multivalent functions.

A function $f \in \mathrm{A}_p$ given by (1.1) is said to be $p-$ valently starlike if it satisfies the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{pf(z)}\right) > 0, \quad (z \in \mathrm{U}).$$

We denote this class of functions by S_p^*. Note that the class S_p^* reduces to $\mathrm{S}_1^* := \mathrm{S}^*$, the class of starlike functions in U, introduced by Robertson [17].

A function $f \in \mathrm{A}_p$ is said to be p-valently convex if it satisfies the condition

$$\mathrm{Re}\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad (z \in \mathrm{U}).$$

We denote by C_p the familiar subclass of A_p. In particular $p=1$, $\mathrm{C}_1 := \mathrm{C}$ the class of convex functions in U, introduced by Robertson [17] (also see [4]).

For $n \geq 2$, Hayman [9] showed the difference of successive coefficients is bounded by an absolute constant i.e.

$$||a_{n+1}| - |a_n|| \leq A.$$

Using different technique, Milin [15] showed that $A < 9$. Ilina [10] improved this to $A < 4.26$. Further, Grispan [8] restricted to $A < 3.61$. For starlike function S^*, Leung [12] proved that the best possible bound is $A = 1$. On the other hand, it is known that for the class S, A cannot be reduced to 1. When $n = 2$, Golusin [5,6], Jenkins [11] and Duren [4] showed that for $f \in \mathrm{S}$, $-1 \leq |a_3| - |a_2| \leq 1.029\ldots$ and that both upper and lower bounds in (1.1) are sharp. When $n = 2$ and $n = 3$, Panigrahi [16] showed that for $f \in \mathrm{C}$, $|a_3| - |a_2| \leq 0.521$ and $|a_4| - |a_3| \leq 0.521$. Also for $f \in \mathrm{S}^*$, $|a_3| - |a_2| \leq 1.25$ and $|a_4| - |a_3| \leq 2$ both the inequalities are sharp.

We now define the following differential operator $\mathrm{D}_{\mu,\delta,p}^{j,\alpha} : \mathrm{A}_p \to \mathrm{A}_p$
by

$$\mathrm{D}_{\mu,\delta,p}^{j,\alpha} f(z) = z^p + \sum_{n=1}^{\infty} [(n+p)^{\alpha} + (n+p-1)(n+p)^{\alpha}\mu]^j C(\delta,n,p) a_{n+p} z^{n+p} \tag{1.2}$$

where

$$C(\delta,n,p)=\frac{\Gamma(n+p+\delta)}{\Gamma(n+1)\Gamma(\delta+p)}.$$

and $j,\alpha\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}, p\in\mathbb{N}, \mu,\delta\geq 0.$

By specializing the parameters j,α,μ,δ and p we obtain the following operators studied earlier by various researchers: Namely,

- If $\alpha=p=1$, $\mu=0$, $\delta=0$ or $\alpha=\delta=0$, $\mu=p=1$, the operator $D_{0,0,1}^{j,1}\equiv \mathrm{D}_{1,0,1}^{j,0}\equiv \mathrm{D}^{j}$ is the popular Salagean operator [19];
- When $j=0$, $p=1$, then $\mathrm{D}_{\mu,\delta}^{0,\alpha}$ which is the Ruscheweyh differential operator (see [18]);
- For $\alpha=0, \delta=0$, $p=1$, then $\mathrm{D}_{\mu,0,1}^{j,0}=\mathrm{D}_{\mu}^{j}$ which is the differential operator studied by Al-Oboudi (see [1]);
- If $\alpha=0$ and $p=1$ then $\mathrm{D}_{\mu,\delta,1}^{j,0}=\mathrm{D}_{\mu,\delta}^{j}$ has been studied by Darus and Ibrahim (see [2]);
- When $p=1$, then $\mathrm{D}_{\mu,\delta,1}^{j,\alpha}=\mathrm{D}_{\mu,\delta}^{j,\alpha}$ which is the generalized differential operator studied by Panigrahi and Murugusundaramoorthy (see [16]).

Motivated by the above concept, in this paper, making use of the differential operator $\mathrm{D}_{\mu,\delta,p}^{j,\alpha}$ we introduce and investigate a new subclass of multivalent functions, as in

Definition 1.1. A function $f\in \mathrm{A}_p$ is said to be in the class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$ if it satisfies the inequality

$$\Re\left(\frac{(1-t)z(\mathrm{D}_{\mu,\delta,p}^{j,\alpha}f(z))'+tz(\mathrm{D}_{\mu,\delta,p}^{j+1,\alpha}f(z))'}{(1-t)\mathrm{D}_{\mu,\delta,p}^{j,\alpha}f(z)+t\mathrm{D}_{\mu,\delta,p}^{j+1,\alpha}f(z)}\right)>0, \qquad (z\in \mathrm{U}) \tag{1.3}$$

where $0\leq t\leq 1, j,\alpha\in\mathbb{N}_0$, $p\in\mathbb{N}$, μ and $\delta\geq 0.$

Note that by taking $t=j=\delta=0$ and $t=\alpha=1$, $j=\mu=\delta=0$ the class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$, reduces the classes S_p^{*} and C_p, respectively.

Remark 1.1. If $t=j=\delta=0$ and $p=1$, then $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$ reduces to the well-known class of starlike functions in U. Similarly, if we let $t=\alpha=p=1$, $j=\mu=\delta=0$ then $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$ reduces to the well-known class of convex functions in U.

The purpose of the present study is to estimate the coefficient differences for the function class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$, when $n=p+1$ and $n=p+2$.

II. Preliminary Results

In order to derive our main results, we have to recall the following preliminary lemmas:

Let P be the family of all functions h analytic in U, for which $Re\{h(z)\}>0$ and

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n, \quad \forall\ z\in \mathrm{U}. \tag{2.1}$$

Lemma 2.1. [4] If $h\in \mathrm{P}$, then $|c_k|\leq 2$, for each $k\geq 1$.

Lemma 2.2. [7] The power series for h given in (2.1) converges in the unit disc U to a function in P if and only if the Toeplitz determinants.

$$D_k=\begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_k \\ c_{-1} & 2 & c_1 & \cdots & c_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \\ c_{-k} & c_{-k+1} & c_{-k+2} & \cdots & 2 \end{vmatrix}, \quad k=1,2,3,\ldots$$

and $c_{-k}=\overline{c_k}$, are all non-negative. These are strictly positive except for $h(z)=\sum_{k=1}^{m}\rho_k h_0 e^{it_k z}$, $\rho_k>0$, t_k real and $t_k\neq t_j$, for $k\neq j$, in this case $D_k>0$ for $k<(m-1)$ and $D_k=0$ for $k\geq m$.

This necessary and sufficient condition due to Caratheodory and Toeplitz can be found in [7].

We may assume without restriction that $c_1>0$ and on using [Lemma 2.2], for $k=2$ and $k=3$ respectively, we get

$$D_2=\begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \\ \overline{c_2} & \overline{c_1} & 2 \end{vmatrix}=\left[8+2\mathrm{Re}\{c_1^2c_2\}-2|c_2|^2-4c_1^2\right]\geq 0,$$

which is equivalent to

$$2c_2=\left\{c_1^2+x(4-c_1^2)\right\},\quad \text{for some}\quad x,\quad |x|\leq 1. \tag{2.2}$$

$$D_3=\begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c_1} & 2 & c_1 & c_2 \\ \overline{c_2} & \overline{c_1} & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & \overline{c_1} & 2 \end{vmatrix}.$$

Then $D_3\geq 0$ is equivalent to

$$(4c_3-4c_1c_2+c_1^3)(4-c_1^2)+c_1(2c_2-c_1^2)^2\leq 2(4-c_1^2)^2-2|2c_2-c_1^2|^2\,. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3=\left\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\right\} \tag{2.4}$$
$$\text{for some real value of } z, \text{ with } |z|\leq 1.$$

III. Main Results

In this section, we prove to estimate the coefficient differences for the function class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$.

Theorem 3.1. Let f given by (1.1) be in the class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$. If $\frac{(p+2)}{(p+3)}A_3\leq A_2\leq\frac{(p+2)}{(p+1)}A_1$, then

$$||a_{p+2}|-|a_{p+1}||\leq\frac{8pA_1^2+(p+1)A_2^2}{4p(p+1)A_1^2A_2}, \tag{3.1}$$

and

$$||a_{p+3}|-|a_{p+2}||\leq\frac{(3p+1)^2A_2^2+8p(p+1)A_3^2}{4p(p+1)^2A_2A_3^2}, \tag{3.2}$$

where

$$A_1=(p+1)^{\alpha j}(1+p\mu)^j(\delta+p)[1+((p+1)^{\alpha}(1+p\mu)-1)t],$$

$$A_2=(p+2)^{\alpha j}(1+(p+1)\lambda)^j\,\frac{(\delta+p)(\delta+p+1)}{2}[1+((p+2)^{\alpha}(1+(p+1)\mu)-1)t],$$

and

$$A_3=(p+3)^{\alpha j}(1+(p+2)\mu)^j\,\frac{(\delta+p)(\delta+p+1)(\delta+p+2)}{6}[1+((p+3)^{\alpha}(1+(p+2)\mu)-1)t].$$

Proof: Let the function $f(z)$ represented by (1.1) be in the class $\mathrm{M}_{\mu,\delta,p}^{j,t}(\alpha)$. By geometric interpretation, there exists a function $h\in\mathrm{P}$ given by (2.1) such that

$$\frac{(1-t)z(\mathrm{D}_{\mu,\delta,p}^{j,\alpha}f(z))'+tz(\mathrm{D}_{\mu,\delta,p}^{j+1,\alpha}f(z))'}{(1-t)\mathrm{D}_{\mu,\delta,p}^{j,\alpha}f(z)+t\mathrm{D}_{\mu,\delta,p}^{j+1,\alpha}f(z)}=h(z). \tag{3.3}$$

Replacing $D^{j,\alpha}_{\mu,\delta,p} f(z)$, $D^{j+1,\alpha}_{\mu,\delta,p} f(z)$, $(D^{j,\alpha}_{\mu,\delta,p} f(z))'$ and $D^{j+1,\alpha}_{\mu,\delta,p} f(z)'$ by their equivalent expressions and the equivalent expression for $h(z)$ in series (3.3), we have

$$(1-t)z(D^{j,\alpha}_{\mu,\delta,p} f(z))' + tz(D^{j+1,\alpha}_{\mu,\delta,p} f(z))' = h(z)\left\{(1-t)D^{j,\alpha}_{\mu,\delta,p} f(z) + tD^{j+1,\alpha}_{\mu,\delta,p} f(z)\right\}.$$

$$\begin{aligned}
&(1-t)z\left\{pz^{p-1} + \sum_{n=1}^{\infty}(n+p)[(n+p)^{\alpha} + (n+p-1)(n+p)^{\alpha}\mu]^{j}\,C(\delta,n,p)a_{n+p}z^{n+p-1}\right\}\\
&\qquad + tz\left\{pz^{p-1} + \sum_{n=1}^{\infty}(n+p)[(n+p)^{\alpha} + (n+p-1)(n+p)^{\alpha}\mu]^{j+1}\,C(\delta,n,p)a_{n+p}z^{n+p-1}\right\}\\
&= (1-t)\left\{z^{p} + \sum_{n=1}^{\infty}\left[(n+p)^{\alpha} + (n+p-1)(n+p)^{\alpha}\mu\right]^{j} C(\delta,n,p)a_{n+p}z^{n+p}\right\}\\
&\qquad + t\left\{z^{p} + \sum_{n=1}^{\infty}\left[(n+p)^{\alpha} + (n+p-1)(n+p)^{\alpha}\mu\right]^{j+1} C(\delta,n,p)a_{n+p}z^{n+p}\right\}\times\left\{1+\sum_{n=1}^{\infty}c_n z^n\right\}
\end{aligned} \tag{3.4}$$

Equating the coefficients of like power of z^{p+1}, z^{p+2} and z^{p+3} respectively on both sides of (3.4), we have

$$(p+1)A_1 a_{p+1} = c_1 + A_1 a_{p+1},$$
$$(p+2)A_2 a_{p+2} = c_2 + c_1 A_1 a_{p+1} + A_2 a_{p+2},$$
$$(p+3)A_3 a_{p+3} = c_3 + A_1 a_{p+1} c_2 + A_2 a_{p+2} c_1 + A_3 a_{p+3},$$

where A_1, A_2 and A_3 are given in the statement of theorem.
After simplifying, we get

$$a_{p+1} = \frac{c_1}{pA_1},\; a_{p+2} = \frac{c_2}{(p+1)A_2} + \frac{c_1^2}{p(p+1)A_2}, \tag{3.5}$$

and

$$a_{p+3} = \frac{c_3}{(p+2)A_3} + \frac{(2p+1)c_1c_2}{p(p+1)(p+2)A_3} + \frac{c_1^3}{p(p+1)(p+2)A_3}.$$

Since,

$$||a_{n+p+1}| - |a_{n+p}|| \le |a_{n+p+1} - a_{n+p}|,$$

we need to consider $|a_{p+2} - a_{p+1}|$ and $|a_{p+2} - a_{p+3}|$.

Taking into account (3.5) and (2.2) we obtain

$$\begin{aligned}
|a_{p+2} - a_{p+1}| &= \left|\frac{c_2}{(p+1)A_2} + \frac{c_1^2}{p(p+1)A_2} - \frac{c_1}{pA_1}\right|\\
&= \left|\frac{1}{(p+1)A_2}\left(\frac{c_1^2}{2} + \frac{x}{2}(4-c_1^2)\right) + \frac{c_1^2}{(p+1)A_2} - \frac{c_1}{pA_1}\right|\\
&= \left|\frac{p+2}{2p(p+1)A_2}c_1^2 - \frac{c_1}{pA_1} + \frac{x}{2(p+1)A_2}(4-c_1^2)\right|.
\end{aligned} \tag{3.6}$$

We can assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c\ (c \in [0;2])$ (see Lemma 2.1). Applying triangle inequality and replacing $|x|$ by η in the right hand side of (3.6) and using the inequality $A_2 \le \frac{(p+2)}{(p+1)}A_1$, it reduces to

$$|a_{p+2} - a_{p+1}| \le \frac{c}{pA_1} - \frac{(p+2)c^2}{2p(p+1)A_2} + \frac{4-c^2}{2(p+1)A_2}\eta \tag{3.7}$$

$$= \chi(c,\eta) \qquad (0 \le \eta = |x| \le 1),$$

where

$$\chi(c,\eta) = \frac{c}{pA_1} - \frac{(p+2)c^2}{2p(p+1)A_2} + \frac{4-c^2}{2(p+1)A_2}\eta. \tag{3.8}$$

We assume that the upper bound for (3.7) occurs at an interior point of the $\{(\eta, c) : \eta\,[0,1]\}$ and $c \in [0,2]$. Differentiating (3.8) partially with respect to η, we get

$$\frac{\partial \chi}{\partial \eta} = \frac{4-c^2}{2(p+1)A_2}. \tag{3.9}$$

From (3.9) we observe that $\frac{\partial \chi}{\partial \eta} > 0$ for $0 < \eta < 1$ and for fixed c with $0 < c < 2$. Therefore $F(c,\eta)$ is an increasing function of η, which contradicts our assumption that the maximum value of χ occurs at an interior point of the set $\{(\eta, c) : \eta \in [0,1]\}$ and $c \in [0,2]$. So, fixed $c \in [0,2]$, we have

$$\max_{0\le\eta\le1} \chi(c,\eta) = \chi(c,1) = \tau(c) \text{ (say).}$$

Therefore replacing μ by 1 in (3.8), we obtain

$$\tau(c) = \frac{c}{pA_1} + \frac{2p-(p+1)c^2}{p(p+1)A_2}, \tag{3.10}$$

$$\tau'(c) = \frac{1}{pA_1} - \frac{2c}{pA_2} \tag{3.11}$$

and

$$\tau''(c) = -\frac{2}{pA_2} < 0 .$$

For optimum value of $\tau(c)$, consider $\tau'(c) = 0$. It implies that $c = \frac{A_2}{2A_1}$. Therefore, the maximum value of $\tau(c)$ is $\frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1^2 A_2}$ which occurs at $c = \frac{A_2}{2A_1}$. from the expression (3.10), we get

$$\tau_{\max} = \tau\left(\frac{A_2}{2A_1}\right) = \frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1^2 A_2}. \tag{3.12}$$

From (3.7) and (3.12), we have

$$|a_{p+2} - a_{p+1}| \le \frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1^2 A_2},$$

which proves the assertion (3.1) of Theorem 3.1.

Using the same technique, we will prove (3.2). From (3.5) and an application of (2.4) we have

$$|a_{p+3} - a_{p+2}| = \left| \frac{c_3}{(p+2)A_3} + \frac{(2p+1)c_1c_2}{p(p+1)(p+2)A_3} + \frac{c_1^3}{p(p+1)(p+2)A_3} - \frac{c_2}{(p+1)A_2} - \frac{c_1^2}{p(p+1)A_2} \right|$$

$$= \left| \begin{array}{l} \frac{1}{4(p+2)A_3}\{c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} \\ \qquad + \frac{(2p+1)c_1}{2p(p+1)(p+2)A_3}\{c_1^2 + x(4-c_1^2)\} + \frac{c_1^3}{p(p+1)(p+2)A_3} \\ \qquad\qquad - \frac{1}{2(p+1)A_2}\{c_1^2 + x(4-c_1^2)\} - \frac{c_1^2}{p(p+1)A_2} \end{array} \right|$$

$$|a_{p+3} - a_{p+2}| = \left| \begin{array}{l} \frac{(p+3)c_1^2}{4(p+2)A_3} - \frac{(p+2)}{2p(p+1)A_2}c_1^2 + \frac{(p^2+3p+1)c_1}{2p(p+1)(p+2)A_3}(4-c_1^2)x \\ \qquad - \frac{c_1(4-c_1^2)x^2}{4(p+2)A_3} + \frac{1}{2(p+2)A_3}(4-c_1^2)(1-|x|^2)z \\ \qquad\qquad - \frac{1}{2(p+1)A_2}(4-c_1^2)x \end{array} \right| \tag{3.13}$$

As earlier, we assume without loss of generality that $c_1 = c$ with $0 \le c \le 2.$ Applying triangle inequality and replacing $|x|$ by η in the right hand side of (3.13) and using the fact that $A_3 \le \frac{p+3}{p+2} A_2$, it reduces to

$$\begin{aligned} |a_{p+3} - a_{p+2}| \le & \frac{(p+3)c^2}{4(p+2)A_3} - \frac{(p+2)}{2p(p+1)A_2} c^2 + \frac{(p^2+3p+1)c}{2p(p+1)(p+2)A_3}(4-c^2)\eta \\ & - \frac{c(4-c^2)\eta^2}{4(p+2)A_3} + \frac{1}{2(p+2)A_3}(4-c^2)(1-\eta^2)z \\ & - \frac{1}{2(p+1)A_2}(4-c^2)\eta \\ = & \xi(c,\eta), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} \xi(c,\eta) = & \frac{(p+3)c^2}{4(p+2)A_3} - \frac{(p+2)}{2p(p+1)A_2} c^2 + \frac{(p^2+3p+1)c}{2p(p+1)(p+2)A_3}(4-c^2)\eta \\ & - \frac{c(4-c^2)\eta^2}{4(p+2)A_3} + \frac{1}{2(p+2)A_3}(4-c^2)(1-\eta^2)z \\ & - \frac{1}{2(p+1)A_2}(4-c^2)\eta. \end{aligned} \tag{3.15}$$

Suppose that $\xi(c,\eta)$ in (3.15) attains its maximum at an interior point (c,η) of $[0,2]\times[0,1].$ Differentiating (3.15) partially with respect to η , we have

$$\begin{aligned} \frac{\partial \xi}{\partial \eta} &= \frac{(p^2+3p+1)c(4-c^2)}{2p(p+1)(p+2)A_3} + \frac{c(4-c^2)\eta}{2(p+2)A_3} - \frac{(4-c^2)\eta}{(p+2)A_3} + \frac{(4-c^2)}{2(p+1)A_2} \\ &= -\frac{(c^2-4)}{2p(p+1)(p+2)A_3}\left[c(p^2+3p+1+p(p+1)\eta) - 2p(p+1)\eta + \frac{(p+2)A_3}{A_2} \right]. \end{aligned}$$

Now $\frac{\partial \xi}{\partial \eta} = 0$ which implies

$$c = \frac{2p(p+1)\left(\eta - \frac{(p+2)A_3}{2p(p+1)A_2} \right)}{p(p+1)\eta + p^2 + 3p + 1} < 0 \qquad (0 < \eta < 1),$$

which is false since $c > 0.$ Thus $\xi(c,\eta)$ attains its maximum on the boundary of $[0,2]\times[0,1].$ Thus for fixed c, we have

$$\max_{0\le\eta\le1} \xi(c,\eta) = \xi(c,1) = \psi(c) \ (say)$$

Therefore, replacing η by 1 in (3.15) and simplifying we get

$$\psi(c) = \frac{(3p+1)c}{p(p+1)(p+2)A_3} + \frac{2}{(p+1)A_2} - \frac{c^2}{pA_2} \tag{3.16}$$

$$\psi'(c) = \frac{(3p+1)}{p(p+1)A_3} - \frac{2c}{pA_2} \text{ and } \psi''(c) = -\frac{2}{pA_2} < 0. \tag{3.17}$$

For an optimum value of $\psi(c)$, consider $\psi'(c) = 0$ which implies $c = \frac{(3p+1)A_2}{2(p+1)A_1}$. Therefore, the maximum value of $\psi(c)$ occurs at $c = \frac{(3p+1)A_2}{2(p+1)A_1}$. From the expression (3.16) we obtain

$$\psi_{\max} = \psi\left(\frac{(3p+1)A_2}{2(p+1)A_1} \right) = \frac{(3p+1)^2 A_2^2 + 8p(p+1)A_3^2}{4p(p+1)^2 A_2 A_3^2}. \tag{3.18}$$

From (3.14) and (3.18), we have

$$|a_{p+3}-a_{p+2}|=\frac{(3p+1)^2A_2^2+8p(p+1)A_3^2}{4p(p+1)^2A_2A_3^2}.$$

The proof of Theorem 3.1 is thus completed.

Taking $t=\alpha=1; \mu=\delta=j=0$ in Theorem 3.1 we get

Corollary 3.2. Let f given by (1.1) be in the class C then

$$||a_{p+2}|-|a_{p+1}||\le\frac{32p+(p+1)(p+2)^2}{8p^2(p+1)^2(p+2)}$$

and

$$||a_{p+3}|-|a_{p+2}||\le\frac{9(3p+1)^2+8p(p+1)(p+3)^2}{2p^2(p+1)^3(p+2)(p+3)^2}$$

Both the inequalities are sharp.

Putting $t=j=\delta=0$ in Theorem 3.1 we get

Corollary 3.3. Let f given by (1.1) be in the class S^*. Then

$$||a_{p+2}|-|a_{p+1}||\le\frac{32p+(p+1)^3}{8p^2(p+1)^2}$$

and

$$||a_{p+3}|-|a_{p+2}||\le\frac{9(3p+1)^2+8p(p+1)(p+2)^2}{2p^2(p+1)^3(p+2)^2}$$

Both the inequalities are sharp.

For $p=1$, Theorem 3.1 reduces to the results obtained in

Corollary 3.4. [16] Let f given by (1.1) be in the class $M_{\mu,\delta}^{j,t}(\alpha)$. If $\frac{3A_3}{4}\le A_2\le\frac{3A_1}{2}$, then

$$||a_3|-|a_2||\le\frac{4A_1^2+A_2^2}{4A_1^2A_2},$$

and

$$||a_4|-|a_3||\le\frac{A_2^2+A_3^2}{A_2A_3^2},$$

where

$$A_1=2^{\alpha j}(1+\mu)^j(\delta+1)[1+(2^\alpha(1+\mu)-1)t],$$

$$A_2=3^{\alpha j}(1+2\mu)^j\frac{(\delta+1)(\delta+2)}{2}[1+(3^\alpha(1+2\mu)-1)t],$$

and

$$A_3=4^{\alpha j}(1+3\mu)^j\frac{(\delta+1)(\delta+2)(\delta+3)}{6}[1+(4^\alpha(1+3\mu)-1)t].$$

Remark 3.1. Here we remark that the results obtained in (corollary 1, [16]) is computationally wrong. The estimates $||a_3|-|a_2||\le\frac{25}{38}$ and $||a_4|-|a_3||\le\frac{25}{38}$ must be $||a_3|-|a_2||\le\frac{25}{48}$ and $||a_4|-|a_3||\le\frac{25}{48}$.

Taking $t=\alpha=p=1; \mu=\delta=j=0$ in Theorem 3.1 we get following

Corollary 3.5. [16] Let f given by (1.1) be in the class C. Then

$$||a_3|-|a_2||\le\frac{25}{48} \text{ and } ||a_4|-|a_3||\le\frac{25}{48}$$

Both the inequalities are sharp.

Putting $t=j=\delta=0$ and $p=1$ in Theorem 3.1 we get following

Corollary 3.6. [16] Let f given by (1.1) be in the class S^*. Then

$$||a_3|-|a_2|| \le \frac{5}{4} \text{ and } ||a_4|-|a_3|| \le 2$$

Both the inequalities are sharp.

References

[1]. F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Math. Sci., 27(2004), 1429-1436.

[2]. M. Darus and R. W. Ibrahim, Generalization of differential operator, J. Math. Stat., 4(2008), 138-144.

[3]. M. Darus and R. W. Ibrahim, New classes containing generalization of differential operator, Appl. Math. Sci., 3(2009), 2507-2515.

[4]. P. L. Duren, Univalent Functions. Grundlehren der Mathematischen Wissenschaften Fundamental Principles of Mathematical Sciences]}, 259. Springer- Verlag, New York, 1983.

[5]. G. M. Golusin, On distortion theorems and coefficients of univalent functions, Rec. Math. [Mat. Sbornik] N.S., 19(1946). 183-202.

[6]. G. M. Golusin, Some questions in the theory of univalent functions, Trudy. Mat. Inst. Steklov., 27(1949), 1-12.

[7]. U. Grenander and G. Szego, Toeplitz forms and their application, Berkeley and Los Angeles: Univ. of California Press, 1958.

[8]. A. Z. Grispin, Improved bounds for the difference of the moduli of adjacent coefficients of univalent functions, in Some Questions in the Modern Theory of Functions, Sib. Inst. Math, (1976), 41-45.

[9]. W. K. Hayman, On successive coefficients of univalent functions, J. London Math. Soc., 38(1963), 228-243.

[10]. L. P. Ilina, On the relative growth of adjacent coefficients of univalent functions, Mat. Zametki, 4(1968) 715-722 (in Russian) = Math. Notes 4(1968), 918-922.

[11]. J. A. Jenkins, On certain coefficients of univalent functions, 1960 Analytic functions pp. 159-194 Princeton Univ. Press, Princeton, N.J.

[12]. Y. J. Leung, Successive coefficients of starlike functions, Bull. London Math. Soc.,10(1978), 193-196.

[13]. R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc., 85(1982), 225-230.

[14]. R. J. Libera and E. J. Zlotkiewicz, Coefficients bounds for the inverse of a function with derivative in p, Proc. Amer. Math. Soc., 87(1983), 251-289.

[15]. I. M. Milin, Adjacent coefficients of univalent functions, Dokl. Acad. Nauk SSSR., 180(1968), 1294-1297 (in Russian)=Soviet Math. Dokl., 9(1969), 762-765.

[16]. T. Panigrahi and G. Murugusundaramoorthy, On successive coefficient estimate for certain subclass of analytic functions, Applied Mathematics E-Notes, 16(2016), 117-124.

[17]. M. S. Robertson, Quasi-subordination and coefficient conjectures, Bull. Amer. Math. Soc., 76(1970), 1-9.

[18]. S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49(1975), 109-115.

[19]. G. S. Salagean, Subclasses of Univalent Functions. Complex Analysis-fth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin, 1983.