IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. I (Nov. - Dec.2016), PP 99-108
www.iosrjournals.org

Concomitant of Order Statistics from Weighted Marshall-Olkin Bivariate Exponential Distribution

M.J.S.Khan[1], Sumit Kumar[2]

[1] *Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, India*
[2] *Department of Applied Statistics, Babasaheb Bhimrao Ambedkar University, Lucknow, India*

***Abstract**: A new class of probability distribution namely Weighted Marshall-Olkin Bivariate Exponential Distribution is considered in this paper. The distribution of concomitant of order statistics and joint distribution of two concomitant of order statistics has been obtained. Further explicit expression for single moment of concomitant of order statistics is obtained. Explicit expression for moment generating function of concomitant of order statistics has been obtained also.*

***Keywords:** Concomitant, Weighted Marshall-Olkin bivariate exponential distribution, moments, order statistics.*

I. Introduction

The concept of concomitant of order statistics was first introduced by David (1973). Let $(X_i, Y_i); i = 1, 2, ..., n$ be a random sample drawn from an arbitrary bivariate distribution with cumulative distribution function (*cdf*) $F(x, y)$ and probability density function (*pdf*) $f(x, y)$. If the marginal X observations in the sample are ordered as $X_{1:n}, X_{2:n}, ..., X_{n:n}$, then the accompanying Y observation in an ordered pair with X observation equal to $X_{r:n}$ is called the concomitant of the r^{th} order statistics $X_{r:n}$ and is denoted by $Y_{[r:n]}$.

The most important use of concomitant arises in the selection procedures, when $k(1 \le k \le n)$ individuals are chosen on the basis of their X values, then the corresponding Y values represent the performance on an associated characteristic, which is hard to measure or can be observed only later. For example; if the top k out of n rams, as judged by their genetic make-up are selected for breeding, then $Y_{[1:n]}, ..., Y_{[n:n]}$ might represent the quality ofthe wool of a corresponding female offspring in the cross. Another application of concomitant of order statistics may be competitive exams in which X may be score in preliminary test and Y may be the corresponding scores in main test.

Jamalizadeh and Kundu (2013) proposed a four parameter Weighted Marshall-Olkin bivariate exponential model denoted by $WMOBE(\alpha, \lambda_1, \lambda_2, \lambda_{12})$ as an improvisation over Marshall-Olkin bivariate exponential model or Marshall-Olkin bivariate Weibull model, providing comparatively better fit in certain cases. They established the properties of this new Weighted Marshall-Olkin bivariate exponential distribution whose marginal has univariate weighted exponential (WE) Distribution. The Weighted Marshall-Olkin bivariate exponential is established as follows:

Univariate weighted exponential distribution with parameters $\alpha > 0$ and $\lambda > 0$ has *pdf* and *cdf* as follows

$$f(x, \alpha, \lambda) = \left(\frac{\alpha + 1}{\alpha}\right) \lambda e^{-\lambda x} (1 - e^{-\alpha \lambda x}); x > 0, \alpha, \lambda > 0 \tag{1.1}$$

$$F(x, \alpha, \lambda) = 1 + \left(\frac{e^{-\lambda x}}{\alpha}\right) [e^{-\alpha \lambda x} - (\alpha + 1)] \tag{1.2}$$

Now, the bivariate random vector (ξ_1, ξ_2) has Marshall-Olkin bivariate exponential distribution if it has the joint *pdf*

[1] *Corresponding author email address:.jahangirskhan@gmail.com*

$$f(\xi_1,\xi_2)=\begin{cases}\lambda_1(\lambda_2+\lambda_{12})e^{-\lambda_1\xi_1}e^{-(\lambda_2+\lambda_{12})\xi_2}; & \xi_1<\xi_2\\ \lambda_2(\lambda_1+\lambda_{12})e^{-\lambda_2\xi_2}e^{-(\lambda_1+\lambda_{12})\xi_1}; & \xi_2<\xi_1\\ \lambda_{12}e^{-(\lambda_1+\lambda_2+\lambda_{12})\xi}; & \xi_1=\xi_2=\xi\end{cases}$$

and is denoted by $MOBE(\lambda_1,\lambda_2,\lambda_{12})$

If U_1,U_2 and U_3 are the independent exponential random variables with parameters $\lambda_1,\lambda_2,\lambda_{12}$ respectively, such that

$(\xi_1,\xi_2)\underline{\underline{d}}[\min(U_0,U_1);\min(U_0,U_2)]$

Then the random vector (X,Y) is said to have $WMOBE$ distribution with parameter $\theta=(\alpha,\lambda_1,\lambda_2,\lambda_{12})$, if

$$X\underline{\underline{d}}\ \frac{\xi_1}{W}<\min(\xi_1,\xi_2)$$

and

$$Y\underline{\underline{d}}\ \frac{\xi_2}{W}<\min(\xi_1,\xi_2)$$

where

$W\sim\exp(\alpha)$

It is denoted by

$(X,Y)\sim WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$

with corresponding joint *cdf* as

$$F(x,y)=\begin{cases}F_1(x,y); & \text{if } 0<x<y\\ F_2(x,y); & \text{if } 0<y<x\\ F_0(x,y); & \text{if } y=x>0\end{cases}$$

where

$$F_1(x,y)=1-\left(\frac{\alpha+\lambda}{\alpha}\right)\left[e^{-(\lambda_1x+\lambda_2y+\lambda_{12}y)}(1-e^{-\alpha x})+\left(\frac{\alpha}{\alpha+\lambda_1}\right)e^{-(\lambda_2y+\lambda_{12}y+\lambda_1x+\alpha x)}-\left\{\frac{\alpha(\lambda_2+\lambda_{12})}{(\alpha+\lambda)(\alpha+\lambda_1)}\right\}e^{-(\alpha+\lambda)y}\right] \quad (1.3)$$

$$F_2(x,y)=1-\left(\frac{\alpha+\lambda}{\alpha}\right)\left[e^{-(\lambda_2y+\lambda_1x+\lambda_{12}x)}(1-e^{-\alpha y})+\left(\frac{\alpha}{\alpha+\lambda_2}\right)e^{-(\lambda_2y+\lambda_{12}x+\lambda_1x+\alpha y)}-\left\{\frac{\alpha(\lambda_2+\lambda_{12})}{(\alpha+\lambda)(\alpha+\lambda_2)}\right\}e^{-(\alpha+\lambda)x}\right] \quad (1.4)$$

$$F_0(x,y)=1-\left(\frac{\alpha+\lambda}{\alpha}\right)\left[e^{-\lambda x)}(1-e^{-\alpha x})+\left(\frac{\alpha}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x}\right] \quad (1.5)$$

where $\lambda=\lambda_1+\lambda_2+\lambda_{12}$,

and the corresponding joint *pdf* of $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$ is given as

$$f(x,y)=\begin{cases}f_1(x,y); & \text{if } 0<x<y\\ f_2(x,y); & \text{if } 0<y<x\\ f_0(x,y); & \text{if } y=x>0\end{cases}$$

where

$$f_1(x,y)=\left(\frac{\alpha+\lambda}{\alpha}\right)\lambda_1(\lambda_2+\lambda_{12})e^{-\lambda_1x}e^{-(\lambda_2+\lambda_{12})y}(1-e^{-\alpha x}) \quad (1.6)$$

$$f_2(x,y)=\left(\frac{\alpha+\lambda}{\alpha}\right)\lambda_2(\lambda_1+\lambda_{12})e^{-(\lambda_1+\lambda_{12})x}e^{-\lambda_2y}(1-e^{-\alpha y}) \quad (1.7)$$

$$f_0(x,y)=\left(\frac{\alpha+\lambda}{\alpha}\right)\lambda_{12}e^{-\lambda x}(1-e^{-\alpha x}) \quad (1.8)$$

Jamalizadeh and Kundu (2013) proved that when $(X,Y) \sim WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$, then $X \sim WE\left(\frac{\alpha+\lambda_2}{\lambda_1+\lambda_{12}},\lambda_1+\lambda_{12}\right)$ having the marginal *pdf* and marginal *cdf* of X as follows:

$$f(x)=\left(\frac{\alpha+\lambda}{\alpha+\lambda_2}\right)(\lambda_1+\lambda_{12})e^{-(\lambda_1+\lambda_{12})x}[1-e^{-(\alpha+\lambda_2)x}],\ \ x>0 \tag{1.9}$$

$$F(x)=1+\frac{e^{-(\lambda_1+\lambda_{12})x}}{(\alpha+\lambda_2)}[(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda_2)x}-(\alpha+\lambda)],\ \ x>0 \tag{1.10}$$

Therefore the conditional *pdf* of $Y \mid X$ is as follows

$$f(y\mid x)=\begin{cases} f_1(y\mid x);\ \text{if}\ 0<x<y \\ f_2(y\mid x);\ \text{if}\ 0<y<x \\ f_0(y\mid x);\ \text{if}\ y=x>0 \end{cases}$$

where

$$f_1(y\mid x)=\frac{\lambda_1(\alpha+\lambda_2)(\lambda_2+\lambda_{12})}{\alpha(\lambda_1+\lambda_{12})}e^{-\lambda_1 x}e^{(\lambda_1+\lambda_{12})x}e^{-(\lambda_2+\lambda_{12})y}(1-e^{-\alpha x})[1-e^{-(\alpha+\lambda_2)x}]^{-1} \tag{1.11}$$

$$f_2(y\mid x)=\frac{\lambda_2(\alpha+\lambda_2)}{\alpha}e^{-\lambda_2 y}(1-e^{-\alpha y})[1-e^{-(\alpha+\lambda_2)x}]^{-1} \tag{1.12}$$

$$f_0(y\mid x)=\frac{\lambda_{12}(\alpha+\lambda_2)}{\alpha(\lambda_1+\lambda_{12})}e^{-\lambda_2 x}(1-e^{-\alpha x})[1-e^{-(\alpha+\lambda_2)x}]^{-1} \tag{1.13}$$

The theory of concomitant of order statistics was introduced by David (1973). Several authors discussed the concomitant of order statistics. Balasubramanian and Beg (1998) discussed the concomitant of order statistics in Gumble's bivariate exponential distribution. In this sequence of study, Begum and Khan (2000) and Begum (2003) obtained the expressions of concomitant of order statistics from Marshall and Olkin's bivariate Weibull distribution and from bivariate Pareto II distribution. Aleem (2006) obtained the expression for concomitant of order statistics from bivariate inverse Rayleigh distribution. Chacko and Thomas (2011) developed the theory of estimation of parameters using concomitant of order statistics from Morgenstern type bivariate exponential distribution. Philip and Thomas (2015) obtained the concomitant of order statistics from extended Farlie-Gumble-Morgenstern bivariate logistic distribution and discussed it's application in point estimation.

In the present article, we have studied the properties of $Y_{[r:n]}$ if the random variables $(X_i,Y_i), i=1,2,...,n$ are independently and identically distributed (*iid*) and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$. Rest of the paperis organized as follows. In section 2, we have obtained the *pdf* and *cdf* of r^{th} order statistics $X_{r:n}$ and we have obtained the joint density of r^{th} order statistics and s^{th} order statistics, $1\le r<s\le n$, when $X \sim WE\left(\frac{\alpha+\lambda_2}{\lambda_1+\lambda_{12}},\lambda_1+\lambda_{12}\right)$ with *pdf* and *cdf* given in (1.9) and in (1.10) respectively. In section 3, we have obtained the density of concomitant first order statistics $Y_{[1:n]}$ and using this density, we have deduced the *pdf* of concomitant of r^{th} order statistics $Y_{[r:n]}$.Subsequently, in section 4, we have computed the moments of concomitant of r^{th} order statistics in explicit and exact form. In section 5, we have obtained the exact expression for moment generating function of concomitant of r^{th} order statistics $Y_{[r:n]}$. Finally, in section 6, we obtained the joint distribution of concomitant of two order statistics $Y_{[r:n]}$ and $Y_{[s:n]}$.

II. Distribution of Order Statistics

Let $X_1,X_2,...,X_n$ be the n *iid* observations from the population having *cdf* $F(x)$ and *pdf* $f(x)$ then the *pdf* of r^{th} order statistics $X_{r:n}$ is given as

$$f_{r:n}(x)=C_{r:n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

Where $C_{r:n}=\dfrac{n!}{(r-1)!(n-r)!}$.

Thus if $X_1, X_2, ..., X_n$ be the n *iid* observations from a population having *pdf* $f(x)$ and *cdf* $F(x)$ given in (1.9) and (1.10) respectively, then the *pdf* of r^{th} order statistics is given as

$$f_{r:n}(x)=C_{r:n}\frac{(\alpha+\lambda)(\lambda_1+\lambda_{12})}{(\alpha+\lambda_2)^n}e^{-(\lambda_1+\lambda_{12})x}[1-e^{-(\alpha+\lambda_2)x}][(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x}]^{n-r}$$
$$\times[(\alpha+\lambda_2)+(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x}-(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x}]^{r-1} \quad (2.1)$$

Putting $r=1$ in (2.1), we get the *pdf* of first order statistics $X_{1:n}$ as

$$f_{1:n}(x)=\frac{n(\alpha+\lambda)(\lambda_1+\lambda_{12})}{(\alpha+\lambda_2)^n}e^{-(\lambda_1+\lambda_{12})x}[1-e^{-(\alpha+\lambda_2)x}][(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x}]^{n-1} \quad (2.2)$$

The joint *pdf* of $X_{r:n}$ and $X_{s:n}$, $(1\le r\le s<n)$ is given as

$$f_{r,s:n}(x_1,x_2)=C_{r,s:n}[1-\bar{F}(x_1)]^{r-1}[\bar{F}(x_1)-\bar{F}(x_2)]^{s-r-1)}[\bar{F}(x_2)]^{n-s}f(x_1)f(x_2)$$

Where $\bar{F}(x)=[1-F(x)]$ and $C_{r,s:n}=\dfrac{n!}{(r-1)![s-(r+1)]!(n-s)!}$.

Thus, if $X_1, X_2, ..., X_n$ be the n *iid* observations from the population having *pdf* $f(x)$ and *cdf* $F(x)$ given in (1.9) and (1.10) respectively, then the joint *pdf* of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r,s:n}(x_1,x_2)=C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha+\lambda_2}\right)^2(\lambda_1+\lambda_{12})^2e^{-(\lambda_1+\lambda_{12})(x_1+x_2)}[1-e^{-(\alpha+\lambda_2)x_1}][1-e^{-(\alpha+\lambda_2)x_2}]$$
$$\times\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\binom{r-1}{l}\binom{s-r-1}{q}(-1)^{l+q}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x_1}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x_1}]^{l+s-r-q+1} \quad (2.3)$$
$$\times[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x_2}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x_2}]^{n-s+q}$$

III. Distribution of Concomitant of Order Statistics

In this section, we have deduced the result for the *pdf* of concomitants of order statistics when random variables $(X_i, Y_i);\ i=1,2,...,n$ are *iid* and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$. First we have obtained the *pdf* of first concomitants of order statistics $Y_{[1:n]}$ and utilizing *pdf* of $Y_{[1:n]}$, we have obtained the *pdf* of r^{th} concomitants of order statistics $Y_{[r:n]}$. The *pdf* of concomitant of first order statistics $Y_{[1:n]}$ is given by

$$g_{[1:n]}(y)=\int_0^y f_1(y|x)f_{1:n}(x)dx+\int_y^\infty f_2(y|x)f_{1:n}(x)dx+f_3(y|x)f_{1:n}(x)$$

The *pdf* of concomitant of first order statistics $Y_{[1:n]}$, when random variables $(X_i, Y_i);\ i=1,2,...,n$ are *iid* and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$ is given by

$$g_{[1:n]}(y)=\frac{n\lambda_1(\lambda_2+\lambda_{12})e^{-(\lambda_2+\lambda_{12})}}{\alpha(\alpha+\lambda_2)^{n-1}}\int_0^y e^{-\lambda_1x}e^{-(\lambda_1+\lambda_{12})x}(1-e^{-\alpha x})[1-e^{-(\alpha+\lambda_2)x}]^{-1}[1-e^{-(\alpha+\lambda_2)x}]$$
$$\times[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x}]^{n-1}dx$$
$$+\frac{n\lambda_2e^{-\lambda_2y}(1-e^{-\alpha y})}{\alpha(\alpha+\lambda_2)^{n-1}}\int_y^\infty[1-e^{-(\alpha+\lambda_2)x}]^{-1}e^{-(\lambda_2+\lambda_{12})x}[1-e^{-(\alpha+\lambda_2)x}][(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})x}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)x}]^{n-1}dx$$
$$+\frac{n\lambda_{12}(\alpha+\lambda)e^{-\lambda y}(1-e^{-\alpha y})}{\alpha(\alpha+\lambda_2)^{n-1}}[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})y}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)y}]^{n-1}$$

Now denoting $A_p = n(\lambda_1+\lambda_{12})+(\alpha+\lambda_2)p$, this implies that

$$g_{[1:n]}(y)=\frac{n\lambda_1(\lambda_2+\lambda_{12})}{\alpha(\alpha+\lambda_2)^{n-1}}\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}(\alpha+\lambda)^{n-p}(\lambda_1+\lambda_{12})^p$$
$$\times\left[\left(\frac{e^{-(\lambda_2+\lambda_{12})y}-e^{-(A_p+\lambda_2)y}}{A_p-\lambda_{12}}\right)-\left(\frac{e^{-(\lambda_2+\lambda_{12})y}-e^{-(A_p+\alpha+\lambda_2)y}}{A_p+\alpha-\lambda_{12}}\right)\right]$$
$$+\frac{n\lambda_2}{\alpha(\alpha+\lambda_2)^{n-1}}\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}(\alpha+\lambda)^{n-p}(\lambda_1+\lambda_{12})^{p+1}\left[\frac{e^{-(A_p+\lambda_2)y}-e^{-(A_p+\alpha+\lambda_2)y}}{A_p}\right]$$
$$+\frac{n\lambda_{12}(\alpha+\lambda)}{\alpha(\alpha+\lambda_2)^{n-1}}\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}(\alpha+\lambda)^{n-p}(\lambda_1+\lambda_{12})^p[e^{-(A_p+\lambda_2)y}-e^{-(A_p+\alpha+\lambda_2)y}] \tag{3.2}$$

It is well known that *cdf* of order statistics are connected by the relation (see David; 1981)

$$F_{r:n}(x)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}F_{1:i}(x);\ 1\le r\le n$$

This relation also holds for *pdf* of concomitant of order statistics as (see Balasubramanian and Beg; 1998). Therefore, the *pdf* of concomitant of r^{th} order statistics $Y_{[r:n]}$ is given as

$$g_{[r:n]}(y)=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}g_{[1:i]}(y)$$

$$g_{[r:n]}(y)=\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_1(\lambda_2+\lambda_{12})}{\alpha(\alpha+\lambda_2)^{i-1}}(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^p$$
$$\times\left[\left(\frac{e^{-(\lambda_2+\lambda_{12})y}-e^{-(A_p+\lambda_2)y}}{A_p-\lambda_{12}}\right)-\left(\frac{e^{-(\lambda_2+\lambda_{12})y}-e^{-(A_p+\alpha+\lambda_2)y}}{A_p+\alpha-\lambda_{12}}\right)\right] \tag{3.3}$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_2}{\alpha(\alpha+\lambda_2)^{i-1}}(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^{p+1}\left[\frac{e^{-(A_p+\lambda_2)y}-e^{-(A_p+\alpha+\lambda_2)y}}{A_p}\right]$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_{12}}{\alpha(\alpha+\lambda_2)^{i-1}}(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^p[e^{-(A_p+\lambda_2)y}-e^{-(A_p+\alpha+\lambda_2)y}]$$

IV. Moment of Concomitant of Order Statistics

In this section, we have deduced the explicit expression for the moment of concomitant of r^{th} order statistics when random variables $(X_i, Y_i);\ i=1,2,...,n$ are *iid* and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$ .Utilizing these results, we can compute means and varainces of concomitant of r^{th} order statistics. The j^{th} moment about origin of concomitant of r^{th} order statistics $Y_{[r:n]}$ is given by

$$\mu^j_{Y_{[r:n]}}=\int_0^\infty y^j g_{[r:n]}(y)dy$$

Using (3.3), we have

$$\mu^{j}_{Y_{[r:n]}} = \sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_1(\lambda_2+\lambda_{12})}{\alpha(\alpha+\lambda_2)^{i-1}}(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^{p}$$
$$\times\Gamma(j+1)\left[\left(\frac{(\lambda_2+\lambda_{12})^{-(j+1)}-(A_p+\lambda_2)^{-(j+1)}}{A_p-\lambda_{12}}\right)-\left(\frac{(\lambda_2+\lambda_{12})^{-(j+1)}-(A_p+\alpha+\lambda_2)^{-(j+1)}}{A_p+\alpha-\lambda_{12}}\right)\right]$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_2}{\alpha(\alpha+\lambda_2)^{i-1}}$$
$$\times(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^{p+1}\Gamma(j+1)\left[\frac{(A_p+\lambda_2)^{-(j+1)}-(A_p+\alpha+\lambda_2)^{-(j+1)}}{A_p}\right]$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}(-1)^{i-n+r+p-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}\frac{i\lambda_{12}}{\alpha(\alpha+\lambda_2)^{i-1}}(\alpha+\lambda)^{i-p}(\lambda_1+\lambda_{12})^{p}[(A_p+\lambda_2)^{-(j+1)}-(A_p+\alpha+\lambda_2)^{-(j+1)}]$$

V. Moment Generating Function of Concomitant of Order Statistics

In this section, we have obtained the exact expression for the moment generating function (*mgf*) of concomitant of r^{th} order statistics when random variables $(X_i, Y_i);\ i=1,2,...,n$ are *iid* and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$. The *mgf* of concomitant of r^{th} order statistics $Y_{[r:n]}$ is given by

$$M_{Y_{[r:n]}}(t)=E[e^{tY_{[r:n]}}]=\int_0^{\infty}e^{ty}g_{Y_{[r:n]}}(y)dy$$

Thus in view of (3.3), we have

$$M_{Y_{[r:n]}}(t)=\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}(-1)^{i-n+r+p-1}\frac{i\lambda_1(\lambda_2+\lambda_{12})}{\alpha(\alpha+\lambda_2)^{i-1}}(\lambda_1+\lambda_{12})^{p}(\alpha+\lambda)^{i-p}$$
$$\times\left[\left(\frac{(\lambda_2+\lambda_{12}-t)^{-1}-(A_p+\lambda_2-t)^{-1}}{A_p-\lambda_{12}}\right)-\left(\frac{(\lambda_2+\lambda_{12}-t)^{-1}-(A_p+\alpha+\lambda_2-t)^{-1}}{A_p+\alpha-\lambda_{12}}\right)\right]$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}(-1)^{i-n+r+p-1}\frac{i\lambda_2}{\alpha(\alpha+\lambda_2)^{i-1}}(\lambda_1+\lambda_{12})^{p+1}(\alpha+\lambda)^{i-p}\left[\frac{(A_p+\lambda_2-t)^{-1}-(A_p+\alpha+\lambda_2-t)^{-1}}{A_p}\right]$$
$$+\sum_{i=n-r+1}^{n}\sum_{p=0}^{i-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{p}(-1)^{i-n+r+p-1}\frac{i\lambda_{12}}{\alpha(\alpha+\lambda_2)^{i-1}}(\lambda_1+\lambda_{12})^{p}(\alpha+\lambda)^{i-p}[(A_p+\lambda_2-t)^{-1}-(A_p+\alpha+\lambda_2-t)^{-1}] \tag{5.1}$$

Equation (5.1) can be utilized to compute the moment about origin of any order.

VI. Joint Density of Concomitants of Two Order Statistics

In this section, the joint density of $Y_{[r:n]}$ and $Y_{[s:n]}$, $1\le r<s\le n,$ when random variables $(X_i, Y_i), i=1,2,...,n$ are *iid* and follows $WMOBE(\alpha,\lambda_1,\lambda_2,\lambda_{12})$, has been deduced. Let $Y_{[r:n]}$ and $Y_{[s:n]}$ be the concomitant of r^th^ and s^th^ order statistics, respectively. Then the joint *pdf* of $Y_{[r:n]}$ and $Y_{[s:n]}$, is given by

$$g_{[r:n]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f(y_1\mid x_1)f(y_2\mid x_2)f_{r,s:n}(x_1,x_2)dx_1dx_2$$

where $f_{r,s:n}(x_1,x_2)$ is the joint *pdf* of two order statistics $X_{r:n}$ and $X_{s:n}$. Since, the joint *pdf* is given for three different conditions. So, in this case the joint density of concomitants of two order statistics will be given as;

$$g_{[r:n]}(y_1,y_2)=I_1(y_1,y_2)+I_2(y_1,y_2)+I_3(y_1,y_2)+I_4(y_1,y_2)+I_5(y_1,y_2)+I_6(y_1,y_2)+I_7(y_1,y_2)$$

where, $I_1(y_1,y_2),...,I_7(y_1,y_2)$ are the joint densities under different mutually exclusive conditions and are define below;

Case I: When $X<Y$

We have three case as

6.1. ***If:*** $0 \le x_1 < y_1 < x_2 < y_2 \le \infty$

Then on using (1.11) and (2.3), we get

$$I_1(y_1,y_2) = g_{[r,s:n]}(y_1,y_2) = C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2 \lambda_1^{\,2}(\lambda_2+\lambda_{12})^2 e^{-(\lambda_2+\lambda_{12})(y_1+y_2)} \sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}$$

$$\times\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_{y_1}^{y_2}\{e^{-\lambda_1 x_2}(1-e^{-\alpha x_2})(\alpha+\lambda)^{n-s+q}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}(\alpha+\lambda)^{l+s-r-q+1}$$

$$\times\sum_{u=0}^{l+s-r-q+1}(-1)^u\binom{l+s-r-q+1}{u}\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)^u\int_0^{y_1}e^{-\lambda_1 x_1}(1-e^{-\alpha x_1})e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}e^{-(\alpha+\lambda_2)ux_1}dx_1\}dx_2$$

after some algebraic simplification, we get

$$g_{[r,s:n]}(y_1,y_2) = C_{r,s:n}\left[\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda}\right]^2 e^{-(\lambda_2+\lambda_{12})(y_1+y_2)}\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}\binom{l+s-r-q+1}{u}$$

$$\times\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+3}(\lambda_1+\lambda_{12})^{u+v}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}$$

$$\times\left[\frac{e^{-[\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u]y_1}}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}-\frac{e^{-[\lambda_1+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u]y_1}}{\lambda_1+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}\right]$$

$$\times\left[\frac{e^{-[\lambda_1+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v]y_1}-e^{-[\lambda_1+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v]y_2}}{\lambda_1+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v}-\frac{e^{-[\lambda_1+\alpha+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v]y_1}-e^{-[\lambda_1+\alpha+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v]y_2}}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(n-s+q)+(\alpha+\lambda_2)v}\right]$$

6.2. ***If:*** $0 \le x_1 < x_2 < y_1 < y_2 \le \infty$

Again, on using (1.11) and (2.3), we get

$$I_2(y_1,y_2) = g_{[r,s:n]}(y_1,y_2) = C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2 \lambda_1^{\,2}(\lambda_2+\lambda_{12})^2 e^{-(\lambda_2+\lambda_{12})(y_1+y_2)} \sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}$$

$$\times\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_{0}^{y_1}\{e^{-\lambda_1 x_2}(1-e^{-\alpha x_2})(\alpha+\lambda)^{n-s+q}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}$$

$$\times(\alpha+\lambda)^{l+s-r-q+1}\sum_{u=0}^{l+s-r-q+1}(-1)^u\binom{l+s-r-q+1}{u}\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)^u\int_0^{x_2}e^{-\lambda_1 x_1}(1-e^{-\alpha x_1})e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}e^{-(\alpha+\lambda_2)ux_1}dx_1\}dx_2$$

on solving which, we get

$$g_{[r,s:n]}(y_1,y_2) = C_{r,s:n}\left[\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda}\right]^2 e^{-(\lambda_2+\lambda_{12})(y_1+y_2)}\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}\binom{l+s-r-q+1}{u}$$

$$\times\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+3}(\lambda_1+\lambda_{12})^{u+v}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}$$

$$\times\left[\frac{1}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}\left\{\frac{e^{-[2\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+2\alpha]y_1}}{2\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+2\alpha}\right.\right.$$

$$\left.-\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)]y_1}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)}\right\}-\frac{1}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}$$

$$\left.\times\left\{\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+\alpha]y_1}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+\alpha}-\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)]y_1}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)}\right\}\right]$$

6.3. ***If:*** $0 \le x_1 < x_2 < y_2 < y_1 \le \infty$

Again, on using (1.11) and (2.3), we get

$$I_3(y_1,y_2)=g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2\lambda_1^2(\lambda_2+\lambda_{12})^2e^{-(\lambda_2+\lambda_{12})(y_1+y_2)}\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}$$

$$\times\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_0^{y_2}\{e^{-\lambda_1x_2}(1-e^{-\alpha x_2})(\alpha+\lambda)^{n-s+q}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}$$

$$\times(\alpha+\lambda)^{l+s-r-q+1}\sum_{u=0}^{l+s-r-q+1}(-1)^u\binom{l+s-r-q+1}{u}\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)^u\int_0^{x_2}e^{-\lambda_1x_1}(1-e^{-\alpha x_1})e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}e^{-(\alpha+\lambda_2)ux_1}dx_1\}dx_2$$

on solving which, we get

$$g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left[\frac{\lambda_1(\lambda_2+\lambda_{12})}{\lambda}\right]^2e^{-(\lambda_2+\lambda_{12})(y_1+y_2)}\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}\binom{l+s-r-q+1}{u}$$

$$\times\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+3}(\lambda_1+\lambda_{12})^{u+v}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\left[\frac{1}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}\right.$$

$$\times\left\{\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+2\alpha]y_2}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+2\alpha}-\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)]y_2}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)}\right\}$$

$$-\frac{1}{\lambda_1+\alpha+(\lambda_1+\lambda_{12})(l+s-r-q+1)+(\alpha+\lambda_2)u}\left\{\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+\alpha]y_2}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)+\alpha}\right.$$

$$\left.\left.-\frac{e^{-[2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)]y_2}}{2\lambda_1+(\lambda_1+\lambda_{12})(l+n-r+1)+(\alpha+\lambda_2)(u+v)}\right\}\right]$$

Case II: When $X>Y$

Again we have three case as

6.4. ***If:*** $0\le y_1<x_1<y_2<x_2\le\infty$

Then on using (1.12) and (2.3), we get

$$I_4(y_1,y_2)=g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2\lambda_2^2(\lambda_1+\lambda_{12})^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})$$

$$\times\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}\left(\frac{\alpha+\lambda}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_{y_2}^{\infty}\{e^{-(\lambda_1+\lambda_{12})x_2}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}$$

$$\times\int_{y_1}^{y_2}e^{-(\lambda_1+\lambda_{12})x_1}e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_1}\right]^{l+s-r-q+1}dx_1\}\,dx_2$$

on solving which, we get

$$g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left[\frac{\lambda_2(\lambda_1+\lambda_{12})}{\alpha}\right]^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}$$

$$\times\binom{l+s-r-q+1}{u}\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+1}(\lambda_1+\lambda_{12})^{u+v}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\left[\frac{e^{-[(\lambda_1+\lambda_{12})(n-s+q+1)+(\alpha+\lambda_2)v]y_2}}{(\lambda_1+\lambda_{12})(n-s+q+1)+(\alpha+\lambda_2)v}\right]$$

$$\times\left[\frac{e^{-[(\lambda_1+\lambda_{12})(l+s-r-q+2)+(\alpha+\lambda_2)u]y_1}-e^{-[(\lambda_1+\lambda_{12})(l+s-r-q+2)+(\alpha+\lambda_2)u]y_2}}{(\lambda_1+\lambda_{12})(l+s-r-q+2)+(\alpha+\lambda_2)u}\right]$$

6.5. ***If:*** $0\le y_1<y_2<x_1<x_2\le\infty$

Again on using (1.12) and (2.3)

$$I_5(y_1,y_2)=g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2\lambda_2^{\,2}(\lambda_1+\lambda_{12})^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})$$

$$\times\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}\left(\frac{\alpha+\lambda}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_{y_2}^{\infty}\Big\{e^{-(\lambda_1+\lambda_{12})x_2}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}$$

$$\times\int_{y_2}^{x_2}e^{-(\lambda_1+\lambda_{12})x_1}e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_1}\right]^{l+s-r-q+1}dx_1\Big\}dx_2$$

on solving which, we get

$$g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left[\frac{\lambda_2(\lambda_1+\lambda_{12})}{\lambda}\right]^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}$$

$$\times\binom{l+s-r-q+1}{u}\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+1}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\frac{(\lambda_1+\lambda_{12})^{u+v}}{(\lambda_1+\lambda_{12})(l+s-r-q+2)+(\alpha+\lambda_2)u}$$

$$\times\left[\frac{e^{-[(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)]y_2}}{(\lambda_1+\lambda_{12})(n-s+q+1)+(\alpha+\lambda_2)v}-\frac{e^{-[(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)]y_2}}{(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)}\right]$$

6.6. ***If:*** $0\le y_2<y_1<x_1<x_2\le\infty$

Again on using (1.12) and (2.3)

$$I_6(y_1,y_2)=g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left(\frac{\alpha+\lambda}{\alpha}\right)^2\lambda_2^{\,2}(\lambda_1+\lambda_{12})^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})$$

$$\times\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{r-1}{l}\binom{s-r-1}{q}\left(\frac{\alpha+\lambda}{\alpha+\lambda_2}\right)^{l+n-r+1}\int_{y_1}^{\infty}\Big\{e^{-(\lambda_1+\lambda_{12})x_2}e^{-(\lambda_1+\lambda_{12})(n-s+q)x_2}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_2}\right]^{n-s+q}$$

$$\times\int_{y_1}^{x_2}e^{-(\lambda_1+\lambda_{12})x_1}e^{-(\lambda_1+\lambda_{12})(l+s-r-q+1)x_1}\left[1-\left(\frac{\lambda_1+\lambda_{12}}{\alpha+\lambda}\right)e^{-(\alpha+\lambda_2)x_1}\right]^{l+s-r-q+1}dx_1\Big\}dx_2$$

on solving which, we get

$$g_{[r,s:n]}(y_1,y_2)=C_{r,s:n}\left[\frac{\lambda_2(\lambda_1+\lambda_{12})}{\alpha}\right]^2e^{-\lambda_2(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}\sum_{u=0}^{l+s-r-q+1}\sum_{v=0}^{n-s+q}(-1)^{l+q+u+v}\binom{n-s+q}{v}$$

$$\times\binom{l+s-r-q+1}{u}\binom{s-r-1}{q}\binom{r-1}{l}(\alpha+\lambda)^{l+n-r-u-v+1}\left(\frac{1}{\alpha+\lambda_2}\right)^{l+n-r+1}\frac{(\lambda_1+\lambda_{12})^{u+v}}{(\lambda_1+\lambda_{12})(l+s-r-q+2)+(\alpha+\lambda_2)u}$$

$$\times\left[\frac{e^{-[(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)]y_1}}{(\lambda_1+\lambda_{12})(n-s+q+1)+(\alpha+\lambda_2)v}-\frac{e^{-[(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)]y_1}}{(\lambda_1+\lambda_{12})(l+n-r+3)+(\alpha+\lambda_2)(u+v)}\right]$$

Case III: When $X=Y$

Then

$$I_7(y_1,y_2)=f_3(y_1\,|\,x_1)\,f_3(y_2\,|\,x_2)\,f_{r,s:n}(x_1,x_2)$$

On using (1.13) and (2.3) we have

$$I_7=C_{r,s:n}\,\frac{\lambda_{12}^{\;2}(\alpha+\lambda)^2}{\alpha^2}e^{-\lambda(y_1+y_2)}(1-e^{-\alpha y_1})(1-e^{-\alpha y_2})\sum_{l=0}^{r-1}\sum_{q=0}^{s-r-1}(-1)^{l+q}\binom{s-r-1}{q}\binom{r-1}{l}$$

$$\times[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})y_1}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)y_1}]^{l+s-r-q+1}[(\alpha+\lambda)e^{-(\lambda_1+\lambda_{12})y_2}-(\lambda_1+\lambda_{12})e^{-(\alpha+\lambda)y_2}]^{n-s+q}$$

On adding $I_1,\ldots,I_7$ we get the required joint density of two concomitant of order statistics.

Acknowledgement

The authors would like to thanks the referee's for carefully reading the paper and for helpful suggestions which greatly improved the paper. First author is also thankful to University Grant Commission for awarding UGC-BSR start up grant ((No. F.30-90/2015(BSR)).

References

[1]. Aleem, M. (2006): Concomitant of Order Statistics from Bivariate Inverse Rayleigh Distribution. *Pak. j. stat. oper. res.,* **2**, 45-48.

[2]. Balasubramanian, K. and Beg, M.I. (1998): Concomitant of Order Statistics in Gumbel's Bivariate Exponential distribution. Sankhya Ser.- *B*, **60**, 399-406.

[3]. Begum, Anjuman A. and Khan, A.H. (2000): Concomitant of order statistics from Marshall and Olkin's bivariate Weibull distribution. *Calcutta Statist. Assoc. Bull.,* ***50*** *(197-198), 65-70.*

[4]. Begum, Anjuman A. (2003): Concomitant of order statistics from bivariate Pareto II distribution. *Int. j. stat. sci.,**2**, 27-35.*

[5]. Chacko, M. and Thomas, P.Y. (2011): Estimation of Parameter of MorgensternType Bivariate Exponential Distribution Using Concomitant of Order Statistics. *Stat. Methodol.,* **8**, 363-376.

[6]. David, H.A. (1973): Concomitant of Order Statistics. *B Int. Statist. Inst.*, **45**, 295-300.

[7]. David, H. A. (1981): *Order statistics*. 2nd edition, *John Wiley, New York.*Jamalizadeh, A. and Kundu, D. (2013): Weighted Marshall-Olkin Bivariate Exponential Distribution. *Statistics.* **47(5)**, 917–92.

[8]. Philip, A. and Thomas, P. Y. (2015): On concomitants of order statistics arising from the extended Farlie- Gumble- Morgenstern bivariate logistic distribution and it's application in estimation. *Stat. Methodol.,* ***25****, 59-73.*