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Rank of Product of Certain Algebraic Classes

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Abstract: *The properties of rank of a finite semigroup were presented by Howie in [1], and this properties is more general in algebraic system such as semigroup, or indeed even a group. We use this concept (properties) to compute the upper rank and the intermediate rank of direct product of a distinct monoid and the quotient group.*
Keywords: *Monoid, Independent set, Quotient group and Cyclic group.*

I. Introduction And Preliminaries

Many authors have studied the rank properties in the context of general algebras since the work of Marczewski in [2]. This property is similar to the concept of dimension in linear algebra. Howie and Ribeiro in [1] considered the following definition of rank for a finite semigroup.

1. $r_1(S)$=max{k : every subset U of cardinality k in S is independent}, this is called the small rank
2. $r_2(S)$= min{ |U| : U⊆S, $\langle U\rangle$=S} This is called the lower rank
3. $r_3(S)$= max{ |U| : U⊆S, $\langle U\rangle$=S, U is independent}. This is called the intermediate rank
4. $r_4(S)$= max{ |U| : U⊆S, U is independent}. This is called the upper rank
5. $r_5(S)$=min{k : every subset U of cardinality k in S generate S}. This is called the larger rank.

1.1 Definition (independent subset)
A subset U of a semigroup S is said to be independent if for all element a belonging to U, a does not belong to the generating subset $U\backslash\{a\}$ of S. That is

$$(\forall a \in U)\ a \notin \langle U\backslash\{a\}\rangle$$

1.2 All the five ranks coincide in certain semigroups. However, there exist semigroups for which all these ranks are distinct.
In this work , we adopt the notations and definition given in [1] and [3]. A monoid is a semigroup with identity element. We shall in section 2 compute the intermediate rank and the upper rank of the direct product of monoid . In section 3, we compute that of the quotient group. Throughout this work , our semigroup S is a monoid.

1.3 REMARK
As the definition of different rank implies, lower intermediate and upper ranks, it has been shown that $r_2(S)\leq r_3(S)\leq r_4(S)$.

SECTION 2
We present in this section the result of the rank of the direct and subdirect product of the monoid. Our intermediate rank $r_3(S)$ is denoted by ρ(S), and the upper rank $r_4(S)$ by R(S), except otherwise stated.
Theorem 2.1
Let A, B be monoids, then R(AxB)≥R(A)+R(B)
Proof
If $a_1, a_2, \ldots., a_k$ and $b_1, b_2, \ldots, b_l$ are maximal sets in A and B, respectively, then
$(a_1, 1), (a_2, 1), \ldots., (a_k, 1)$, and $(1, b_1), (1, b_2), \ldots, (1, b_l)$ are independent in A and B.
Also ,
$\{(a_1, 1), (a_2, 1), \ldots., (a_k, 1), (1, b_1), (1, b_2), \ldots, (1, b_l)\}$ are independent set in AxB. Then R(AxB)≥R(A)+R(B).
Similarly, for monoids A, B, C, we have that for independent sets $(c_1, c_2, \ldots., c_t)$ in C and C is not a subset of A or B, we would have
$\{(a_1, 1,1), (a_2, 1,1), \ldots., (a_k, 1,1), (1, b_1, 1), (1, b_2, 1), \ldots, (1, b_l, 1), (1,1, c_1), \ldots, (1,1, c_t)\}$ is independent subset in AxBxC. Moreover , From product set , we have R(AxBxC)=R(A)xR(B)xR(C) and R(AxBxC)≥R(A)+R(B)+R(C)
■
Corollary 2.2
If our monoids is distinct, then R(BxA)>R(B)+R(A_{K+1})
Proof

Let A_1, A_2, …,A_m be distinct monoids, then a typical element in A_1x A_2x … xA_m is (a_1,a_2, …,a_m), $a_i \in A_i$. An independent set in A_i, is $(a_{i_1}, a_{i_2}, \dots, a_{i_r})$. Also, for the subdirect product
$\{(a_{i_1}, 1, \dots, 1), (1, a_{i_2}, 1, \dots, 1), \dots, (1, 1, \dots, 1, a_{i_r})$ are independent set in
(A_1x A_2x … xA_m)
Let m=k
R(A_1x A_2x … xA_k)≥ R(A_1)+R(A_2)+ …+ R(A_k)
For m=k+1
R(A_1x A_2x … xA_{k+1})≥ R(A_1)+R(A_2)+ … +R(A_{k+1})
Let A_1x A_2x … xA_k be B
Then R(B×A)≥R(B)+R(A_{k+1})
■
Theorem2.3
Then intermediate rank ρ is given by
R(B×A)≥R(B)+R(A_{k+1})
Proof
The proof is straightforward from theorem (2.2) for the same distinct monoid.

SECTION 3
The collection of all cosets of normal subgroups form a group usually referred to as quotient group. We now compute the rank of this quotient group in this section

REMARK 3.1
The notion of a quotient group is fundamental for group theory and indeed is one of the most important concepts in mathematics. We therefore repeat some of the relevant points
1. The elements of G/N (G is a group and N is a normal subgroup) are the distinct coset of N, the law of composition being multiplication of subset (or addition of cosets when G is written additively.)
2. The identity (neutral) element in the group N, regarded as one of the cosets.
3. It is immaterial whether we use right or left coset since Nt=tN, because N is normal for t∈G.
4. Recall that the representative of a particular coset is not unique.
Theorem3.2
For any quotient group G/N (the group G is finite) where G_1 and G_2 are distinct in G/N,
R(G_1x G_2)≥R(G_1)x R(G_2) and ρ(G_1x G_2)≥ ρ (G_1)x ρ (G_2)
Proof:
Let G be a group and N⊴G, then G/N is a group.

$$G/N = \{Nx_0, Nx_1, \dots, Nx_t\}$$

Put $Nx_0 \equiv N$ $(i.e\ x_0 \equiv e)$
$G/N \cong G^* \{g_0, g_1, \dots, g_t\}$ **.** Let t be the minimum rank of independent set of G^*
By Lagranges theorem, It is well known that

$$|G^*| = \frac{|G|}{|N|}$$

Thus, the rank of any quotient group $G^* < rank\ of\ G$.
If for $G^*{}_1, G^*{}_2, \dots, G^*{}_t$ is a set of respective quotients group of G modulo $N_1, \dots, N_t$ respectively, we have
$G^*{}_1 \times G^*{}_2 \times \dots \times G^*{}_t = {}^{G_1}/_{N_1} \times {}^{G_2}/_{N_2} \times \dots \times {}^{G_t}/_{N_t}$

$$R(G^*{}_1 \times G^*{}_2 \times \dots \times G^*{}_t) = R({}^{G_1}/_{N_1} \times {}^{G_2}/_{N_2} \times \dots \times {}^{G_t}/_{N_t})$$

$$< R(G_1 \times \dots \times G_t)$$

$$R(Q(G_1) \times \dots \times Q(G_t)) \le R(G_1 \times \dots \times G_t)$$

Each $G^* = H(G) \cong Q(G)$
Rewriting this for intermediate rank we have

$$\rho(H(G)) < \rho(G)$$

■
The notions of independence in Abelian group G is compare to that of subsemigroup of a group G. We make use of the definition in [6]
Corollary 3.3
For a cyclic group H of order $P_i^{\alpha_i}$,
The rank $R(K_a) \ge R(H_1) + R(H_2) + \dots + R(H_k)$
Proof
Let $n = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_k^{\alpha_k} = n_1 n_2 \dots n_k$. Where $n_i = P_i^{\alpha_i}$, and $q_i = \frac{\alpha_i}{P_i^{\alpha_i}}$

$$K_a = H_1 \times H_2 \times \ldots \times H_k = \langle a^{q_i} \rangle$$

$H_i = \langle a^{q_i} \rangle$ is a cyclic group of order ${P_i}^{\alpha_i}$

$$R(K_a) = R(H_1 \times H_2 \times \ldots \times H_k) \geq R(H_1) + R(H_2) + \cdots + R(H_k)$$

Rank $r(H_i) = r(\langle a^{q_i} \rangle)$=

$$R(K_a) \geq R(H_1) + R(H_2) + \cdots + R(H_k) \geq r(H_1) + r(H_2) + \cdots + r(H_k) = K.$$

$r(H_i)$ is the lower rank of the cyclic group H.

■

REMARK 3.4

1. For any commutative semigroup S and T, the rank R(SxT)= rank(S)+rank(T) [5]
2. The rank of the direct product of any algebraic classes is computed based on the defining structure as shown above.

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