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Reciprocal Series of K–Fibonacci Numbers with Subscripts in Linear Form

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Abstract: *The aim of this paper is to find a formula that allows to find the infinite sum ofreciprocals of certain k–Fibonacci numbers whose subscripts are in linear form.Particularizing this formula, we are able to obtain other formulas previously foundby other authors for the case of both classical Fibonacci and Pell sequences.*
Keywords:*Binet identity, k–Fibonacci numbers, k–Lucas numbers.*

I. Introduction

One of the more studied sequences is the Fibonacci sequence [1, 2], and it hasbeen generalized in many ways [3]. Here, we use the following one-parametergeneralization of the Fibonacci sequence [4,5].

1.1 Definition.

For any integer number $k \geq 1$, the k–Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$,is defined recurrently by $F_{k,0}=0,\ F_{k,1}=1$ and $F_{k,n+1}=k\,F_{k,n}+F_{k,n-1}$

Note for k = 1 the classical Fibonacci sequence $F=\{F_n\}=\{0,1,1,2,3,5,8...\}$ is obtained and for k = 2 it isthe Pell sequence $P=\{P_n\}=\{0,1,2,5,12,29,70...\}$.

The characteristic equation of the definition is $r^2=k\,r+1$whose solutionsare $\sigma_1=\dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2=\dfrac{k-\sqrt{k^2+4}}{2}$, that verify $\sigma_1\sigma_2=-1,\ \sigma_1+\sigma_2=k,\ \sigma_1-\sigma_2=\sqrt{k^2+4},\ \sigma^2=k\,\sigma+1,\ \sigma_1>0,\ \sigma_2<0$.

For the properties of the k-Fibonacci numbers, see [4, 5].In particular, the Binet Identity is $F_{k,n}=\dfrac{\sigma_1^n-\sigma_2^n}{\sigma_1-\sigma_2}$ and the convolution formula, $F_{k,n+m}=F_{k,n+1}F_{k,m}+F_{k,n}F_{k,m-1}$

Finally, we define the k-Fibonacci numbers of negative index as $F_{k,-n}=(-1)^{n+1}F_{k,n}$

1.2 Definition.

For any integer number $k \geq 1$, the k–Lucas sequence [6], say $\{L_{k,n}\}_{n\in N}$,is defined recurrently by $L_{k,0}=2,\ L_{k,1}=k$ and $\mathrm{L}_{k,n+1}=k\,L_{k,n}+L_{k,n-1}$

Note for k = 1 the classical Lucas sequence is obtained $L=\{L_n\}=\{2,1,3,4,7,11...\}$ and for k = 2 it is the Pell-Lucas sequence $L_2=PL=\{PL_n\}=\{2,2,6,14,34,82...\}$.

The Binet Identity for the k-Lucas numbers is $L_{k,n}=\sigma_1^n+\sigma_2^n$. The k-Lucas numbers are related to the k-Fibonacci numbers by the relation $L_{k,n}=F_{k,n-1}+F_{k,n+1}$

II. Preliminary

In this section we will prove some lemmas that we will need for the proof of the main theorem of this paper.

2.1 Lemma 1.

For $m \geq 1$,

$$F_{k,2m}=F_{k,m}\cdot L_{k,m} \tag{1}$$

Proof. Applying the Binet formula, $F_{k,m}\cdot L_{k,m}=\dfrac{\sigma_1^m-\sigma_2^m}{\sigma_1-\sigma_2}\cdot(\sigma_1^m+\sigma_2^m)=\dfrac{\sigma_1^{2m}-\sigma_2^{2m}}{\sigma_1-\sigma_2}=F_{k,2m}$

2.2 Lemma 2.
It is

$F_{k,a+4b}+F_{k,a}=F_{k,a+2b}L_{k,2b}$ (2)

Proof. Applying the convolution formula and $F_{k,-n}=(-1)^{n+1}F_{k,n}$, the Left Hand Side (LHS) of this equation is

$$LHS=F_{k,a+2b+2b}+F_{k,a+2b-2b}=F_{k,a+2b+1}F_{k,2b}+F_{k,a+2b}F_{k,2b-1}+F_{k,a+2b+1}F_{k,-2b}+F_{k,a+2b}F_{k,-2b-1}$$
$$=F_{k,a+2b+1}F_{k,2b}+F_{k,a+2b}F_{k,2b-1}-F_{k,a+2b+1}F_{k,2b}+F_{k,a+2b}F_{k,2b+1}=F_{k,a+2b}(F_{k,2b-1}+F_{k,2b+1})=F_{k,a+2b}L_{k,2b}$$

2.3 Lemma 3.

$$F_{k,2(2a+1)+(2p+1)}+F_{k,2p+1}=F_{k,2a+1}L_{k,2(a+p+1)} \quad (3)$$

Proof. Using the Binet formulas for $F_{k,n}$ and $L_{k,n}$, and taking into account $\sigma_1\sigma_2=-1$, it is

$$RHS=\frac{1}{\sqrt{k^2+4}}\left(\sigma_1^{2a+1}-\sigma_2^{2a+1}\right)\left(\sigma_1^{2a+2p+2}+\sigma_2^{2a+2p+2}\right)$$
$$=\frac{1}{\sqrt{k^2+4}}\left(\sigma_1^{4a+2p+3}+\sigma_1^{2a+1}\sigma_2^{2a+1+2p+1}-\sigma_1^{2a+1+2p+1}\sigma_2^{2a+1}-\sigma_2^{4a+2p+3}\right)$$
$$=\frac{1}{\sqrt{k^2+4}}\left(\sigma_1^{4a+2p+3}-\sigma_2^{4a+2p+3}-\sigma_2^{2p+1}+\sigma_1^{2p+1}\right)=LHS$$

2.4 Lemma 4.

$$L_{k,a}L_{k,2b}-(k^2+4)F_{k,a}F_{k,2b}=2L_{k,a-2b} \quad (4)$$

It is enough to apply the Binet formulas for $F_{k,,n}$ and $L_{k,n}$

2.5 Lemma 5.

$$L_{k,a+2b}F_{k,a}-L_{k,a}F_{k,a+2b}=-2(-1)^aF_{k,2b} \quad (5)$$

Proof.

$$LHS=\frac{1}{\sqrt{k^2+4}}\left(\left(\sigma_1^{a+2b}+\sigma_2^{a+2b}\right)\left(\sigma_1^{a}-\sigma_2^{a}\right)-\left(\sigma_1^{a}+\sigma_2^{a}\right)\left(\sigma_1^{a+2b}-\sigma_2^{a+2b}\right)\right)$$
$$=\frac{1}{\sqrt{k^2+4}}\left(\sigma_1^{2a+2b}-\sigma_2^{2a+2b}-(-1)^a\sigma_1^{2b}+(-1)^a\sigma_2^{2b}-\sigma_1^{2a+2b}+\sigma_2^{2a+2b}-(-1)^a\sigma_1^{2b}+(-1)^a\sigma_2^{2b}\right)$$
$$=-2(-1)^a\frac{\sigma_1^{2b}-\sigma_2^{2b}}{\sqrt{k^2+4}}=-2(-1)^aF_{k,2b}$$

III. Main Result

In this section we will prove the main theorem of this paper.

3.1 Main theorem.
Let us suppose *p* and *r* are integer numbers. Then

$$\sum_{j=0}^{2n}\frac{1}{F_{k,(2p+1)(2j+1)+4r}+F_{k,2p+1}}=\frac{L_{k,(2p+1)(2n+1)}}{L_{k,2r}L_{k,2p+1}F_{k,(2p+1)(2n+1)+2r}} \quad (6)$$

First we prove the following equality:

$$\sum_{j=0}^{2n}\frac{1}{F_{k,(2p+1)(2j+1)+4r}+F_{k,2p+1}}=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,(2p+1)(2n+1)+2r}}{F_{k,(2p+1)(2n+1)+2r}}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right) \quad (7)$$

Proof. We will proceed by induction on *2n*.

If $n=0$, we must prove that $\frac{1}{F_{k,2p+1+4r}+F_{k,2p+1}}=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,2p+1+2r}}{F_{k,2p+1+2r}}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right)$.

FromLemma2,with $a=2p+1$ and $b=r$, $LHS=\frac{1}{F_{k,2p+1+2r}L_{k,2r}}$ while

$$RHS=\frac{L_{k,2p+1+2r}L_{k,2r}-(k^2+4)F_{k,2r}F_{k,2p+1+2r}}{2F_{k,2p+1+2r}L_{k,2r}L_{k,2p+1}}$$

From Lemma 4 with $a=2p+1+2r$ and $b=r$, $L_{k,2p+1+2r}L_{k,2r}-(k^2+4)F_{k,2p+1+2r}F_{k,2r}=2L_{k,2p+1}$. Hence $RHS=\frac{1}{F_{k,2p+1+2r}L_{k,2r}}=LHS$

Assuming Main Theorem is true for *n*, we must prove it is true for *n + 1*.

If S_n is the sum of (7) and a_n the summands, then for *n + 1* it is $S_{2n+2}=S_{2n}+a_{2n+1}+a_{2n+2}$

$$=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,(2p+1)(2n+1)+2r}}{F_{k,(2p+1)(2n+1)+2r}}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right)+\frac{1}{F_{k,(2p+1(4n+3)+4r}+F_{k,2p+1}}+\frac{1}{F_{k,(2p+1(4n+5)+4r}+F_{k,2p+1}}$$

Therefore, we must prove

$$\frac{1}{F_{k,(2p+1)(4n+3)+4r}+F_{k,2p+1}}+\frac{1}{F_{k,(2p+1)(4n+5)+4r}+F_{k,2p+1}}$$

$$=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,(2p+1)(2n+3)+2r}}{F_{k,(2p+1)(2n+3)+2r}}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}-\frac{L_{k,(2p+1)(2n+1)+2r}}{F_{k,(2p+1)(2n+1)+2r}}+(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right)$$

$$=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,(2p+1)(2n+3)+2r}}{F_{k,(2p+1)(2n+3)+2r}}-\frac{L_{k,(2p+1)(2n+1)+2r}}{F_{k,(2p+1)(2n+1)+2r}}\right)$$

Let $q=(2p+1)(2n+1)+2r$ be. Then we must prove

$$\frac{1}{F_{k,2q+2p+1}+F_{k,2p+1}}+\frac{1}{F_{k,2q+3(2p+1)}+F_{k,2p+1}}=\frac{1}{2L_{k,2p+1}}\left(\frac{L_{k,q+2(2p+1)}}{F_{k,q+2(2p+1)}}-\frac{L_{k,q}}{F_{k,q}}\right) \tag{8}$$

From Lemma 3 with $2a+1=q$ it is $F_{k,2q+2p+1}+F_{k,2p+1}=F_{k,q}L_{k,q+2p+1}$

From Lemma 2 with $a=2p+1$ and $2b=q+2p+1$ it is $F_{k,a+4b}+F_{k,a}=F_{k,2q+3(2p+1)}+F_{k,2p+1}=F_{k,2p+1+q+2p+1}L_{k,q+2p+1}=F_{k,q+2(2p+1)}L_{k,q+2p+1}$

Then, in the equation (8) $LHS=\frac{1}{F_{k,q}L_{k,q+2p+1}}+\frac{1}{F_{k,q+2(2p+1)}L_{k,q+2p+1}}=\frac{F_{k,q+2(2p+1)}+F_{k,q}}{F_{k,q}L_{k,q+2p+1}F_{k,q+2(2p+1)}}$.

FromLemma3, $F_{k,2a+1+2(2p+1)}+F_{k,2a+1}=F_{k,2p+1}L_{k,2(p+a+1)}$,with $2a+1=q$,

$F_{k,q+2(2p+1)}+F_{k,q}=F_{k,2p+1}L_{k,2\left(p+\frac{q-.1}{2}+1\right)}=F_{k,2p+1}L_{k,2p+q+1}$ and therefore $LHS=\frac{F_{k,2p+1}}{F_{k,q}F_{k,q+2(2p+1)}}$

On the other hand, in the equation (8) it is $RHS=\frac{L_{k,q+2(2p+1)}F_{k,q}-L_{k,q}F_{k,q+2(2p+1)}}{2L_{k,p+1}F_{k,q+2(2p+1)}F_{k,q}}$.

From Lemma 5, with $a=q,\ c=2p+1$, it is $L_{k,q+2(2p+1)}F_{k,q}-L_{k,q}F_{k,q+2(2p+1)}=F_{k,a+2c}F_{k,a}-L_{k,a}F_{k,a+2c}=2F_{k,2(2p+1)}$.

Taking into account $F_{k,2m}=F_{k,m}L_{k,m}$, finally $RHS=\frac{F_{k,2(2p+1)}}{L_{k,2p+1}F_{k,q+2(2p+1)}F_{k,q}}=\frac{F_{k,2p+1}}{F_{k,q+2(2p+1)}F_{k,q}}=LHS$ and the formula (7) is proven.

If $a=(2p+1)(2n+1)+2r,\ b=r$ in the equation (4), the numerator of the RHS of the equation (6) is $L_{k,(2p+1)(2n+1)+2r}L_{k,2r}-(k^2+4)F_{k,(2p+1)(2n+1)+2r}F_{k,2r}=2L_{k,(2p+1)(2n+1)}$ and Main theorem of this paper is at last proven.

3.2 Lemma 6.

It is verified

$$\lim_{n\to\infty}\frac{L_{k,n}}{F_{k,n}}=\sqrt{k^2+4} \tag{9}$$

Proof. In [3, 4] it is proven $\lim_{n\to\infty}\frac{F_{k,n+1}}{F_{k,n}}=\sigma_1$. Then $\lim_{n\to\infty}\frac{L_{k,n}}{F_{k,n}}=\lim_{n\to\infty}\frac{F_{k,n+1}+F_{k,n-1}}{F_{k,n}}=\sigma_1+\frac{1}{\sigma_1}=\sigma_1-\sigma_2=\sqrt{k^2+4}$

Consequently $\sum_{j=0}^{\infty}\frac{1}{F_{k,(2p+1)(2j+1)+4r}+F_{k,2p+1}}=\frac{1}{2L_{k,2p+1}}\left(\sqrt{k^2+4}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right)$

3.3 Corollary 1.

In this Corollary, we will use the equation (7) or the (6), as appropriate.

Giving integer values to *r*, *p*, and *k*, these formulas are particularized as follows:

1. $r=0: \sum_{j=0}^{2n} \frac{1}{F_{k,(2p+1)(2j+1)}+F_{k,2p+1}} = \frac{L_{k,(2p+1)(2n+1)}}{2L_{k,2p+1}F_{k,(2p+1)(2n+1)}}$

a. $p=0: \sum_{j=0}^{2n} \frac{1}{F_{k,2j+1}+1} = \frac{1}{2k}\frac{L_{k,2n+1}}{F_{k,2n+1}} \to \sum_{j=0}^{\infty} \frac{1}{F_{k,2j+1}+1} = \frac{\sqrt{k^2+4}}{2k}$

$k=1: \sum_{j=0}^{\infty} \frac{1}{F_{2j+1}+1} = \frac{\sqrt{5}}{2}$ [7]

$k=2: \sum_{j=0}^{\infty} \frac{1}{P_{2j+1}+1} = \frac{\sqrt{2}}{2}$

b. $p=1: \sum_{j=0}^{2n} \frac{1}{F_{k,6j+3}+(k^2+1)} = \frac{1}{2(k^3+3k)}\frac{L_{k,6n+3}}{F_{k,6n+3}} \to \sum_{j=0}^{\infty} \frac{1}{F_{k,6j+3}+(k^2+1)} = \frac{\sqrt{k^2+4}}{2(k^3+3k)}$

$k=1: \sum_{j=0}^{\infty} \frac{1}{F_{6j+3}+2} = \frac{\sqrt{5}}{8}$

$k=2: \sum_{j=0}^{\infty} \frac{1}{P_{6j+3}+5} = \frac{\sqrt{2}}{14}$

2. $r=1: \sum_{j=0}^{2n} \frac{1}{F_{k,(2p+1)(2j+1)+4}+F_{k,2p+1}} = \frac{L_{k,(2p+1)(2n+1)}}{(k^2+2)L_{k,2p+1}F_{k,(2p+1)(2n+1)+2}}$

a. $p=0: \sum_{j=0}^{2n} \frac{1}{F_{k,2j+5}+1} = \frac{L_{k,2n+1}}{2(k^2+2)\,k\,F_{k,2n+3}} = \frac{1}{2k}\left(\frac{L_{k,2n+3}}{F_{k,2n+3}} - \frac{(k^2+4)k}{k^2+2}\right)$

$\to \sum_{j=0}^{\infty} \frac{1}{F_{k,2j+5}+1} = \frac{1}{2k}\left(\sqrt{k^2+4} - \frac{(k^2+4)k}{k^2+2}\right)$

$k=1: \sum_{j=0}^{\infty} \frac{1}{F_{2j+5}+1} = \frac{1}{2}\left(\sqrt{5}-\frac{5}{3}\right)$

$k=2: \sum_{j=0}^{\infty} \frac{1}{P_{2j+5}+1} = \frac{1}{2}\left(\sqrt{8}-\frac{8}{3}\right)$

b. $p=1: \sum_{j=0}^{2n} \frac{1}{F_{k,6j+7}+(k^2+1)} = \frac{L_{k,6n+3}}{(k^2+2)(k^3+3k)F_{k,6n+5}} = \frac{1}{2(k^3+3k)}\left(\frac{L_{k,6n+5}}{F_{k,6n+5}} - \frac{(k^2+4)k}{k^2+2}\right)$

$\to \sum_{j=0}^{\infty} \frac{1}{F_{k,6j+7}+(k^2+1)} = \frac{1}{2(k^3+3k)}\left(\sqrt{k^2+4} - \frac{(k^2+4)k}{k^2+2}\right)$

$k=1: \sum_{j=0}^{\infty} \frac{1}{F_{6j+7}+2} = \frac{1}{8}\left(\sqrt{5}-\frac{5}{3}\right)$

$k=2: \sum_{j=0}^{\infty} \frac{1}{P_{6j+7}+2} = \frac{1}{14}\left(\sqrt{2}-\frac{4}{3}\right)$

3.4 Corollary 2.

If $S_{\infty}(k,r,p) = \sum_{j=0}^{\infty} \frac{1}{F_{k,(2p+1)(2j+1)+4r}+F_{k,2p+1}}$, then $S_{\infty}(k,r,p+h) = \frac{L_{k,2p+1}}{L_{k,2p+1+2h}} S_{\infty}(k,r,p)$

For instance: from Corollary 1, $S_{\infty}(2,1,1) = \sum_{j=0}^{\infty} \frac{1}{P_{6j+7}+5} = \frac{1}{14}\left(\sqrt{2}-\frac{4}{3}\right)$ therefore

$S_{\infty}(2,1,4) = \frac{L_{2,3}}{L_{2,9}} S_{\infty}(2,1,1) \to \sum_{j=0}^{\infty} \frac{1}{P_{18j+13}+985} = \frac{1}{2786}\left(\sqrt{2}-\frac{4}{3}\right)$

3.5 Corollary 3.

If $p = 0$ then $\sum_{j=0}^{\infty}\frac{1}{\mathrm{F}_{\mathrm{k,2j+1+4r}}+1}=\frac{1}{2k}\left(\sqrt{k^2+4}-(k^2+4)\frac{F_{k,2r}}{L_{k,2r}}\right)$ and

$$\sum_{j=0}^{\infty}\frac{1}{\mathrm{F}_{\mathrm{k,2j+1+4(r+h)}}+1}=\frac{1}{2k}\left(\sqrt{k^2+4}-(k^2+4)\frac{F_{k,2(r+h)}}{L_{k,2(r+h)}}\right).$$

So $\sum_{j=0}^{\infty}\left(\frac{1}{F_{k,2j+1+4r}+1}-\frac{1}{F_{k,2j+1+4(r+h)}+1}\right)=\frac{k^2+4}{2k}\left(\frac{F_{k,2(r+h)}}{L_{k,2(r+h)}}-\frac{F_{k,2r}}{L_{k,2r}}\right)$ andthen

$\sum_{j=0}^{2n-1}\frac{1}{F_{k,2j+1+4r}+1}=\frac{k^2+4}{2k}\left(\frac{F_{k,2(r+n)}}{L_{k,2(r+n)}}-\frac{F_{k,2r}}{L_{k,2r}}\right)$, that is $\sum_{j=0}^{2n-1}\frac{1}{F_{k,2j+1+4r}+1}=\frac{k^2+4}{2k}\frac{F_{k,2(r+n)}L_{k,2r}-F_{k,2r}L_{k,2(r+n)}}{L_{k,2(r+n)}L_{k,2r}}$

Finally, applying Lemma 5 with $a = 2r$ and $c = n$, it is

$$\sum_{j=0}^{2n-1}\frac{1}{F_{k,2j+1+4r}+1}=\frac{k^2+4}{2k}\frac{F_{k,2n}}{L_{k,2(r+n)}L_{k,2r}} \quad (10)$$

In particular, for $k = 1$ and $r = 0$ it is $\sum_{j=0}^{2n-1}\frac{1}{F_{2j+1}+1}=\frac{5F_{2n}}{2L_{2n}}$.

In this form we have found the formula for the sum of an odd number of terms of the indicated form, while from the formula (1) (a) of Corollary 1 it is $\sum_{j=0}^{2n}\frac{1}{F_{2j+1}+1}=\frac{L_{2n+1}}{2F_{2n+1}}$

From the difference of these last two formulas and applying the Binet formulas, we get $L_{2n}L_{2n+1}=F_{4n+1}+1$

References

[1] A.F. Horadam, A generalized Fibonacci sequence, *Math. Mag. 68,* (1961), 455-459.
[2] S. Vajda, *Fibonacci & Lucas Numbers, and the Golden Section, Theory andApplications*.(Ellis Horwood limited, 1989).
[3] V.E. Hoggat, *Fibonacci and Lucas numbers,* (Palo Alto, CA: Houghton-Mifflin, 1969).
[4] S. Falcon, A. Plaza, On the Fibonacci k-numbers,*Chaos, Solit. &Fract. 32(5),* (2007) 1615–24.
[5] S. Falcon, A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, *Chaos, Solit. &Fract. 33(1),* (2007) 38–49.
[6] S. Falcon, On the k–Lucas numbers, *Int. J. Contemp. Math. Sciences, 6(21),* 2011, 1039–1050
[7] Robert P. Backstrom, On Reciprocal Series Related to Fibonacci Numbers with Subscripts in Arithmetic Progression, *The Fibonacci Quarterly, 19(1),* 1981, 14–21