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Generalist Species Predator – Prey Model and Maximum Sustainable Yield

Mohammed Yiha Dawed[1], Purnachandra Rao Koya[2],
Temesgen Tibebu Mekonen[3]
[1,2]*School of Mathematical and Statistical Sciences, Hawassa University, Hawassa, Ethiopia*
[3]*Department of Mathematics, Debere Birhan University, Debere Birhan, Ethiopia*

Abstract:*Generalist species Predator – Prey Model and Maximum sustainable Yield have been discussed in this study.Both prey and predator populations are considered to follow logistic law of growth. The model possibly concerns with the prey population as Tilapia fish and the predator population as Whale fish.Furthermore, intra – specific competition among predator population is also included.The conversion efficiency of the predator is proportional to its capturing efficiency that makes the model more real and practical. Theorems on global stability of interior equilibrium points and non-existence of limit cycle are proved applying Bendixson - Dulac criteria and the Lyapunov theory. The perturbation analysis, leading order systems and stability of the interior equilibrium point are included. Prey harvesting, predator harvesting and maximum sustainable yield have been discussed and the results are supported by numerical simulations.*

Keywords:*Bounded solution***,** *Preypredator,Perturbation analysis, Populationharvesting,Global stability, Lyapunov function.*

I. Introduction

Unlike specialist predator-prey model, generalist species predator-prey model is a model in which the predatorhasother alternative sources offood [6]. The present system, containing two species:predator and prey, is an extension of the classical predator prey model [2, 10]. Holling Type I predator response function and intra-specific competition among predatorshave been considered and included in the model.

The capture efficiency of predator population is proportional to its conversion efficiency.The proportionality constant is considered to be a constant as one unit in order to make the model simpleas it is described in the existing model system [1].But, in the present study the proportionality constant is considered to be a variable varying in the open interval$(0, \; 1)$ and thus the model is made more realistic.The proposed model has seven dimensioned parameters while the corresponding scaled model has four dimensionless parameters. Whale fish and Tilapia fish could be examples that satisfy the proposed model of predator and prey respectively.

In this study uniqueness, positivity and boundedness of solution of the model, stability analysis of the co-existence equilibrium points,prey and predator harvestingstogether with intra-specific competition among predators themselvesand numerical simulations of the models are presented and discussed.Global stability of interior equilibrium points of the models are discussedapplying Bendixson – Dulac criteria and Lyapunov theory. Moreover non – existence of limit cycle in the positive quadrant is justified. We also verified in the leading order system that the trivial equilibrium point is unstable saddle node and is also a degenerate case.

II. The model

In this section, thealready existing and the newly proposed models have beenintroduced and briefly described. The proposed model is designed and developed having itsbase on the existing model [1]. The present modelnot only overcomes draw backs of the existing model but alsocorrectly addresses a realistic situation. Of course, the classical predator prey model forms a basis for species interaction systems and remains as a strong pillar reference for researchers working in this area.

2.1 The Existing Model

The specialist predator prey model with the inclusion of intra – specific competitionsamong predators themselvesis given in [1] and can be expressed as

$$dx/dt = r\,x\,[1-(x/k)] - p\,x\,y \quad (1)$$
$$dy/dt = -d\,y + p\,x\,y - \mu\,y^2 \quad (2)$$

Prey population grows logistically. The parameter μ represents intra – specific competition coefficient of the predator population. The parameter p in (1) represents capture efficiency coefficient while the same in (2) represents conversion efficiency coefficient. That is, both the capture and the conversion efficiency coefficients are considered to be equal.The selection of equality simplifies analysis of the model but does not represent a real situation and thus has become a drawback of this model.

2.2 The Modified Model

Here, the model (1) – (2)is modified so as to overcome the stated drawbackand can be stated as

$$dS/dT = \alpha_1\, S\,[1-(S/K_1)] - \alpha_2\, S\, R \quad (3)$$

$$dR/dT = \beta_1\, R\,[1-(R/K_2)] + \alpha_3\, S\, R - \mu\, R^2 (4)$$

The system of two equations (3) – (4) represent generalist predator prey model. Here (i)$\alpha_3 = \delta\alpha_2$, $0 < \delta < 1$ (ii) $S(T)$is the size of prey population with carrying capacity K_1 (iii) $R(T)$is the size of predator population with carrying capacityK_2 (iv)$\alpha_1$is intrinsic growth rate of prey population, (v)$\alpha_2$is the rate at which the predator and prey meet or capture efficiency coefficient (vi)$\beta_1$is intrinsic growth rate of predator population (vii)$\alpha_3$is the rate at which the predator population growsor conversion efficiency coefficient (viii) μ is the intra specific competition rate among predator population itself and (ix) the quantities $\alpha_1,\ \alpha_2,\ \alpha_3,\ \beta_1,\ \mu$ are all represent positive parameters.

It is assumed that the predator population had alternative choice for food. That is, the predator population does not depend on prey alone for food but has few alternative choices too available.Moreover, the conversion and capture efficiency coefficients $\alpha_2$and $\alpha_3$being dependent they are linearly proportional to each other.This consideration has made the present model moreimprovedthanthe already existing one.

2.3 Scaling the system of equations of the modified model

The main objective in scaling is to reduce the number of parameters and to make them unit less. Furthermore, it simplifies the model equations.For manipulating the technique of scaling, good knowledge and understanding of the mathematical equations that governs the system is a prerequisite. In the process of scaling we attempt to select intrinsic reference variables or scales so that each term in the dimensional equations transforms into the product of a constant dimensional factor and a dimensionless factor of unit of order of magnitude.

Two time scales connecting with growth rates are introduced through $T_S = (1/\alpha_1)$and$T_R = (1/\beta_1)$.On the biological grounds, the growth time of prey T_S is considered to runfaster in comparison with that of the predator T_R and thus $T_S < T_R$. Furthermore, the transformation equations for time scale $T = (1/\alpha_1)\, t$ and that for the population scales $S = K_1\, x$, $R = K_2\, y$ areconsidered.Inserting these transformationsinto the model (3) – (4), we obtain the corresponding scaled equations as

$$dx/dt = x\,(1-x) - \alpha\, x\, y \equiv F_1(x,y) \quad (5)$$

$$dy/dt = \delta\, y\,(1-y) + \beta\, x\, y - \sigma\, y^2 \equiv F_2(x,y) (6)$$

In the scaled model (5) – (6), the notations$\alpha = (\alpha_2/\alpha_1)K_2,\ \delta = (\beta_1/\alpha_1),\ \beta = (\alpha_3/\alpha_1)K_1$and $\sigma = \mu\, K_2$ are all used for the purpose of representingdimensionless parameters.

Theimposition of initial values at zerofor both the variables x and y i.e., $F_1(0,0) = F_2(0,0) = 0$ leads totwo interpretations: (i) The limiting and functional values of both the functions $F_1$and F_2 at origin are zero and hence they both are continues at origin (ii) Both the functions F_1 and F_2 are continues in the positive quadrant $R_+{}^2 = \{(x,y): x > 0, y > 0\}$.

A solution with non-negative initial value exists and is unique. Furthermore, it stays non-negative [9]. Now, boundedness of the solution is shown in what follows in the form of a theorem.

2.4 Boundedness of solution of the modified model

Theorem–1: All solutions $(x(t), y(t))$ of the system of model equations (5) – (6) together with positive initial condition (x_0, y_0) are bounded within the regionA $= \{(x,y): 0 \le x\,(t) \le 1, 0 \le y\,(t) \le \beta + 1\}$.

Proof: Boundedness argument for x:from the first equation of the model (5) – (6), it is true that $dx/dt \le x\,(1-x)$. Using partial fraction method, the equality solution of $dx/dt \le x\,(1-x)$ is $x(t) = \{[x_0/(1-x_0)]/\{e^{-t} + [x_0/(1-x_0)]\}\}$where $x_0 = x(0) > 0$.

Boundedness argument for y:from the second equation $dy/dt = \delta\, y\,(1-y) + \beta\, x\, y - \sigma\, y^2$of the model (5) – (6) it is true that $dy/dt \le \delta\, y\,(1-y) + \beta\, x\, y \le \delta\, y\,(1-y) + \beta\,(1) y = \delta\, y\,(1-y) + \beta\, y$. Using partial fraction method, the equality solution of $dy/dt \le \delta\, y\,(1-y) + \beta\, y$ has the form $y(t) = \{[\beta+1]/\{ce^{-(\beta+1)\delta t} + 1\}\}$ where c is integration constant. Hence, $dy/dt \le \delta\, y\,(1-y) + x\, y \Rightarrow y \le 1 + \beta\ \forall t \ge 0$.

Therefore all solutions of the model system with positive initial value in R_2^+ are bounded in the region A.

2.5Qualitative analysis of the modified Model

In this section equilibrium point of co – existence for the model (5) – (6) is obtained. Also analyses of its stability and simulation studies are made. Upon equating the right hand sides of the model equations to zero, the co – existence equilibrium point of the system is obtained as$(x^*,\ y^*) = \{[(\delta(1-\alpha)+\sigma)/(\delta+\alpha\beta+\sigma)],\ \ [(\beta+\delta)/(\delta+\alpha\beta+\sigma)]\}$. It can be pointed out here that this equilibrium point is valid if and only if $\alpha < [1 + (\sigma/\delta)]$. Additionally the three points (0, 0),

$(1, \ 0)$ and $\{0, \ \delta/(\delta+\sigma)\}$ are also axial equilibrium points of the model and they all are valid for all permissible parametric values.

2.5.1 Local stability of the co – existence equilibrium point

The local and asymptotic stability of co-existence equilibrium point can be studied by constructing Jacobian matrix $J(x,y)$for the system and finding eigenvalues of that matrix at this equilibrium point.
It is also appropriate here to recall that the co-existence equilibrium point is said to be stable if $tr(J)$ is negative while $Det(J)$ is positive. Here, $tr(J)$ and $Det(J)$ represent respectively trace and determinants of the matrix J. Further, an equilibrium point is said to be locally and asymptotically stable if the solution curves of the system those start near to this point essentially go towards this point for all forward times.
The Jacobian matrix $J\,(x, \ y)$for the system (5) – (6) takes the form

$$J(x, \ y) = \begin{bmatrix} 1-2x-\alpha y & -\alpha x \\ \beta y & \delta - 2y(\delta+\sigma)+\beta x \end{bmatrix} \qquad (7)$$

The second order square matrix (7), at co-existence equilibrium point$(x^*, \ y^*)$, reduces to the form as

$$J(x^*, \ y^*) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Here in the matrix $J(x^*, \ y^*)$ the four elements $a_1, a_2, \ a_3$and $a_4$are notations and they are used to represent the following expressions:

$$a_1 = \{[\alpha\delta - (\sigma+\delta)]/[\delta+\alpha\beta+\sigma]\}$$
$$a_2 = -\alpha\{[\delta(1-\alpha)+\sigma]/[\delta+\alpha\beta+\sigma]\}$$
$$a_3 = \beta[(\beta+\delta)/(\delta+\alpha\beta+\sigma)]$$
$$a_4 = \delta + (\beta+\delta)\{[\alpha\beta/[\sigma+\delta(1-\alpha)]^2] - \ [2/(\delta+\alpha\beta+\sigma)](1+\beta)\}$$

2.5.2 Conditions for Local stability of the co-existence equilibrium point

It has already been stated that the local stability of the co-existence equilibrium point $(x^*, \ y^*)$ requires two requirements.
The first requirement is that the eigenvalues of the Jacobian matrix must be negative. That is$Tr(J) = a_1 + a_4$ is a negative quantity. This requirement results in forcing a condition on the model parameters as

$$\delta + \beta\{\alpha(\beta+\delta)/[\sigma+\delta(1-\alpha)]^2\} + [(\alpha\delta - 3\delta - 3\sigma)/(\delta+\alpha\beta+\sigma)] < 0$$

The second requirement is that the determinant of the Jacobian matrix must be positive. That is $Det(J) = a1a4 - a2a3$is a positive quantity. This requirement results in forcing a condition on the model parameters: Both the expressions $\{2+\beta[2-\alpha(\beta+\delta)]\} - \alpha\beta$ and $\sigma + \delta - \alpha\delta$ are positive.
Moreover, the other two equilibria$(0, \ 0)$and $(1, \ 0)$are unstable for all permissible values of the parameters involved. However, the fourth equilibrium point $\{0, \ [\delta/(\delta+\sigma)]\}$is stable as long as the parameters satisfy the conditions $\delta < \sigma$ and $\alpha\beta\delta > [(\alpha+\delta)(\sigma-\delta)]$ but is unstable otherwise.

2.5.3 Global stability of the co-existence equilibrium point

The global stability of the co-existence equilibrium point $(x^*, \ y^*)$of the model equations (5) – (6) is stated in the form of a theorem and proved as follows:
Theorem–2 If the co-existence equilibrium point $(x^*, \ y^*)$is locally asymptotically stable in the positive xy-plane region then it will alsobe globally asymptotically stable in the same region.
Proof: Consider that $h(x, \ y) = (1/xy)$be a Dulac's positive function in the positive quadrant. Also let us define two other functions as $h_1(x, \ y) = x(1-x) - \alpha xy$ and $h_2(x, \ y) = \delta y(1-y) + \beta xy - \sigma y^2$.Then $\emptyset(x, \ y) = (\partial/\partial x)(h\,h_1) + (\partial/\partial y)(h\,h_2) = -[(\delta+\sigma)/x]$ and hence it is a negative function of its arguments.Here, $\emptyset(x, \ y)$does not change sign and is not identically zero in the positive quadrant of the xy – plane. Thus, by Bendixson-Dulac criterion the interior equilibrium point is globally asymptotically stable and moreover the system has no limit cycle in this region.

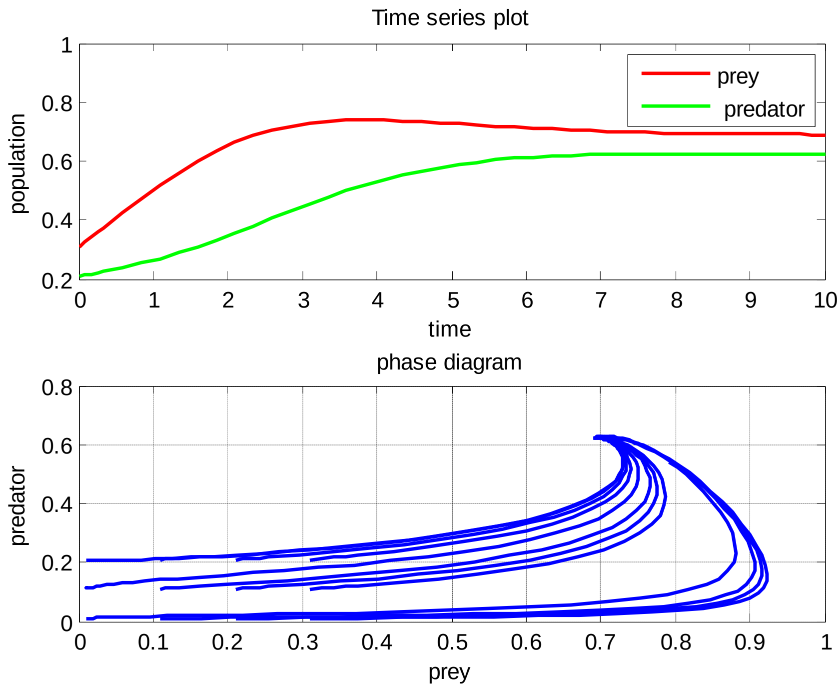

Figure 1: Time series plot and phase diagram of the model (5) – (6) for the parametric values$\alpha = 0.5,\ \beta = 0.7,\ \sigma = 0.9,\ \delta = 0.2.$

The time series plot in Figure 1 illustrates that the population sizes of prey and predator converge to theirrespective equilibrium values 0.689 and 0.620. Further, the phase portrait of this system with different initial values indicates that the solution curves of the model system go towards the interior equilibrium point $(0.689,\ \ 0.620)$. Hence, it is evident pictorially that this interior equilibrium point is globally asymptotically stable.

III. Persistence of the Model

The system of model equations(5) – (6) is said to be persistence if all its variables too persist i.e., all populations in the system are eventually bounded away from zero [5, 8].As it is already described, the following hold true:
(i) The initial value $x(0) = x_0 > 0$ leads to $\lim_{t\to\infty} x(t) \quad = 1$
(ii) The initially value $y(0) = y_0 > 0$ leads to $\lim_{t\to\infty} y(t) \quad = 1 + \beta > 0$
Therefore, both variables $x(t)$and $y(t)$of the model are bounded or persist and hence the system also persists.

IV. Perturbation Analysis

Having been scaled the model equations, their approximate solutions can be derived by systematically exploiting the sizes of dimensionless parameters. This procedure is well known as a perturbation theory. The two important methods those are very frequently used in solving the perturbed equations are (i) regular perturbation and (ii) singular perturbation methods. Singular perturbation method is used if the model exhibits different characters when the small parameter is set equal to zero as well as whenit is set different from zero [3].

However, in the present case regular perturbation method is applied because the small parameter δ does not appear together with the highest derivative in the model equations. Also it is possible to consider that $(1/\alpha_1) \ll (1/\beta_1)$and this relation reflects in the parameter δ to be a very small positive quantity.
Thus, the system has approximate solution inthe form of Taylor like expansion in terms of the small parameterδ,i.e., $\delta \ll 1$; as,

$$x(t) = X_0(t) + \delta X_1(t) + \delta^2 X_2(t) + \cdots$$
$$y(t) = Y_0(t) + \delta Y_1(t) + \delta^2 Y_2(t) + \cdots$$

In the series expansion $X_0(t)$ and $Y_0(t)$ are the leading order terms and the remainingterms $\delta X_1(t)$, $\delta^2 X_2(t), \ldots, etc.$and $\delta Y_1(t),\ \delta^2 Y_2(t), \ldots, etc.$are higher order correction terms those are small as expected.
Similarly, for the inner perturbation expansion we obtain the leading order equations, as

$$dX_0/dt \ = \ X_0(1 - X_0) - \alpha X_0 Y_0 \quad (8)$$
$$dY_0/dt \ = \ \beta X_0 Y_0 - \sigma\, Y_0^2 \ (9)$$

Theorem–3: All solutions $(X_0(\tau),\quad Y_0(\tau))$ of the system of model equations (8) – (9) with positive initial condition $(a_0,\ b_0)$ are bounded within the region $B = \{(X_0(\tau),\quad Y_0(\tau)): 0 \leq X_0(\tau) \leq 1, 0 \leq Y_0(\tau) \leq \beta/\sigma\}$.
Proof: As it is already stated above, a solution with non-negative initial value exists and is unique. Furthermore, it remains non-negative for all further times [9].
Now let us develop an appropriate argument for boundednessof X_0. From the first equation $dX_0/dt = X_0(1 - X_0) - \alpha X_0 Y_0$ of the model (8) – (9), it holds true that $dX_0/dt \leq X_0(1 - X_0)$. However, usingthe method of partial fractions, the analytic solution of the perfect equation $dX_0/dt = X_0(1 - X_0)$ can be computed as $X_0 = \{[a_0/(1 - a_0)]/\{e^{-t} + [a_0/(1 - a_0)]\}\}$ where $a_0 = X_0(0)$ is a positive quantity. The expression for X_0 gives that $\lim_{t\to\infty} X_0 = 1$ and hence the variable is bounded.
Just similar to the above, here let us develop an appropriate argument for boundedness of Y_0. From the second equation $dY_0/dt = \beta X_0 Y_0 - \sigma Y_0^2$ of the model (8) – (9) it holds true that $dY_0/dt \leq \beta X_0 Y_0 - \sigma Y_0^2 \leq \beta Y_0 - \sigma Y_0^2$, since upper bound of the variable X_0 is 1. Using the method of partial fractions, the solution for the perfect equation $dY_0/dt = \beta Y_0 - \sigma Y_0^2$ can be obtained as $Y_0(t) = \left[\beta/\left(\sigma + Ke^{-\beta t}\right)\right]$ where K is integral constant. The expression for Y_0 gives that $\lim_{t\to\infty} Y_0 = (\beta/\sigma)$ and hence the variable is bounded.
Having shown that both the variables X_0 and Y_0 are bounded above with the respective boundaries 1 and (β/σ) it can be concluded that all solutions of the model equations with any positive initial values selected from R_2^+ are bounded in the region B.

4.1 Qualitative analysis

It appears that it may notbe possible or at least quite difficult to obtain an analytic solutions for both the variables X_0 and Y_0. But, it is possible to solve for them numerically. Hence, the numerical solutions will be considered as an alternative to the correspondinganalytical ones.
The leading order equation has two axial equilibrium points $E_0(0,\quad 0)$ and $E_1(1,\quad 0)$ and one co-existence equilibrium point $E_2({X_0}^*, {Y_0}^*)$ where ${X_0}^* = [\sigma/(\sigma + \alpha\beta)]$ and ${Y_0}^* = [\beta/(\sigma + \alpha\beta)]$. All the three equilibrium points are physically valid and meaningful for all the permissible parametric values. The linearization technique gives us the Jacobian matrix $J({X_0}^*, {Y_0}^*)$ as

$$J({X_0}^*, {Y_0}^*) = \begin{bmatrix} 1 - 2{X_0}^* - \alpha {Y_0}^* & -\alpha {X_0}^* \\ \beta {Y_0}^* & \beta {X_0}^* - 2\sigma {Y_0}^* \end{bmatrix}$$

4.2 Local stability of the equilibrium points

Here it can be shown that (i) both the equilibrium points $E_0(0,\quad 0)$ and $E_1(1,\quad 0)$ are unstable nodes but (ii) the co-existence equilibrium point $E_2(x^*,\quad y^*)$ is stable provided that the inequality conditions on the parameters $0 < \left(\sigma/\sqrt{\alpha}\right) < \beta < 1$ are satisfied.
The Jacobian matrix $J({X_0}^*, {Y_0}^*)$ evaluated at the equilibrium point $E_0(0,\quad 0)$ takes the form as

$$J(0,\quad 0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues of this matrix $J\ (0,\quad 0)$ can found to be $\lambda_1 = 1$ and $\lambda_2 = 0$; which indicate a degenerate case. Also since the eigenvalues of $J(0,\quad 0)$ is equal to one and is a positive value, i.e., $TrJ\ (0,\quad 0) = 1 > 0$ the trivial equilibrium point $E_0(0,\quad 0)$ is an unstable saddle node.
The Jacobian matrix $J\ ({X_0}^*, {Y_0}^*)$ evaluated at the equilibrium point $E_1(1,\quad 0)$ takes the form as

$$J\ (1,\quad 0) = \begin{bmatrix} -1 & -\alpha \\ 0 & \beta \end{bmatrix}$$

The eigenvalues of this matrix $J\ (1,\quad 0)$ are found to be $\lambda_1 = -1$ and $\lambda_2 = \beta$; which are real but opposite in sign. Therefore, the axial equilibrium point $E_1(1,\quad 0)$ is unstable.
The Jacobian matrix $J\ ({X_0}^*, {Y_0}^*)$ evaluated at the equilibrium point $E_2(x^*,\quad y^*)$ takes the form as

$$J(X_0^*,\quad Y_0^*) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Here,notations for the matrix elements are used as follows:
$a = 1 - 2[\sigma/(\sigma + \alpha\beta)] - \alpha[\beta/(\sigma + \alpha\beta)]$
$b = -\alpha[\beta/(\sigma + \alpha\beta)]$

$$c = \beta\ [\beta/(\sigma + \alpha\beta)]$$

$$d = \beta\ [\sigma/(\sigma + \alpha\beta)] - 2\sigma[\beta/(\sigma + \alpha\beta)]$$

In order that the co-existence equilibrium point $E_2(x^*,\quad y^*)$ is stable, the two requirements those are to be satisfied are: Trace of the matrix $J(X_0^*,\quad Y_0^*)$ is negative while its determinant is positive. The former requirement $Tr(J) = a + d < 0$ leads to the condition $0 < \beta < 1$ while the latter $Det(J) = ad - bc > 0$ leads to $\alpha > (\sigma/\beta)^2$. Thus, both stability conditionsimply that the model parameters satisfy the relations as $0 < \left(\sigma/\sqrt{\alpha}\right) < \beta < 1$.
Theorem–4 If the co-existence equilibrium point $(X_0^*,\quad Y_0^*)$ of the model equations (8) – (9) is locally asymptotically stable in the positive X_0Y_0 −plane region then it is also globally asymptotically stable in the same region.

Proof: Let $F(X_0,\ Y_0) = (1/X_0Y_0) > 0$be a Dulac's function in the positive quadrant. Also define two other functions as $F_1(X_0,\ Y_0) = X_0(1 - X_0) - \alpha X_0Y_0$ and $F_2(X_0,\ Y_0) = \beta X_0Y_0 - \sigma\, Y_0^2$. Then, the function $\psi\,(X_0,\ Y_0)$ defined to be equal to $(\partial/\partial X_0)(F\,F_1) + (\partial/\partial Y_0)(F\,F_2)$ is simplified to have an expression $[-(1/Y_0) - (\sigma/X_0)]$ and is a negative quantity. Equivalently it can be expressed as $\psi\,(X_0,\ Y_0) = (\partial/\partial X_0)(F\,F_1) + (\partial/\partial Y_0)(F\,F_2) = [-(1/Y_0) - (\sigma/X_0)] < 0$. However, it can be observed that the function $\psi\,(X_0,\ Y_0)$does not change sign and is not identically zero in the positive quadrant of the X_0Y_0 – plane. Thus, according to Bendixson-Dulac criterion this interior equilibrium point $(X_0^*,\ Y_0^*)$is globally asymptotically stable and the system has no limit cycle in that region.

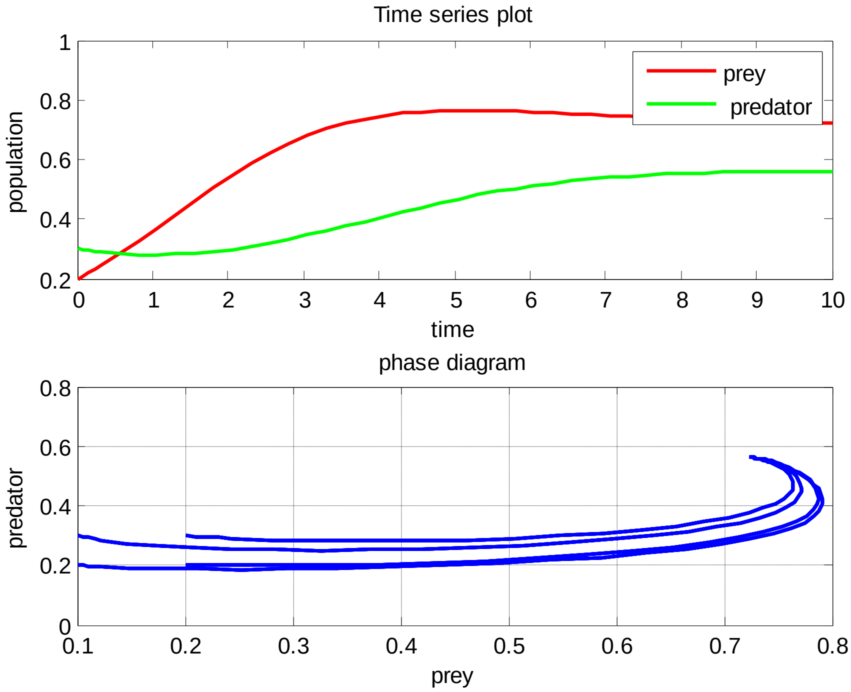

Figure 2: Numerical solution of leading order equation with $\alpha = 0.5, \beta = 0.7, \sigma = 0.9$.

The time series plot of the system (8) – (9) given in figure 2 illustrates that the population sizes converge to their finite equilibrium values. The predator population size is greater than the prey population size which is normally expected. The Phase portrait of the system corresponding to different initial values indicates that the solution curves of the leading order equation converge towards the interior equilibrium point$(0.721, 0.561)$. This observation supports that the interior equilibrium point is globally asymptotically stable.

V. Modeling with the inclusion of Harvesting

A further harvesting variable E where $0 < E < E_{max}$ is introduced in to the model which is called a fishing effort or Fishing mortality. If$E_{max} \geq 1$ then the stock would be driven to extinction [7]. Here, harvesting of either prey or the predator are considered and analyzed the effects.

5.1 Modeling with the inclusion of Prey Harvesting

Consider that the prey is Tilapia fish with population size S while the predator is Whale Fish with the size R. Both the populations are further assumed to grow following logistic function in the similar way as it is described in the model equations (3)- (4). A further variable E is introduced into the prey equation which is called fishing effort[4]. Assume that the catch of fish per unit effort is proportional to the availably amount of fish S. Thus, a generalist species predator prey model with prey harvesting and intra – specific competition among predators has of the form

$$(dS/dT) = \alpha_1\, S\,[1 - (S/K_1)] - \alpha_2 SR - qES \qquad (10)$$

$$(dR/dT) = \beta_1\, R\,[1 - (R/K_2)] + \alpha_3 SR - \mu\, R^2 \qquad (11)$$

Here, q is the proportionality constant known as 'catch ability' coefficient and it describes how easily the fish can be harvested. Then, the term qE corresponds to the mortality or reduction of prey population caused due to harvesting and has the same dimension as α_1.

The scaled versions of the equations (10) – (11) using the new scaled parameters $T = (1/\alpha_1)t,\ S = K_1\ x,\ R = K_2\ y$ take the form as

$$(dx/dt) = x(1-x) - \emptyset_1 xy - e_1\ x \qquad (12)$$
$$(dy/dt) = \emptyset_2\ y(1-y) - \emptyset_3 xy - \emptyset_4 y^2 \qquad (13)$$

It may be observe that the dimensionless parameters are represented by the notations $\emptyset_1 = (\alpha_2/\alpha_1)K_2,\ \ \emptyset_2 = (\beta_1/\alpha_1),\ \ \emptyset_3 = (\alpha_3/\alpha_1)K_1,\ \ \emptyset_4 = (\mu/\alpha_1)K_2ande_1 = (q/\alpha_1)E$. The boundedness of solution of this system is stated in the form of a theorem and proved as follows:

Theorem – 5: All solutions $(x,\ \ y)$ of the model equations (12) – (13) together with a positive initial condition (x_0, y_0) are bounded within the region$C = \{(x, y): 0 \le x\ (t) \le 1,\ \ \ 0 \le y\ (t) \le \phi_3 + 1\}$.

Proof:As it is already sated that a solution with non-negative initial values exists and is unique. Furthermore, the solution remains non-negative during entire evolution of time [9].

Now let us develop an appropriate boundedness argument for x:from the first equation $(dx/dt) = x(1-x) - \emptyset_1 xy - e_1 x$ of the model (12) – (13), it holds true that $dx/dt \le x\,(1-x)$. On applying the method of partial fractions, the analytical solution for the perfect equation $dx/dt \le x\,(1-x)$ can be obtained as $x(t) = \{[c_0/(1-c_0)]/\{e^{-t} + [c_0/(1-c_0)]\}\}$ where$c_0 = x(0) > 0$ is the integral constant. The foregoing expression gives that $\lim_{t\to\infty} x(t) = 1$ and hence the variable $x(t)$ is bounded.

Similarly, let us also develop an appropriate argument for boundedness of the variable y: from the second equation $(dy/dt) = \emptyset_2\ y(1-y) - \emptyset_3 xy - \emptyset_4 y^2$ of the model (12) – (13) it holds true without loss of generality that $dy/dt \le \emptyset_2\ y(1-y) - \emptyset_3 y$: Here we have discarded the last term and substituted the value $x = 1$ as it is the upper bound of the variable. Using the method of partial fractions the solution of the perfect equation can be obtained as $y(t) = \left\{[\emptyset_3 + 1]/\left\{c_1 e^{-(\emptyset_3+1)\emptyset_2 t} + 1\right\}\right\}$ where c_1 is an integral constant. The foregoing expression gives that $\lim_{t\to\infty} y(t) \le 1 + \emptyset_3$ for all $t \ge 0$ and hence the variable $y(t)$ is bounded.

Therefore, it can be concluded that all solutions $(x,\ \ y)$ of the model equations together with any positive initial value selected from R_2^+ are bounded in the region C.

5.2 Co-existence equilibrium point

Let the co-existence equilibrium point for the model (12) – (13) be represented by$(a^*,\ \ b^*)$. Here the notations used in the coordinates stand for the following expressions:

$$a^* = [\emptyset_2(1 - \emptyset_1 - e_1) + \emptyset_4(1 - e_1)]/[\emptyset_2 + \emptyset_1\emptyset_3 + \emptyset_4]$$
$$b^* = [\emptyset_2 + \emptyset_3(1 - e_1)]/[\emptyset_2 + \emptyset_1\emptyset_3 + \emptyset_4]$$

The co-existence point exists only when the inequality conditions on the parameters (i)$\emptyset_2 + \emptyset_4 > \emptyset_1\emptyset_2 + e_1\emptyset_2 + e_1\emptyset_4$and (ii)$\emptyset_2 + \emptyset_3 > e_1\emptyset_3$ hold true.

5.3 Maximum sustainable yield

The sustainable yield during the equilibrium state is denoted by $y(e_1)$andit can beexpressed following the procedure given in[1] as $y(e_1) = e_1 a^* = e_1\,[\emptyset_2(1 - \emptyset_1 - e_1) + \emptyset_4(1 - e_1)]/[\emptyset_2 + \emptyset_1\emptyset_3 + \emptyset_4]$. Here $(a^*,\ \ b^*)$ denotes locally and asymptotically stable interior equilibrium point.

The maximum sustainable yield of the population is achieved at$e_{1msy} = (1/2)[(\emptyset_2 + \emptyset_4 - \emptyset_1\emptyset_2)/(\emptyset_4 + \emptyset_2)]$ and it is resulted fromsolving$(dy/de_1) = 0$.It is also important to note the limits as $0 < e_{1msy} < (q/\alpha_1)$. The maximum sustainable yield msy is defined as

$$msy = -A_1\left(e_{1msy}\right)^2 + A_2 e_{1msy}\ \equiv f\left(e_{1msy}\right)$$

Here the notations stand for $A_1 = \emptyset_2 + \emptyset_4$ and$A_2 = \emptyset_2(1 - \emptyset_1) + \emptyset_4$. The function$f$ is an arbitrary function of its argument.It is possible to rewrite to the form

$$f\left(e_{1msy}\right) = \begin{cases} 0 & ,for\ \ e_{1msy} = 0\ \ or\ \ (A_2/A_1) \\ 2A_2^2/A_1 & ,for\ \ e_{1msy} = (A_2/2A_1) \end{cases}$$

It is straight forward to notice that the maximum sustainable yield is $4A_2\ e_{1msy}$.

Theorem – 6: If the co-existence equilibrium point $(a^*,\ \ b^*)$of (12) – (13) is locally asymptotically stable in the positive xy – plane region, then it is also globally asymptotically stable in the sameregion.

Proof: Consider Dulac's function in the positive quadrant as $G(x,\ \ y) = (1/xy) > 0$ and also two other functions defined as $G_1(x,\ \ y) = x(1-x) - \phi_1 xy - e_1 x$ and $G_2(x,\ \ y) = \phi_2 y(1-y) - \phi_3 xy - \phi_4 y^2$. Then, the function $\Omega(x,\ \ y)$ defined by $= (\partial/\partial x)(GG_1) + (\partial/\partial y)(GG_2)$ takes a negative expression as$[-(1/y) - (\emptyset_2/x) - (\emptyset_4/x)] < 0$. Hence, $\Omega(x,\ \ y)$ does not change its negative sign and is not identically zero in the positive quadrant of xy – plane. Thus, by Bendixson ‑ Dulac criterion the interior equilibrium point is globally asymptotically stable and the system has no limit cycle in that region.

Figure 3 illustrates that the population sizes converges to finite equilibrium values. The maximum yield is 0.157at$e_1 = 0.455$ and the sameis zero at$e_1 = 0.909$ when the values are adjusted to three significant digits.

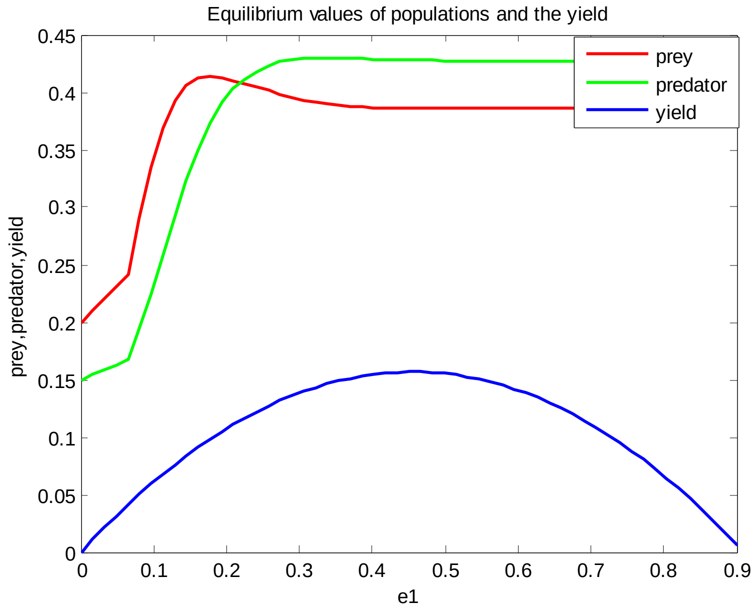

Figure 3: Populations and the yield as a function of effort$e_1$for $\phi_1 = 0.5,\ \phi_2 = 0.2,\ \phi_3 = 0.7,\ \phi_4 = 0.9$

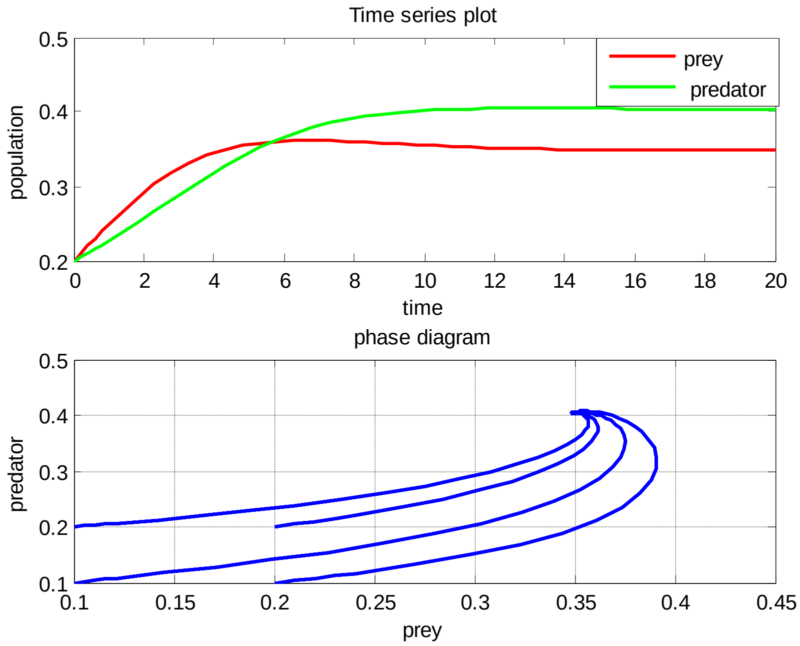

Figure 4: Time series plot and phase diagram of population biomass when the target population is prey for the parametric values $\phi_1 = 0.5,\ \phi_2 = 0.2,\ \phi_3 = 0.7,\ \phi_4 = 0.9\ ,\ e_1 = 0.45$.

The time series plot of the system (12) – (13) in Figure 4 indicates that population size converges to finite equilibrium value. The phase portrait of this system in this figure corresponding to different initial values indicates that the interior equilibrium point (0.348, 0.389) is globally asymptotically stable.

5.4 Modeling with the inclusion of predator harvesting

We also consider that the predator or Whale fish population grows naturally according to logistic function and we also include harvesting term. A further variable E is introduced and is called *fishing effort*. We assume that the catch of fish per unit effort is proportional to the available amount of Whale fish R. Thus, a generalist predator prey model together with intra specific competition and predator harvesting is

$$(dS/dT) = \alpha_1\, S\,[1 - (S/K_1)] - \alpha_2 SR \quad (14)$$

$$(dR/dT) = \beta_1 R [1 - (R/K_2)] + \alpha_3 SR - \mu R^2 - \eta E R \qquad (15)$$

Here η the proportionality constant known as *catch ability* coefficient that describes how easily the fish can be harvested. Then, the factor ηE corresponds to the fishing mortality or exclusion caused due to harvesting and the factor has the same dimension as β_1. The scaled form of the equations (14) – (15) using the transformations $T = (1/\alpha_1)\, t,\ S = K_1\, x$ and $R = K_2\, y$ reduces to

$$(dx/dt) = x(1 - x) - \varphi_1 xy \qquad (16)$$

$$(dy/dt) = \varphi_2\, y(1 - y) + \varphi_3 xy - \varphi_4 y^2 - e_2\, y. \qquad (17)$$

Here the notations used, to represent parametric expressions, are $\varphi_1 = (\alpha_2/\alpha_1)K_2,\ \varphi_2 = (\beta_1/\alpha_1),\ \varphi_3 = (\alpha_3/\alpha_1)K_1,\ \varphi_4 = (\mu/\alpha_1)K_2$ and $e_2 = (\eta/\alpha_1)E$.

Theorem–7: All solutions (x, y) of the model equations (16) – (17) together with any positive initial condition (x_0, y_0) are bounded with in a region $D = \{(x, y): 0 \le x\,(t) \le 1, 0 \le y\,(t) \le \varphi_3 + 1\}$.

Proof: As it is already described, a solution with non-negative initial value exists and is unique. Furthermore, the solution remains non-negative [9].

Boundedness argument for x: from the first equation(16) of the model it holds true that $dx/dt \le x\,(1 - x)$. Using partial fraction method, the equality solution can be obtained as $x(t) = \{[x_0/(1 - x_0)]/\{e^{-t} + [x_0/(1 - x_0)]\}\}$ where $x_0 = x(0) > 0$. Also, $\lim_{x\to\infty} x(t) = 1$. Hence, $dx/dt \le x\,(1 - x)$ implies that $x(t) \le 1,\ \forall t \ge 0$. Hence x is bounded.

Boundedness argument for y: from the second equation (17) of the model it holds true that $dy/dt \le \varphi_2\, y(1 - y) + \varphi_3 y$, where we have set $x(t) = 1$. Using themethod of partial fractions, the analytical solution of the perfect equation has been found to be of the form $y(t) = \{[\varphi_3 + 1]/\{ce^{-(\varphi_3+1)\varphi_2 t} + 1\}\}$, where c is any integral constant. Also, $\lim_{t\to\infty} y(t) = 1 + \varphi_3$. Hence, $dy/dt \le \varphi_2\, y(1 - y) + \varphi_3 y$ implies that $y \le 1 + \varphi_3\ \forall t \ge 0$. Hence y is bounded.

Therefore it can be concluded that all solutions of the model equations with any positive initial value taken from R_2^+ are bounded in the region D.

The co-existence equilibrium point of (16) – (17) is denoted by $(c^*,\ d^*)$ where the coordinates represent the expressions $c^* = 1 - \varphi_1 [\varphi_2 + \varphi_3 - e_2]/[\varphi_2 + \varphi_1\varphi_3 + \varphi_4]$ and $d^* = [\varphi_2 + \varphi_3 - e_2]/[\varphi_2 + \varphi_1\varphi_3 + \varphi_4]$. This equilibrium point exists if the two inequality conditions on the parameters (i)$\varphi_2 + \varphi_1 e_2 + \varphi_4 > \varphi_1\varphi_2$ and (ii)$\varphi_2 + \varphi_3 > e_2$ hold good.

Maximum sustainable yield

The Maximum sustainable yield or *msy* in the equilibrium situation is denoted by $y(e_2)$ and is defined as $y(e_2) = e_2 d^* = e_2 [\varphi_2 + \varphi_3 - e_2]/[\varphi_2 + \varphi_1\varphi_3 + \varphi_4]$ where $(c^*,\ d^*)$ is locally asymptotically stable interior equilibrium point of the model equations(16) – (17). The Maximum sustainable yield or *msy* is achieved at $e_2 = [(\varphi_2 + \varphi_3)/2]$ and this result follows from $dy/de_2 = 0$. Hence, the maximum sustainable yield is given by

$msy = e_{2msy}\, [\varphi_2 + \varphi_3 - e_{2msy}]/[\varphi_2 + \varphi_1\varphi_3 + \varphi_4] = B_1 e_{2msy} - B_2 (e_{2msy})^2 \equiv g(e_{2msy}$. Here $B_1 = (\varphi_2 + \varphi_3)(\varphi_2 + \varphi_3\varphi_1 + \varphi_4)^{-1}$, $B_2 = (\varphi_2 + \varphi_3\varphi_1 + \varphi_4)^{-1}$ and g is an arbitrary function of its argument. Possibly the function g can be expressed in the form as

$$g(e_{2msy}) = \begin{cases} 0 & , for\ e_{2msy} = 0\ or\ B_1/B_2 \\ {B_1}^2(2 - B_2)/4B_2 & , for\ e_{2msy} = B_1/2B_2 \end{cases}$$

It is also observed that the maximum sustainable yield occurs when $(B_1/2)(2 - B_2)e_{2msy}$ and $0 < B_2 < 2$.

Theorem–8 If the co-existence equilibrium pint $(c^*,\ d^*)$ of (16) –(17) is locally asymptotically stable in any region of the positive quadrant then it is also globally asymptotically stable in the same region.

Proof: Consider Dulac's function in the positive quadrant as $H(x,\ y) = (1/xy) > 0$ and also two other functions as $H_1(x, y) = x(1 - x) - \varphi_1 xy$ and $H_2(x,\ y) = \varphi_2 y(1 - y) - \varphi_3 xy - \varphi_4 y^2 - e_2 y$. Then, the function $X(x, y)$ defined by $(\partial/\partial x)(HH_1) + (\partial/\partial y)(HH_2)$ takes a negative value as $[-(1/y) - (\varphi_2/x) - \varphi 4x < 0$. That is, it always holds that $Xx,y < 0$. Thus, by Bendixson–Dulac criterion the interior equilibrium point is globally asymptotically stable and the system has no limit cycle in this region.

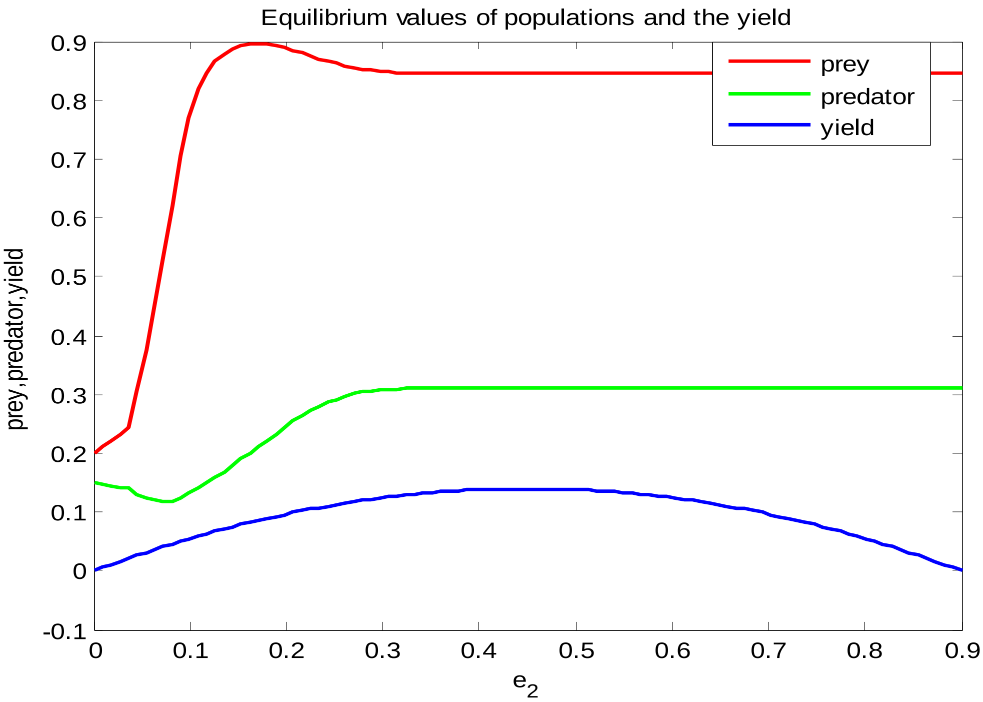

Figure 5 indicates that population converges to finite equilibrium values. The maximum yield is 0.139 at $e_2 = 0.450$ and it is zero at $e_2 = 0.909$ in 3significant digits. It has negative impact on the growth of the predator population while positive impact on the source prey

Figure 5: Equilibrium values of populations and the yield as a function of effort e_2 for the parametric values $\phi_1 = 0.5,\ \phi_2 = 0.2,\ \phi_3 = 0.7,\ \phi_4 = 0.9$.

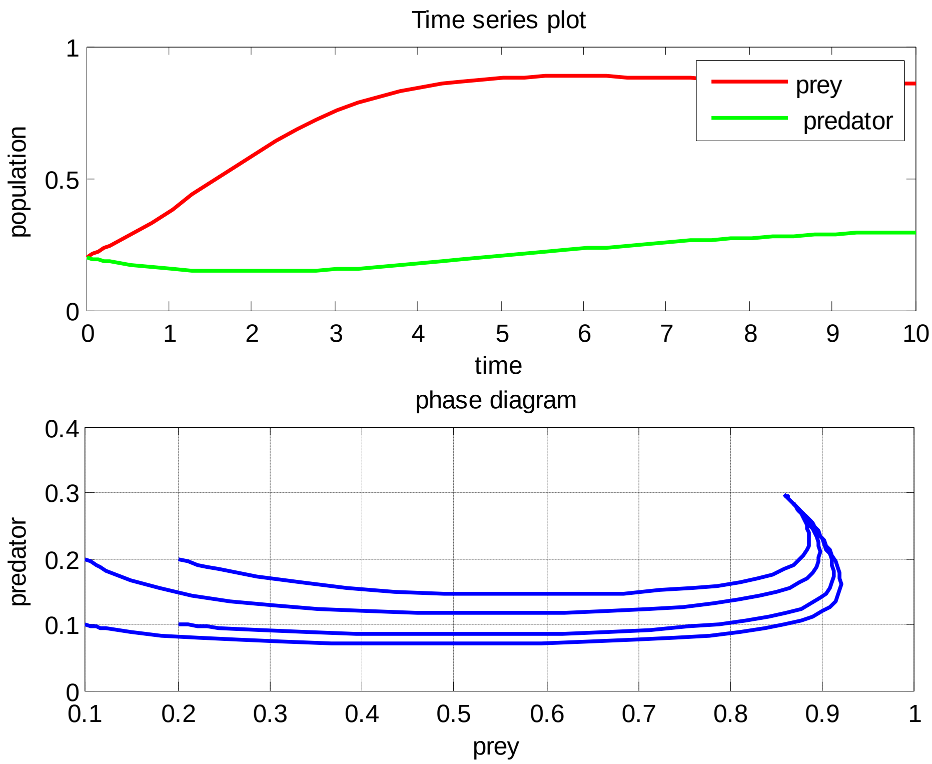

Figure 6: Time series plot and phase diagram of population biomass when the target population is prey where $\varphi_1 = 9.5,\ \varphi_3 = 6.5,\ \varphi_4 = 32,\ \varphi_2 = 0.2, e_2 = 0.5$

The time series plot of the system (14) – (15) in figure 6 indicates that population converges to finite equilibrium values. The phase portrait of this system in Figure 6 corresponding to different initial values indicates that the solution curves of the system go towards the interior equilibrium point (0.3862, 0.276). More over this equilibrium point is globally asymptotically stable.

VI. Lyapunov function and stability of interior equilibrium points

It is well known that the use of Lyapunov function is a powerful tool for determining global stability of an equilibrium point. The Lyapunov theory is used to draw conclusions about the nature of trajectories of a system of differential equations, especially non-linear, without finding the actual trajectories or solving the differential equations.

A system is said to be globally asymptotically stable if for every trajectory$\chi'(t) = F(\chi)$, we have $\chi(t) \to \chi_e$ as $t \to \infty$ where $\chi_e$is an equilibrium point of the system.

Theorem–9: Consider a positive definite function $W(x, \ \ y) = [x - x^* - x^* \ln(x/x^*)] + M[y - y^* -$ $y*$ln$yy*$ about the interior equilibrium point $x*,y*$ of the system (5) – (6) where M is some constant. The interior equilibrium point $(x^*, \ \ y^*)$ is globally asymptotically stable.

Proof: It can beshown that(dW/dt) is a negative definite function**.**Using the chain rule of total differentiation, the differential term (dW/dt) can be expressed in terms of partial derivatives as $[(\partial W/\partial x)(dx/dt) +$ $M\partial W\partial ydydt$ and reduces to the form $1{-}x{*}xdxdt{+}M1{-}y{*}ydydt$ after replacing the partial differential coefficients of W with the respective expressions.

Further, on using (5) – (6) it can be obtained that $dW/dt = -(x - x^*)^2 - \alpha(x - x^*)(y - y^*) - M\delta(y -$ $y{*}2{+}M\beta x{-}x{*}y{-}y{*}{-}M\sigma y{-}y{*}2.$ Up on setting the arbitrary constant as $M{=}\alpha\beta$, the foregoing expression issimplified to the form as $dW/dt = -(x - x^*)^2 - [(\alpha\delta/\beta) + (\alpha\sigma/\beta)](y - y^*)^2$ and is a negative expression. AsdW/dt is a negative definite function and thus it is a Lyapunov function. Therefore the co-existence equilibrium point $(x^*, \ \ y^*)$is globally asymptotically stable.

Theorem – 10: Consider a positive definite function $U(X_0, \ \ Y_0) \ = [X_0 - a^* - a^* \ln(X_0/a^*)] + N[Y_0 - b^* -$ $b*$ln$Y0b*$ about the interior equilibrium point $a*,b*$ of the system of equations(8) – (9) where N is some constant. The interior equilibrium point $(a^*, \ \ b^*)$ is globally asymptotically stable.

Theorem – 11: Consider a positive definite function $V(x, \ \ y) \ = [x - c^* - c^* \ln(x/c^*)] + Q[y - d^* -$ $d*$ln$yd*$ about the interior equilibrium point $c*,d*$ of the system of equations(12) – (13) where Q is some constant. The interior equilibrium point $(c^*, \ \ d^*)$ is globally asymptotically stable.

Theorem – 12: Consider a positive definite function $H(x, y) \ = [x - m^* - m^* \ln(x/m^*)] + Y[y - n^* -$ $n*$ln$yn*$about the interior equilibrium point $m*,n*$ of the system (16) – (17) where Y is some constant. The interior equilibrium point $(m^*, \ \ n^*)$ is globally asymptotically stable.

The Theorems 10, 11 and 12 also can be proved applying the same technique that has been followed in proving Theorem 9.

VII. Result and Discussion

It has been shown that, the locally asymptotically stable interior equilibrium points of the four models (5) – (6), (8) – (9), (12) – (13) and (16) – (17) are also globally asymptotically stable. This result is verified using both Bendixson – Dulac criterion and the Lyapunov function theory. The four positive definite functions constructed about interior equilibrium point of each system are all shown to be negative definite functions and hence they are Lyapunov functions.

Furthermore, no limit cycle is formed in the positive quadrant for any of the four models.

In the predator harvesting, the point at which maximum sustainable yield attain depends on the parameter φ_2 and φ_3 i.e. on the carrying capacity of the prey population and its intrinsic growth rate considering other parameters constant. Furthermore, the yield attains a maximum value when$e_{2msy} = [(\varphi_2 + \varphi_3)/2] < \eta/\alpha_1$.The maximum sustainable yield of predator harvesting is computed to be$(B_1/2)(2 - B_2)e_{2msy}$ andis valid for $0 < B_1 < 2$. On the other hand, the yield of prey harvesting attains maximum when $e_{1msy} = (1/2)[(\emptyset_2 + \emptyset_4 - \emptyset_1\emptyset_2)/(\emptyset_2 + \emptyset_4)] < q/\alpha_1$. The maximum sustainable yield of prey harvesting is $4(\emptyset_2(1 - \emptyset_1) + \emptyset_4)\, e_{1msy}$ and it is independent of the carrying capacity $K_1$of the prey population. In this study it is verified that Tilapia fish and Whale fishcan be harvested independently. In each case the population value converges to a finite positive value.

All simulations of the models show reasonable results relating to (i) global stability of the interior equilibrium points,(ii) The yield per unit effort of populationbiomass (iii) the impact of prey harvesting on the source prey and its predator, and also (iv) the impact of predator harvesting on the predator and its source.

References

[1]. Paritosh Bhattacharya, Maximum sustainable yield policy in prey predator system – A study, 2nd international conference on computing communication and sensor network (CCSN – 2013) proceeding published by International journal of computer application (IJCA).

[2]. Dawed, M.Y., Koya, P. R., and Goshu, A. T. "Mathematical modeling of population growth: The case of Logistic and Von Bertalanffy models", Open Journal of Modeling and Simulation, Vol. 2 (2014), 113-126. http://dx.doi.org/10.4236/ojmsi.2014.24013

[3]. Hrald E.Krogstad*, Mohammed Yiha Dawedand Tadele Tesfa Tegeghe, Alternative analysis of the Michaelis – Menten equations, Teaching Mathematics and its Applications Advance Access published June 23, 2011, Published by Oxford University Press on behalf of The Institute of Mathematics and its Applications.

[4]. Christina Kuttler, Mathematical Ecology, summer semester, 2010.

[5]. H. I. Freedman and S. T. R. Pinho. Persistence and extinction in a mathematical model of cell populations affected by radiation. Periodica Mathematica Hungarica Vol. 56 (1), 2008, Pp. 2535 DOI: 10.1007/s10998-008=5025-2

[6]. Temesgen Tibebu Mekonen. Bifurcation analysis on the dynamics of a generalist predator – prey system. International Journal of Ecosystem. 2012, 2 (3), 38 – 43. DOI: 10.5923/j.ije.20120203.02

[7]. Michael G. Neubert. Marine Reserve and optimal harvesting, 2003, Vol. 6, 843 – 849, doi:10.46/j.1461-0248,2003.00493.x, Blackwell Publishing Ltd. / CNRS

[8]. H.I.Freedman, PAUL WALTMAN, Persistence in Models of Threeinteracting Predator Prey Populations, Department of Mathematics of Iowa, Iowa City, Iowa 52242 and Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322,Elsevier Science Publishing Co.,Inc.,1984,52 Vanderbilt, New York, NY 10017

[9]. Xiuxiang Liu, Yijun Lou, "Global Dynamics of Predator Prey Model", J. Math. Anal. Appl. 371 (2010) 323–340.

[10]. Mohammed Yiha Dawed, Purnachandra Rao Koyaand Temesgen Tibebu Mekonen,Analysis ofPrey – Predator System with Prey Population Experiencing Critical Depensation Growth Function,Vol(3), Issue 6, Pages: 327-334.doi: 10.11648/j.ajam.20150306.23