IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 6 Ver. VI (Nov. - Dec.2016), PP 23-33
www.iosrjournals.org

$(\in,\in\vee q)$ -Q-Fuzzy Subgroups and Normal Subgroups

D. Hazarika[1], K. D. Choudhury[2]
[1]*Department of Mathematic, DHSK College. Dibrugarh,Assam. India.*
[2]*Department of Mathematics, Assam University,Sichar, Assam. India.*

Abstract: *In this paper, the notions of* $(\in,\in\vee q)$ *-Q-fuzzy subgroup and normal subgroup are introduced and some of their properties are investigated.*

Keywords: *Q-fuzzy subgroup and normal subgroup, Fuzzy point,* $(\in,\in\vee q)$ *-Q-fuzzy subgroup.*

I. Introduction

The idea of fuzzy subgroups was initiated by Rosenfield [1]. Goguen [9] generalised the notion of fuzzy subset of X to that of an L-fuzzy subset namely a function from X to a lattice L. Muthuraj *et al.* [10] introduced the notion of Q-fuzzy set. Solairaju and Nagarajan [3] brought in the concept of Q-fuzzy groups. Priya, Ramachandran and Nagalakshmi [7] extended this idea to Q-fuzzy normal subgroups. The concept of "belongs to' and ' quasi coincident with' between fuzzy point and fuzzy set was introduced by Bakhat and Das [6] with the study of $(\in,\in\vee q)$ -fuzzy subgroups and $(\in,\in\vee q)$-fuzzy subrings. Herein, $(\in,\in\vee q)$-Q-fuzzy subgroup and normal subgroup are defined and some results obtained.

II. Preliminaries

Definition2.1 : A mapping $\mu: G\times Q\to[0,1]$ where *G* is a group and *Q* a non empty set, is called a Q-fuzzy set in *G*. For any *Q*-fuzzy set μ in *G* and $t\in[0,1]$, the set $U(\mu,t)=\{\mu(x,q')\geq t, q'\in Q\}$ is called the upper cut of μ.

Definition2.2 : A *Q*-fuzzy set μ in a group G is called a *Q*-fuzzy subgroup if $\forall x,y\in G \text{ and } q'\in Q$, $\mu(xy^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q')\}$

Example2.3 : Let $(\mathbf{Z},+)$ be the additive group and Q denotes the set of integers. Define

$$\mu:\mathbf{Z}\times Q\to[0,1] \, with \, \mu(x,q')=\begin{cases}=0.7 \text{ if } x \text{ is even}\\ =0.3 \text{ if } x \text{ is odd}\end{cases}$$

for all *q´* in Q.

Then μ is a *Q-fuzzy* subgroup of $\mathbf{Z}$.

Solution : (i) Let *x, y* be even, The $x-y$ is even. Therefore $\mu(x-y,q)=0.7$ and

$\min\{\mu(x,y),\mu(y,q)\}=\min\{0.7.0.7\}=0.7$ so that $\mu(x-y,q)=\min\{\mu(x,q),\mu(y,q)\}$

(ii) Let *x ,y* be odd. Then *x-y* is even. . Therefore $\mu(x-y,q)=0.7$ and

$\min\{\mu(x,q),\mu(y,q)\}=\min\{0.3.0.3\}=0.3$ so that $\mu(x-y,q)>\min\{\mu(x,q),\mu(y,q)\}$

(iii) Let *x* be even (odd) and *y* odd (even). Therefore, *x-y* must be odd. Now

$\mu(x-y,q)=0.3 \text{ and } \min\{\mu(x,q),\mu(y,q)\}=\min\{0.7,0.3\}=0.3$ so that

$\mu(x-y,q')=\min\{\mu(x,q'),\mu(y,q')\}$. Thus, $\forall x,y\in(Z,+)$ and $q'\in Q$,

$\mu(x-y,q')\geq\min\{\mu(x,q'),\mu(y,q')\}$. Hence μ is a *Q-fuzzy* subgroup of Z.

Example2.4 : Let us take the multiplicative group G where $G=\{1,-1,i,-i\}$. We define $\mu:G\times Q\to[0,1]$,where Q denotes the set of real numbers, by setting $\mu(1,q')=0.8,\mu(-1,q')=0.5,\mu(i,q')=0.3=\mu(-i,q')$. Then μ is a *Q-fuzzy* subgroup of *G*.

Solution: Clearly,

$\mu\left(1(-1)^{-1},q'\right)=\mu(-1,q')=0.5,\min\{\mu(1,q'),\mu(-1,q')\}=0.5. So,\mu\left(1(-1)^{-1},q'\right)=\min\{\mu(1,q'),\mu(-1,q')\}$

$\mu\left(1(1^{-1}),q'\right)=\mu(1,q')=0.8\text{ and }\min\{\mu(1,q'),\mu(-1,q')\}=0.8. So,\mu\left(1(1^{-1}),q'\right)=\min\{\mu(1,q'),\mu(1,q')\}$

$\mu\left(1(i^{-1}),q'\right)=\mu(-i,q')=0.3\text{ and }\min\{\mu(1,q'),\mu(i,q')\}=0.3. So,\mu\left(1(i^{-1}),q'\right)=\min\{\mu(1,q'),\mu(i,q')\}$

$\mu\left(1(i^{-1}),q'\right)=\mu(i,q')=0.3\text{ and }\min\{\mu(1,q'),\mu(-i,q')\}=0.3. So,\mu\left(1(i^{-1}),q'\right)=\min\{\mu(1,q'),\mu(-i,q')\}$

$\mu\left((-1)(-1)^{-1},q'\right)=\mu(1,q')=0.8\text{ and }\min\{\mu(-1,q'),\mu(-1,q')\}=0.5. So,\mu\left((-1)(-1)^{-1},q'\right)\geq\min\{\mu(-1,q'),\mu(-1,q')\}$

$\mu\left((-1)i^{-1},q'\right)=\mu(i,q')=0.3\text{ and }\min\{\mu(-1,q'),\mu(i,q')\}=0.3. So,\mu\left((-1)i^{-1},q'\right)=\min\{\mu(-1,q'),\mu(i,q')\}$

$\mu\left((-1)(-i)^{-1},q'\right)=\mu(-i,q')=0.3\text{ and }\min\{\mu(-1,q'),\mu(-i,q')\}=0.3. So,\mu\left((-1)(-i)^{-1},q'\right)=\min\{\mu(-1,q'),\mu(-i,q')\}$

$\mu\left(i(i^{-1}),q'\right)=\mu(1,q')=0.8\text{ and }\min\{\mu(i,q'),\mu(i,q')\}=0.3. So,\mu\left(i(i^{-1}),q'\right)\geq\min\{\mu(i,q'),\mu(i,q')\}$

$\mu\left(i(-i)^{-1},q'\right)=\mu(-1,q')=0.5\text{ and }\min\{\mu(i,q'),\mu(-i,q')\}=0.3. So,\mu\left(i(-i)^{-1},q'\right)\geq\min\{\mu(i,q'),\mu(-i,q')\}$

$\mu\left((-i)(-i)^{-1},q'\right)=\mu(1,q')=0.8\text{ and }\min\{\mu(-i,q'),\mu(-i,q')\}=0.3. So,\mu\left((-i)(-i)^{-1},q'\right)\geq\min\{\mu(-i,q'),\mu(-i,q')\}$

$\text{Thus},\forall x,y\in G\text{ and }q'\in Q,\mu\left(xy^{-1},q'\right)\geq\min\{\mu(x,q'),\mu(y,q')\}$. Hence, μ is a Q-fuzzy subgroup of □ .

Definition2.5 : A *Q-fuzzy* set μ of a group *G* is called a *Q-fuzzy* normal subgroup of *G* if $\forall x,y\in G$ and $q'\in Q$.

$\mu\left(xyx^{-1},q'\right)\geq\min\{\mu(x,q'),\mu(y,q')\}$

Or equivalently $\mu(xy,q')=\mu(yx,q')$

Definition2.6: Let μ be a Q-fuzzy set in a group G. Let us define,

$${}_x f:G\times Q\to G\times Q$$
$${}_x f(a,q')=(xa,q')$$

A Q-fuzzy left coset ${}_x\mu$ is defined as ${}_x\mu={}_x f(\mu)$. Likewise, the Q-fuzzy right coset is defined as $\mu_x=f_x(\mu)$.

It can be readily seen that, ${}_x\mu(y,q')=\mu(x^{-1}y,q')$ and $\mu_x(y,q')=\mu(yx^{-1},q')\forall(y,q')\in G\times Q.$

Definition2.7 : Let *G* be a group and *Q* a nonempty set. A *Q-fuzzy* point $(x,q')_t$ is a function defined as

$$(x,q')_t:G\times Q\to[0,1]\text{ where}(x,q')_t(y,q')=\begin{cases}t & \text{if }(x,q')=(y,q')\\ 0 & \text{if }(x,q')\neq(y,q')\end{cases}$$

A *Q-fuzzy* point $(x,q')_t$ is said to belong to *Q-fuzzy* set μ i.e. $(x,q')_t \in \mu$ if $\mu(x,q') \ge t$ and a *Q-fuzzy* point $(x,q')_t$ is said to be quasi coincident with a *Q-fuzzy* set μ written as $(x,q')_t\, q\mu$ if $\mu(x,q')+t>1$. If $(x,q')_t \in \mu, or (x,q')_t\, q\mu$, we write $(x,q')_t \in\vee q\mu$.

III. $(\in,\in\vee q)$-Q-fuzzy subgroup

Definition3.1 : Let G be a group. A *Q-fuzzy* subset $\mu: G\times Q \to [0,1]$ is called $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G if $(x,q')_t \in \mu, (y,q')_t \in \mu \Rightarrow (xy^{-1},q')_{m(t,s)} \in\vee q\mu$ where $m(t,s)=\min\{t,s\}$.

Theorem3.2: Intersection of two $(\in,\in\vee q)$ subgroups of -Q- fuzzy a group G, is again a $(\in,\in\vee q)$-Q-fuzzy subgroup of G.

Proof: Let $\mu and\ \nu$ be two $(\in,\in\vee q)$-Q-fuzzy subgroups of a group G.

Let, $(x,q')_{t,} \in \mu\cap\nu, (y,q')_s \in (\mu\cap\nu) where, t,s \in [0,1]$

$$So, (x,q')_t \in \mu \wedge (x,q')_t \in \nu, (y,q')_s \in \mu \wedge (y,q')_s \in \nu$$
$$\Rightarrow (x,q')_t, (y,q)_s \in \mu \wedge (x,q')_t, (y,q')_s \in \nu$$
$$\Rightarrow (xy^{-1},q')_{m(t,s)} \in\vee q\mu \wedge (xy^{-1},q')_{m(t,s)} \in\vee q\nu$$
$$\Rightarrow \{(xy^{-1},q')_{m(t,s)} \in \mu \vee (xy^{-1},q')q\mu\} \wedge \{(xy^{-1},q')_{m(t,s)} \in \nu \vee (xy^{-1},q')q\nu\}$$
$$\Rightarrow \{(xy^{-1},q') \in \mu \wedge (xy^{-1},q') \in \nu\} \vee \{(xy^{-1},q')q\mu \wedge (xy^{-1},q')q\nu\}$$
$$\Rightarrow (xy^{-1},q') \in (\mu\cap\nu) \vee (xy^{-1},q')q(\mu\cap\nu)$$
$$\Rightarrow (xy^{-1},q) \in\vee q(\mu\cap\nu)$$

Hence the proof.

Remark3.3: i) The result can be extended to a family of $(\in,\in\vee q)$-Q-fuzzy subgroups.

ii) However, the union of two $(\in,\in\vee q)$-Q-fuzzy subgroups of a group G is not necessarily a $(\in,\in\vee q)$-Q-fuzzy subgroups of G.

Theorem3.4: A *Q-fuzzy* subset μ in a group *G* is a *Q-fuzzy* subgroup of *G* if and only if μ is a $(\in,\in)$ *-Q-fuzzy* subgroup of *G*.

Proof: Let μ be a *Q-fuzzy* subgroup of *G*. Let *x, y*$\in G$ such that $(x,q')_t \in \mu, (y,q')_t \in \mu$ where $t,s \in [0.1]$. Then $\mu(x,q') \ge t, \mu(y,q') \ge s$.

Now $\mu(xy^{-1},q') \ge \min\{\mu(x,q'),\mu(y,q')\} \ge \min\{t,s\} = m(t,s) \Rightarrow (xy^{-1},q')_{m(t,s)} \in \mu \Rightarrow \mu$ is a $(\in,\in)$ *-Q-fuzzy* subgroup of G.

Conversely, let μ be a $(\in,\in)$-Q-fuzzy subgroup of G. Let *x,y*$\in G$. Let $\mu(x,q')=t, \mu(y,q')=s$ where $t,s \in [0.1]$.Then,

$\mu(x,q') \ge t, \mu(y,q') \ge s \Rightarrow (x,q')_t \in \mu, (y,q')_t \in \mu$, where μ is a $(\in,\in)$ *-Q-fuzzy* subgroup of G.
So, $(xy^{-1},q')_{m(t,s)} \in \mu(xy^{-1},q') \ge m\{t,s\} = \min\{\mu(x,q'),\mu(y,q')\} \Rightarrow \mu$ is a *Q-fuzzy* subgroup of *G*.

Remark3.5: If μ is $(\in,\in)$-Q-fuzzy subgroup of *G* then it is also a $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of *G*.

Theorem3.6: If μ is a (q,q)-Q-fuzzy subgroup of *G* then μ is also $(\in,\in)$-*Q-fuzzy* subgroup of *G*.

Proof: Let μ be a (q,q)-Q-fuzzy subgroup of *G*. Let *x, y*$\in G$ such that $(x,q')_t \in \mu, (y,q')_t \in \mu$ where $t,s\in[0.1]$.Then, $\mu(x,q')\geq t, \mu(y,q')\geq s \Rightarrow (x,q')+\delta > t, \mu(y,q')+\delta > s,$ for any $\delta > 0$ $\Rightarrow \mu(x,q')+1-t+\delta>1, \quad \mu(y,q')+1-s+\delta>1, \Rightarrow (x,q')_{(1-t+\delta)}\, q\mu, (y,q')_{(1-s+\delta)}\, q\mu$ .But μ is a (q,q) -*Q-fuzzy* subgroup of G. So,

$$(xy^{-1},q')_{m(1-t+\delta,1-s+\delta)}\, q\mu \Rightarrow \mu(xy^{-1},q')+m(1+\delta-t,1+\delta-s)>1$$

$$\Rightarrow \mu(xy^{-1},q')+1+\delta-M(t,s)>1 \Rightarrow \mu(xy^{-1},q')>M(t,s)-\delta \geq m(t,s)$$

where δ is arbitrary.

$\Rightarrow (xy^{-1},q')_{m(t,s)} \in \mu \Rightarrow \mu$ is a $(\in,\in)$ -*Q-fuzzy* subgroup of G.

Theorem3.7: A Q-fuzzy subgroup μ in G is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of G if and only if $\mu(xy^{-1},q')\geq \min\{\mu(x,q'),\mu(y,q'),0.5\}\,\forall x,y\in G$

Proof : Let μ be a $(\in,\in\vee q)$ -Q- fuzzy subgroup of G.

Case1: Let $\min\{\mu(x,q'),\mu(y,q')\}<0.5$.

Then, $\min\{\mu(x,q'),\mu(y,q'),0.5\}=\min\{\mu(x,q'),\mu(y,q')\}$. If possible, let $\mu(xy^{-1},q')<\min\{\mu(x,q'),\mu(y,q')\}$. Let us choose a real number *t* such that

$\mu(xy^{-1},q')<t<\min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow \mu(x,q')>t, \mu(y,q')>t \Rightarrow (x,q')_t \in \mu, (y,q')_t \in \mu$. But $\mu(xy^{-1},q')<t \Rightarrow (xy^{-1},q')_t \notin \mu$ and $\mu(xy^{-1},q')+t<2t<2\min\{\mu(x,q'),\mu(y,q')\}<1$, a contradiction to the fact that μ is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of G. Thus we must have $\mu(xy^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q')\}=\min\{\mu(x,q'),\mu(y,q'),0.5\}\ \forall x,y\in G.$

Case –II : Let $\min\{\mu(x,q'),\mu(y,q')\}\geq 0.5\,\forall x,y\in G$. Then $\min\{\mu(x,q'),\mu(y,q'),0.5\}=0.5$ If possible, let $\mu(xy^{-1},q')<\min\{\mu(x,q'),\mu(y,q'),0.5\}=0.5$. Therefore $\mu(x,q')\geq 0.5$ and $\mu(y,q')\geq 0.5 \Rightarrow (x,q')_{0.5}\in\mu$, and $(y,q)_{0.5}\in\mu$, But $\mu(xy^{-1},q')<0.5 \Rightarrow (xy^{-1},q')_{0.5}\notin\mu$ and so $\mu(xy^{-1},q')+0.5<0.5+0.5=1$, a contradiction to the fact that μ is a $(\in,\in\vee q)$ -*Q fuzzy* subgroup of G.

Hence we have, $\mu(xy^{-1},q')\geq 0.5=\min\{\mu(x,q'),\mu(y,q'),0.5\}$.

Conversely, let $\mu(xy^{-1},q')\geq\min\{\mu(x,q'),q(y,q'),0.5\}$

Let $\forall x,y\in G$ such that $(x,q')_t\in\mu$ and $(y,q')_s\in\mu$ where $t,s\in[0,1]$. Then $\mu(x,q')\geq t$ and $\mu(y,q')\geq s \Rightarrow \min\{\mu(x,q'),\mu(y,q')\}\geq m(t,s)$. But

$\mu(xy^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q'),0.5\}\geq m(t,s,0.5)$

If $m(t,s)\leq 0.5$ then $m(t,s,0.5)=m(t,s)$.*So*, $\mu(xy^{-1},q')\geq m(t,s)\Rightarrow (xy^{-1},q')_{m(t,s)}\in\mu$.

If $m(t,s)>0.5$ then $m(t,s,0.5)=0.5$ .So, $\mu(xy^{-1},q')\geq 0.5 \Rightarrow \mu(xy^{-1},q')+m(t,s)\geq$ $0.5+m(t,s)>1$.

$\Rightarrow (xy^{-1},q')_{m(t,s)}\, q\mu . So (xy^{-1},q')_{m(t,s)} \in\vee q\mu$. Therefore, μ is a $(\in,\in\vee q)-$ *Q fuzzy* subgroup of G.

Theorem3.8 : If the *Q-fuzzy* subgroup μ of G is a $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G and $\mu(xy^{-1},q')<0.5$ $\forall x\in G$, then μ is also a $(\in,\in)$ *-Q-fuzzy* subgroup of G.

Proof: Since, μ is $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G, $\forall x,y\in G$,

$\mu(xy^{-1},q')\geq \min\{\mu(x,q'),\mu(y,q')\} \Rightarrow 0.5>\mu(xy^{-1},q')\geq \min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow \mu(x,q')<0.5$ and $\mu(y,q')<0.5$. Now let $(x,q')_t,(y,q')_s\in\mu$ where $t,s\in[0,1]$.

Then $\mu(x,q')\geq t, \mu(y,q')\geq s$ *i.e.* $t<0.5, s<0.5 \Rightarrow m(t,s)<0.5$.

$\because \mu$ is a $(\in,\in\vee q)$ *Q-fuzzy* subgroup of G, $(x,q')_t\in\mu,(y,q')_s\in\mu$

$\Rightarrow \mu(xy^{-1},q')\geq m(t,s)$ *or* $\mu(xy^{-1},q')+m(t,s)>1$. Since, $m(t,s)<0.5$ we must have in both situations, $\mu(xy^{-1},q')\geq m(t,s)$. Therefore μ is also a $(\in,\in)$ *-Q-fuzzy* subgroup of *G*.

Theorem 3.9: Let μ be $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G and $g\in G$ .Then, ${}_g\mu_{g^{-1}}$ is also a $(\in,\in\vee q)$ -Q-fuzzy subgroup of G.

Proof: Let $(x,q')_t\in {}_g\mu_{g^{-1}}, (y,q')_s\in {}_g\mu_{g^{-1}}$ where $t,s\in[0,1]$. Then,

$({}_g\mu_{g^{-1}})(x,q')\geq t, ({}_g\mu_{g^{-1}})(y,q')\geq s \Rightarrow \mu(g^{-1}xg,q')\geq t, \mu(g^{-1}yg,q')\geq s \Rightarrow (g^{-1}xg,q')_t\in\mu,(g^{-1}yg,q')\in\mu$

Since, μ is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of G,

$\mu((g^{-1}xg)(g^{-1}yg)^{-1},q')\geq m(t,s)\vee \mu((g^{-1}xg)(g^{-1}yg)^{-1},q')+m(t,s)>1......(A)$

Now,

$\mu((g^{-1}xg)(g^{-1}yg)^{-1},q')=\mu((g^{-1}xg)((yg)^{-1}g),q')=\mu((g^{-1}xg)(g^{-1}y^{-1}g),q')=\mu(g^{-1}(xgg^{-1}y^{-1})g,q')$

$=\mu(g^{-1}(xy^{-1})g,q')=({}_g\mu_{g^{-1}})(xy^{-1},q')$

Therefore from (A),

$({}_g\mu_{g^{-1}})(xy^{-1},q')\geq m(t,s)\vee({}_g\mu_{g^{-1}})(xy^{-1},q')+m(t,s)>1$

Therefore, ${}_g\mu_{g^{-1}}$ is also a $(\in,\in\vee q)$ -Q-fuzzy subgroup of G.

IV. Homomorphism of $(\in,\in\vee q)$ *-Q-fuzzy* subgroup

Theorem4.1 : Let of be a homomorphism. If μ' be a $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of $f: G\times Q\rightarrow G'\times Q$ then $f^{-1}(\mu')$ is a $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G.

Proof : We recall that $f^{-1}(\mu')$ as defined as $(f^{-1}(\mu'))(x,q')=\mu'(f(x),q')$ $\forall(x,q)\in G\times Q$ where μ' is a $(\in,\in\vee q)$ *-Q-fuzzy* subgroup of G' .Let $x,y\in G$

Then, $(x,q')_t,(y,q')_s \in f^{-1}(\mu')\forall t,s\in[0.1]$ implies

$$(f^{-1}(\mu'))(x,q')\geq t,(f^{-1}(\mu'))(y,q')\geq s$$
$$\Rightarrow \mu'(f(x),q')\geq t,\mu'(f(y),q')\geq s$$
$$\Rightarrow (f(x),q')_t \in \mu',(f(y),q')_s \in \mu'$$
$$\Rightarrow \left(f(x)(f(y))^{-1},q'\right)_{m(t,s)} \in \mu' \ or \ \left(f(x)(f(y))^{-1},q'\right)_{m(t,s)} q\mu'$$

(Since μ' of a $(\in\in\vee q)$ *-Q-fuzzy* of subgroup of G)

$$\Rightarrow \mu'\left(f(x)(f(y))^{-1},q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)(f(y))^{-1},q'\right)+m(t,s)>1$$
$$\Rightarrow \mu'\left(f(x)f(y^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)f(y^{-1}),q'\right)+m(t,s)>1$$
$$\Rightarrow \mu'\left(f(xy)^{-1},q'\right)\geq m(t,s) \ or \ \mu'\left(f(xy)^{-1},q'\right)+m(t,s)>1$$
$$\Rightarrow \left(f^{-1}(\mu')\right)(xy^{-1},q')\geq m(t,s) \ or \ \left(f^{-1}(\mu')\right)(xy^{-1},q')+m(t,s)>1$$
$$\Rightarrow (xy^{-1},q')_{m(t,s)} \in f^{-1}(\mu') \ or (xy^{-1},q')_{m(t,s)} qf^{-1}(\mu')$$

$\Rightarrow f^{-1}(\mu')$ is a $(\in,\in\vee q)$ -Q-fuzzy of subgroup of G

Theorem4.2 : Let $f:G\times Q\to G'\times Q$ be an epimorphism, where G and G´ are two groups, and Q, a non-empty set. If $f^{-1}(\mu')$ is a $(\in,\in\vee q)$ *-Q-fuzzy* of subgroup of G where μ' is a *Q-fuzzy* subgroup of G',then μ' is also a $(\in,\in\vee q)$ *-Q-fuzzy* of subgroup of G'.

Proof : Let $u,v\in G' s.t (u,q')_t,(v,q')\in\mu'$ where $t,s\in[0.1]$. Now a f being onto $\exists x,y\in G, s.t. f(x)=u, f(y)=v$. Since μ' is a *Q-fuzzy* subset of G,

$$\mu'(u,q')\geq t \ , \ \mu'(v,q')\geq s$$
$$\Rightarrow \mu'(f(x),q')\geq t \ , \ \mu(f(y),q')\geq s$$
$$\Rightarrow \left(f^{-1}(\mu')\right)(x,q')\geq t,\left(f^{-1}(\mu')\right)(y,q')\geq s$$
$$\Rightarrow (x,q')_t \in f^{-1}(\mu') \ of \ (y,q')_s \in f^{-1}(\mu')$$

where $f^{-1}(\mu^{-1})$ is a $(\in,\in\vee q)$ *-Q-fuzzy* of subgroup of *G*.

$$\therefore (xy^{-1},q')_{m(t,s)} \in f^{-1}(\mu') \ or (xy^{-1},q')_{m(t,s)} qf^{-1}(\mu')$$
$$\Rightarrow \left(f^{-1}(\mu')\right)(xy^{-1},q')\geq m(t,s) \ or \ \left(f^{-1}(\mu')\right)(xy^{-1},q')+m(t,s)>1$$
$$\Rightarrow \mu'\left(f(xy^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(xy^{-1}),q'\right)+m(t,s)>1$$
$$\Rightarrow \mu'\left(f(x)f(y^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)f(y^{-1}),q'\right)+m(t,s)>1$$
$$\Rightarrow \mu'\left(f(x)(f(y))^{-1},q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)(f(y))^{-1},q'\right)+m(t,s)>1$$
$$\Rightarrow \mu'(uv^{-1},q')\geq m(t,s) \ or \ \mu'(uv^{-1},q')+m(t,s)>1$$
$$\Rightarrow \mu'(uv^{-1},q')_{m(t,s)} \in \mu' \ or \ \mu'(uv^{-1},q')_{m(t,s)} q\mu'$$

$\Rightarrow \mu'$ is a $(\in,\in\vee q)$ *-Q-fuzzy* of subgroup of G'.

5.0 $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup

Definition5.1 : A *Q-fuzzy* set $\mu: G\times Q\to[0.1]$ where *G* is a group and *Q* a non-empty set is called $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G if $(x,q')_t\in\mu,(y,q')_s\in\mu\Rightarrow(xyx^{-1},q')_{m(t.s)}\in\vee q\mu$ where $m(t.s)=\min\{t,s\}$.

Theorem5.2: The intersection of two $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroups of G is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G.

Proof: Similar to that of Theorem 3.2.

Remark 5.3:i) The result can be extended to a family of $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroups of G.

ii) However, the union of two $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroups of G is not necessarily a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G.

Theorem5.4 : A *Q-fuzzy* subset μ in a group *G* is a *Q-fuzzy* normal subgroup of *G* if and only if μ is a $(\in,\in)$ *-Q-fuzzy* normal subgroup of *G*.

Proof: Let μ be a *Q-fuzzy* normal subgroup of *G*. Let $x,y\in G,\ q'\in Q, s.t.(x,q')_t\in\mu,(y,q')_s\in\mu$, where $t,s\in[0.1]$.Then, $\mu(x,q')\geq t$ and $\mu(y,q')\geq s.$

Now, $\mu(xyx^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q')\}\geq\min\{t,s\}=m(t,s)$

$\Rightarrow(xyx^{-1},q')_{m(t,s)}\in\mu\Rightarrow\mu$ is a $(\in,\in)$ *-Q-fuzzy* normal subgroup of G.

Conversely, let μ be a $(\in,\in)$*-Q-fuzzy* normal subgroup of *G*. Let $x,y\in G,\ q'\in Q, s.t.\mu(x,q')=t$ $and\,\mu(y,q')=s$ where $t,s\in[0.1]$.

Then $\mu(x,q')\geq t$ and $\mu(y,q')\geq s.$

$\Rightarrow(x,q')_t\in\mu$ and $(x,q')_s\in\mu.$ Since μ is a $(\in,\in)$*-Q-fuzzy* normal subgroup of *G*, we have $(xyx^{-1},q')_{m(t,s)}\in\mu$

$\Rightarrow\mu(xyx^{-1},q')\geq m\{t,s\}=\min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow\mu$ is a *Q-fuzzy* normal subgroup of *G* .

Theorem5.5 : If μ is a (q,q) - *Q-fuzzy* normal subgroup of *G* then μ is a $(\in,\in)$ *-Q-fuzzy* normal subgroup of *G*.

Proof : Let μ be a (q,q) - *Q-fuzzy* normal subgroup of *G*.

Let $x,y\in G$ and $q\in Q, s.t.(x,q')_t\in(y,q')_s\in\mu$ where $t,s\in[0.1]$.

Then, $\mu(x,q')\geq t$, $\mu(y,q')\geq s$.

$\Rightarrow\mu(x,q')\delta>t,\mu(y,q')+\delta>s$ for any $\delta>0$

$\Rightarrow\mu(x,q')+1-t+\delta>1,\mu(y,q')+1-s+\delta>1$

$\Rightarrow(x,q')_{(1-t+\delta)}q\mu,(y,q')_{(1-t+\delta)}q\mu$

Since μ is a (q,q)- *Q-fuzzy* normal subgroup of *G*, $(xyx^{-1},q')_{m(1-t+\delta,1-s+\delta)}\tilde{q}\mu$ where $m(1-t+\delta,1-s+\delta)=\min\{1-t+\delta,1-s+\delta\}$

$\Rightarrow \mu(xyx^{-1},q')+m(1-t+\delta,1-s+\delta)>1$

$\Rightarrow \mu(xyx^{-1},q')+1+\delta-M(t,s)>1$ where $M(t,s)=\max\{t,s\}$

$\Rightarrow \mu(xyx^{-1},q')>M(t,s)-\delta$

$\Rightarrow \mu(xyx^{-1},q')\geq M(t,s)$ as δ is arbitrary.

$\Rightarrow \mu(xyx^{-1},q')\geq m(t,s)$ as $M(t,s)\geq \mathrm{m}(t,s)$

$\Rightarrow \mu(xyx^{-1},q')_{m(t,s)}\in\mu$

$\Rightarrow \mu$ is a $(\in,\in)$-*Q-fuzzy* normal subgroup of G .

Theorem5.6 : A *Q-fuzzy* normal subgroup μ of G is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of G if and only if $\mu(xyx^{-1},q')\geq \min\{\mu(x,q'),\mu(y,q'),0.5\}\ \forall x,y\in G.$

Proof : Let μ be a $(\in,\in\vee q)$ -*Q-fuzzy* normal (QFN) subgroup of G.

Case I : Let, $\min\{\mu(x,q'),\mu(y,q')\}<0.5\ \forall x,y\in G$.

Then, $\min\{\mu(x,q'),\mu(y,q'),0.5\}=\min\{\mu(x,q'),\mu(y,q')\}$

If possible, let $\mu(xyx^{-1},q')<\min\{\mu(x,q'),\mu(y,q')\}$. Then, $\exists$ a real number *t* such that,

$\mu(xyx^{-1},q')<t<\min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow \mu(x,q')>t,\mu(y,q')>t$

$\Rightarrow (x,q')_t\in\mu,(y,q')_s\in\mu$

Now $\mu(xyx^{-1},q')<t$

$\Rightarrow (xyx^{-1},q')_t\in\mu$

and $\mu(xyx^{-1},q')+t<2t<2\min\{\mu(x,q'),\mu(y,q')\}<2\times 0.5=1$,a contradiction to the fact that μ is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of G. Hence, we must have,

$\mu(xyx^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q')\}=\min\{\mu(x,q'),\mu(y,q'),0.5\}$

Case II : Let $\min\{\mu(x,q'),\mu(y,q')\}\geq 0.5\ \forall x,y\in G$.

Then, $\min\{\mu(x,q'),\mu(y,q'),0.5\}=0.5$

If possible, let $\mu(xyx^{-1},q')<\min\{\mu(x,q'),\mu(y,q'),0.5\}=0.5$

$\therefore \mu(x,q')\geq 0.5$, $\mu(y,q')\geq 0.5$

$\Rightarrow (x,q')_{0.5}\in(y,q')_{0.5}\in\mu$

Now $\mu(xyx^{-1},q')<0.5$

$\Rightarrow (xyx^{-1},q')_{0.5}\notin\mu$

and $\mu(xyx^{-1},q)+0.5<0.5+0.5=1$, a contradiction to the fact that μ is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of *G*. Hence, we must have,

$\mu(xyx^{-1},q')\geq 0.5=\min\{\mu(x,q'),\mu(y,q'),0.5\}$

Conversely, let, $\mu(xyx^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q'),0.5\}$

Let $\mu(x,q')_t\in\mu,(y,q')_s\in\mu$ where $t,s\in[0.1]$.

Then, $\mu(x,q')\geq t,\mu(y,q')\geq s$

$\Rightarrow\min\{\mu(x,q'),\mu(y,q')\}\geq m(t,s)$

But $\mu(xyx^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q'),0.5\}\geq \mathrm{m}(t,s,0.5)$

If $\mathrm{m}(t,s,)\leq 0.5$ then $m(t,s,0.5)=m(t,s)s.t.$

$\mu(xyx^{-1},q')\geq m(t,s)\Rightarrow(xyx^{-1},q')_{m(t,s)}\in\mu$

If $m(t,s)>0.5$ then $m(t,s,0.5)=0.5s.t.$

$\mu(xyx^{-1},q')\geq 0.5\Rightarrow\mu(xyx^{-1},q')+m(t,s)\geq 0.5+m(t,s)>1$

$\Rightarrow(xyx^{-1},q')_{m(t,s)}\,q\mu$

Thus, $(xyx^{-1},q')_{m(t,s)}\in q\mu\ \Rightarrow\mu$ is a $(\in,\in\vee q)$ *-Q-fuzzy* normal (QFN) subgroup of *G*.

Theorem5.7 : of *Q-fuzzy* subgroup μ of *G* is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of *G* and $\mu(xyx^{-1},q')<0.5\quad\forall x,y\in G$ then μ is also a $(\in,\in)$ *-Q-fuzzy* normal subgroup of *G*.

Proof : Since μ as a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of *G*, $\forall x,y\in G$

$\mu(xyx^{-1},q')\geq\min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow 0.5>\min\{\mu(x,q'),\mu(y,q')\}$

$\Rightarrow\mu(x,q')<0.5$ and $\mu(y,q')<0.5$

Let $(x,q')_t\in\mu,(y,q')_s\in\mu$ where $t,s\in[0.1]$.

Then $\mu(x,q')\geq t,\mu(y,q')\geq s$

$\Rightarrow t<0.5,s<0.5$

$\Rightarrow m(t,s)<0.5$

Since μ is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of *G*, $\forall x,y\in G$

$(x,q')_t\in\mu,(y,q')_s\in\mu\Rightarrow\mu(xyx^{-1},q')\geq m(t,s)$ or $\mu(xyx^{-1},q)+m(t,s)>1$

Since, $m(t,s)<0.5$ we must have $\mu(xyx^{-1},q')\geq m(t,s)$ so that μ is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of *G*.

Theorem 5.8: Let μ be $(\in,\in\vee q)$-Q-fuzzynormal subgroup of G and $g\in G$.Then, ${}_g\mu_{g^{-1}}$ is also a ($\in,\in\vee q)$ -Q-fuzzy normal subgroup of G.

Proof: Let $(x,q')_t\in{}_g\mu_{g^{-1}},(y,q')_s\in{}_g\mu_{g^{-1}}$ where $t,s\in[0,1]$. Then,

$({}_g\mu_{g^{-1}})(x,q')\geq t,({}_g\mu_{g^{-1}})(y,q')\geq s\Rightarrow\mu(g^{-1}xg,q')\geq t,\mu(g^{-1}yg,q')\geq s\Rightarrow(g^{-1}xg,q')_t\in\mu,(g^{-1}yg,q')\in\mu$

Since, μ is a $(\in,\in\vee q)$ -Q-fuzzy normal subgroup of G,

$$\mu\left((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q'\right)\geq m(t,s)\vee\mu\left((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q'\right)+m(t,s)>1......(B)$$

Now,

$$\mu\left((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q'\right)=\mu\left((g^{-1}xg)((g^{-1}yg)((xg)^{-1}g),q'\right)=\mu\left((g^{-1}xg)(g^{-1}yg)(g^{-1}x^{-1}g),q'\right)$$

$$=\mu\left(g^{-1}(xgg^{-1}ygg^{-1}x^{-1})g,q'\right)=\mu\left(g^{-1}(xyx^{-1})g,q'\right)=({}_g\mu_{g^{-1}})(xyx^{-1},q')$$

Therefore from (B),

$$({}_g\mu_{g^{-1}})\left(xyx^{-1},q'\right)\geq m(t,s)\vee({}_g\mu_{g^{-1}})\left(xyx^{-1},q'\right)+m(t,s)>1$$

Therefore, ${}_g\mu_{g^{-1}}$ is also a $(\in,\in\vee q)$ -Q-fuzzy normal subgroup of G.

V. Homomorphism of $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup

Theorem6.1: $f:G\times Q\to G'\times Q$ be a homomorphism, where G,G' are two groups and Q, a non empty set. If μ' is a $(\in,\in\vee q)$ *-Q-fuzzy* normal sub group of G' then $f^{-1}(\mu')$ is also a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G.

Proof : Let us recall that $f^{-1}(\mu')$ is defined as

$\left(f^{-1}(\mu')\right)(x,q')=\mu'\left(f(x),q'\right)\quad\forall(x,q')\in G\times Q$ where μ' is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G´.

Let $x,y\in G$ *and* $q'\in Q$ .Then, $(x,q')_t,(y,q')_s\in f^{-1}(\mu')\quad\forall t,s\in[0.1]$

implies $\left(f^{-1}(\mu')\right)(x,q')\geq t,\left(f^{-1}(\mu')\right)(y,q')\geq s$

$\Rightarrow\mu'\left(f(x),q'\right)\geq t,\mu'\left(f(y),q'\right)\geq s$

$\Rightarrow\left(f(x),q'\right)_t\in\mu',\left(f(y),q'\right)_s\in\mu'$ where μ' is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G'.

So, $\left(f(x)f(y)f(x)^{-1},q'\right)_{m(t,s)}\in\mu'$ *or* $\left(f(x)f(y)f(x)^{-1},q'\right)_{m(t,s)}q\mu'$

$$\Rightarrow\mu'\left(f(x)f(y)f(x)^{-1},q'\right)\geq m(t,s)\ \ or\ \ \mu'\left(f(x)f(y)f(x)^{-1},q'\right)+m(t,s)>1$$

$$\Rightarrow\mu'\left(f(x)f(y)f(x^{-1}),q'\right)\geq m(t,s)\ \ or\ \ \mu'\left(f(x)f(y)f(x^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow\mu'\left(f(xy)f(x^{-1}),q'\right)\geq m(t,s)\ \ or\ \ \mu'\left(f(xy)f(x^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow\mu'\left(f(xyx^{-1}),q'\right)\geq m(t,s)\ \ or\ \ \mu'\left(f(xyx^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow\left(f^{-1}(\mu')\right)\left(xyx^{-1},q'\right)\geq m(t,s)\ \ or\ \ \mu'\left(f^{-1}(\mu')\right)\left(xyx^{-1},q'\right)+m(t,s)>1$$

$$\Rightarrow\left(xyx^{-1},q'\right)_{m(t,s)}\in f^{-1}(\mu')\ or\ \left(xyx^{-1},q'\right)_{m(t,s)}qf^{-1}(\mu')$$

$\Rightarrow\left(xyx^{-1},q'\right)_{m(t,s)}\in\vee qf^{-1}(\mu')\Rightarrow f^{-1}(\mu')$ is $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G.

Theorem6.2: $F:G\times Q\to G'\times Q$ be a homomorphism, where G and G´ are two groups and Q, a non empty set. If $f^{-1}(\mu')$ is a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G' then μ' is also a $(\in,\in\vee q)$ *-Q-fuzzy* normal subgroup of G'.

Proof : Let $u,v\in G'$ *s.t.* $(u,q')_t,(u,q')_s\in\mu'$ where $t,s\in[0.1]$.

Now f being onto, $\exists u,v \in G' \ s.t. \ f(x)=u \ and \ f(y)=v$ .Since μ' is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of G', $\mu'(u,q')\geq t, \mu'(v,q')\geq s$

$$\Rightarrow \mu'\left(f(x),q'\right)\geq t, \mu'\left(f(y),q'\right)\geq s$$

$$\Rightarrow \left(f^{-1}(\mu')\right)(x,q')\geq t, \left(f^{-1}(\mu')\right)(y,q')\geq s$$

$\Rightarrow (x,q')_t \in f^{-1}(\mu'), (y,q')_s \in f^{-1}(\mu')$ where $f^{-1}(\mu')$ is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of *G*.

So, $(xyx^{-1},q')_{m(t,s)} \in f^{-1}(\mu') \ or \ (xyx^{-1},q')_{m(t,s)} \, q f^{-1}(\mu')$

$$\Rightarrow \left(f^{-1}(\mu')\right)(xyx^{-1},q')\geq m(t,s) \ or \ \mu'\left(f^{-1}(\mu')\right)(xyx^{-1},q')+m(t,s)>1$$

$$\Rightarrow \mu'\left(f(xyx^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(xyx^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow \mu'\left(f(xy)\,f(x^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(xy)\,f(x^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow \mu'\left(f(x)\,f(y)\,f(x^{-1}),q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)\,f(y)\,f(x^{-1}),q'\right)+m(t,s)>1$$

$$\Rightarrow \mu'\left(f(x)\,f(y)\,f(x)^{-1},q'\right)\geq m(t,s) \ or \ \mu'\left(f(x)\,f(y)\,f(x)^{-1},q'\right)+m(t,s)>1$$

$$\Rightarrow \mu'(uvu^{-1},q')\geq m(t,s) \ or \ \mu'(uvu^{-1},q')+m(t,s)>1$$

$\Rightarrow (uvu^{-1},q')_{m(t,s)} \in\vee q\mu' \ \Rightarrow \mu'$ is a $(\in,\in\vee q)$ -*Q-fuzzy* normal subgroup of G´.

References

[1]. A. Rosenfield, Fuzzy Groups, J. Math. Anal. Appl. 35 (1971), 512-517.

[2]. A. Solairaju and R. Nagarajan, A New Structure and Construction of Q-fuzzy groups, Advances in Fuzzy Mathematics, Vol. 4(1), (2009), pp 23-29.

[3]. A. Solairaju and R. Nagarajan, Q-fuzzy left R-subgroups of near rings w.r.t. T-norms, Antarctica Journal of Mathematics. 5, (1-2), 2008.

[4]. K. H. Kim, Y. B. Jun, Normal fuzzy r-subgroups in near rings, Fuzzy Sets and Systems, 121 (2001) 341-345.

[5]. K. H. Kim, Y. B. Jun, On fuzzy R-subgroups of near rings, J. fuzzy math. 8 (3) (2000) 549-558.

[6]. S. K. Bhakat and P. Das, $(\in,\in\vee q)$ -fuzzy subgroup, Fuzzy Sets and Systems, 80 (1996), 359-368.

[7]. T. Priya, T. Ramachandran, K. T. Nagalakshmi, On Q-Fuzzy Normal Subgroups.

[8]. V. Murali, Fuzzy Points of equivalent subsets, Inform Sci, Vol. 158 (2004), 277-288.

[9]. J.A. Goguen, L-Fuzzy sets, J. Math.Anal.Appl.18, (1967) 145 -174.

[10]. R. Muthuraj,*et al*. Anti Q-fuzzy BG-ideals in BG- Algebra, International Journal of Computer Applications, (0975- 8887) Volume 4 No-11,August 2010.