Solar System Effects in Schwarzschild Space Time

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Abstract: *Review works Solar System Effects in Schwarzschild de-Space Time. The Schwarzschild space-time describes the gravitational field of a spherically symmetric mass. Based on this space-time we calculate Solar system effects like gravitational redshift, light deflection, gravitational time delay, perihelion shift, geodetic precession.*
Keywords: *Schwarzschild Space-time, gravitational redshift, light deflection, gravitational time delay, perihelion shift, geodetic precession.*

I. Introduction

Until now all gravitational effects in the Solar system and in binary systems are well described by means of the Einstein general theory of relativity [2][3]. Dark matter is introduced to obtain the gravitational field needed to describe observations like the galactic rotation curves, gravitational lensing, or the structure of the cosmic microwave background. Dark energy is invoked to explain the observed accelerated expansion of the universe by means of an additional energy momentum component. In general relativity, the repulsion necessary to obtain an accelerated expansion of the universe can be provided by the inclusion of a vacuum energy. Instead of postulating unknown forms of matter and energy another line of resolving these cosmological issues consists of a modification of e gravitational field. We present in the Schwarzschild space-time, on a Doppler measurement, used to determine the velocity of the Pioneer 10 and 11 spacecraft. We like to emphasize that the observational basis of the pioneer anomaly is by far not of the same quality as the Solar system effects. Therefore, we are looking for effects related to a space-time metric.

II. Schwarzschild Space-time

We calculate Solar system effects for the spherically symmetric Schwarzschild metric[11]

$$ds^2 = \alpha(r)dt^2 - \alpha^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\emptyset^2) \qquad (1.1)$$

Where,

$$\alpha(r) = 1 - \frac{2M}{r} \qquad (1.2)$$

and M the mass of the source. It is an exterior solution of the Einstein field equations for a spherical mass distribution

$$R_{\mu v} - \frac{1}{2}g_{\mu v}R = 8\pi G T_{\mu v} \qquad (1.3)$$

III. Gravitational Redshifst

Since Schwarzschild is a stationary space-time there is a time like Killing vector so that in coordinates adapted to the symmetry the ratio of the measured frequency of a light ray crossing different positions is given by

$$\frac{v}{v_0} = \sqrt{\frac{g^{00}(r)}{g^{00}(r_0)}} \quad \text{Where,}$$

$$g_{00}^{(0)} = 1 - \frac{2M}{r}$$

For small masses M this simplifies to $\quad \frac{v}{v_0} = 1 - \frac{M}{r} + \frac{M}{r_0}$

IV. Deflection of Light

We choose the motion of point particles and light rays in the Schwarzschild space-time is described by the Lagrangian

$$2\pounds = \alpha(r)\dot{t}^2 - \alpha^{-1}(r)\dot{r}^2 - r^2\dot{\phi}^2 \qquad (1.4)$$

Where, the dot stands for differentiation with respect to the affine parameter λ , and where from the very beginning the motion is restricted to the $\theta = \frac{\pi}{2}$ plane. Since the metric (1.1) is independent of t and ϕ the quantities

$$E \equiv \alpha(r)\frac{dt}{d\lambda} , L \equiv r^2\frac{d\phi}{d\lambda}, \tag{1.5}$$

Are conserved along the orbit of the particle and define its energy and angular momentum, respectively.

For light, the Lagrangian (1.4) vanishes. Multiplication of (1.4) by $\frac{\alpha}{L^2}$ and substitution of the angular momentum yields

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{\alpha(r)}{r^2}\right]^{-1/2} \tag{1.6}$$

Where, $b \equiv \frac{L}{E}$ which in Schwarzschild space-time can be interpreted as impact parameter. The is related to increasing/decreasing r. The distance of closest approach $\pm$ defined by

$$\frac{d\phi}{dr}\Big|_{r=r_0} = \left[\frac{1}{b^2} - \frac{\alpha(r_0)}{r_0^2}\right] = 0 \tag{1.7}$$

Relates b to r_0. when we replace b to r_0 in (1.6)

$$\frac{d\phi}{dr} = \pm \frac{1}{r^2}\left[\frac{1}{r_0^2}\left(1 - \frac{2M}{r_0}\right) - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{1.8}$$

V. Gravitational Time Delay

We consider the standard measurement of the time delay of light where a radar signal is sent from the Earth to pass close to the Sun and reflect off another planet or a spacecraft. Let $r_\oplus$ and r_R be thedistance between the Sun and the Earth and the reflector, respectively. The time interval between emission and return of a pulse as measured by a clock on the Earth is

$$\Delta T = 2t(r_\oplus, r_0) + 2t(r_R, r_0) \tag{1.9}$$

Where, $t(r_1, r_2)$ is the elapsed coordinate time along the signal's trajectory between distances r_1 and r_2 to the Sun. The calculation of $t(r_1, r_2)$ requires to represent t as a function of r along the path of the pulse. From (1.5) and (1.6) we obtain

$$\frac{dt}{dr} = \pm \frac{1}{ab}\left[\frac{1}{b^2} - \frac{\alpha(r)}{r^2}\right]^{-1/2} \tag{1.10}$$

and, thus,

$$t(r, r_0) = \int_{r_0}^{r} \frac{dr}{\alpha b(r)}\left[\frac{1}{b^2} - \frac{\alpha(r)}{r^2}\right]^{-1/2} \tag{1.11}$$

Where, we can replace b by r_0 using (1.7)

In the recent Cassini experiment [16] one has not measured the time delay but, instead, the relative change in the frequency

$$y = \frac{v(t) - v_0}{v_0} = \frac{d}{dt}\Delta t, \tag{1.12}$$

Where, v_0 is the frequency of the radio waves emitted on the Earth .These radio waves reach Cassini with a frequency v' and are sent back with the same frequency v'. The frequency of this signal reaching the Earth is $v(t)$.

VI. Perihelion Shift

The perihelion is based on the modification of the geodesic equation with $u = \frac{1}{r}$ one obtains

$$\frac{d^2u}{d\varphi^2} = \frac{M}{L^2} - u + 3Mu^2 \tag{1.13}$$

Now we obtain

$$\delta\varphi = 6\pi \frac{Gm}{a(1-\epsilon^2)c^2}$$

VII. Geodesic Precession

We want to calculate the geodesic precession by introducing a coordinate system rotating with angular velocity ω.

We choose the new angular coordinate ϕ is as

$$d\varphi = d\phi - \omega dt \tag{1.14}$$

And gives in the equation plane $\theta = \pi/2$ the metric

$$ds^2 = (\alpha - r^2\omega^2)\left(dt - \frac{r^2\omega}{\alpha - r^2\omega^2}d\varphi\right)^2 - \alpha^{-1}dr^2 - \frac{r^2\alpha}{\alpha - r^2\omega^2}d\varphi^2 \tag{1.15}$$

The canonical form of the metric is

$$ds^2 = e^{2\Psi}(dt - \omega_i dx^i)^2 - h_{ij}dx^i dx^j \tag{1.16}$$

With

$$e^{2\Psi} = \alpha - r^2\omega^2, \omega_1 = \omega_2 = 0, \omega_3 = \frac{r^2\omega}{\alpha - r^2\omega^2}, h_{11} = \alpha^{-1}, h_{33} = \frac{r^2\alpha}{\alpha - r^2\omega^2} \tag{1.17}$$

With the indices 1,2,3 referring to r, θ, φ respectively. For a given $r = r_0$ we can choose an ω so that $\partial\Psi/\partial r = 0$. since Ψ is related to the acceleration ,this means that the line $r = r_0$ is geodesic. This corresponds to a modified Kepler frequency

$$\omega^2 = \frac{M}{r^3} \tag{1.18}$$

Also obtained in[11] and implies

$$e^{2\Psi} = 1 - \frac{3M}{r},$$

$$\omega_1 = \omega_2 = 0,$$

$$\omega_3 = \frac{r^2\omega}{1 - \frac{3M}{r}},$$

$$h_{11} = \alpha^{-1},$$

$$h_{33} = \frac{r^2\alpha}{1 - \frac{3M}{r}} \tag{1.19}$$

The rotation rate of a gyroscope at affixed position in the rotating coordinate system is given by[11]

$$\Omega^2 = \frac{e^{2\Psi}}{8}h^{ik}h^{jl}(\omega_{i,j} - \omega_{j,i})(\omega_{k,l} - \omega_{l,k}) \tag{1.20}$$

and points in the negative direction relative to the orbit. We obtain

$$\Omega = \sqrt{\frac{M}{r^3}} \tag{1.21}$$

VIII. Doppler Tracking of Satellites on Escape Trajectories

We will want to calculate the two-way Doppler tracking of a satellite moving on a geodesic radially away from the Sun. This amounts to an investigation of whether the presence of a cosmological constant might be made responsible for the anomalous acceleration observed for the Pioneer spacecraft[4]. For simplicity, we take an observer at a fixed radial distance from the Sun and a spacecraft moving radially .so that the Sun, the Earth and space craft lie on one line.

The measurement is carried out as follows: On the Earth a radio signal with frequency v_0 is sent to the spacecraft at position r ,where it will arrive redshifted according to

$$\tilde{v} = \sqrt{\frac{\alpha(r_\oplus)}{\alpha(r)}}v_0 \tag{1.22}$$

Due to the velocity of the spacecraft the measured is in addition Doppler shifted

$$v' = \frac{1}{\sqrt{1-(v^r)^2}}(1 - v^r)\tilde{v} \tag{1.23}$$

Where, $v^r = \theta^r(v)$ the measured radial velocity (v is the 4-velocity of the spacecraft and θ^r is the normalized basis 1-form in r −direction, $g^{\mu v}\theta_\mu^r \theta_v^r = -1$; in coordinates, $\theta_\mu^r = \alpha^{-1/2}(0,1,0,0)$. This is also the frequency sent back to Earth, where it arrives with the frequency

$$v'' = \frac{1}{\sqrt{1-(v^r)^2}}(1-v^r)\sqrt{\frac{\alpha(r_\oplus)}{\alpha(r)}}v' \qquad (1.24)$$

The two-way Doppler tracking will then yield

$$\frac{v''-v_0}{v_0} = -2\frac{v^r}{1+v^r} \approx -2v^r \qquad (1.25)$$

As expected, the gravitational redshift drops out. Only the motion of the spacecraft influences the result. Therefore, what is finally needed is the solution for the radial velocity v obtained from the geodesic equation

$$0 = \frac{d^2 r}{dt^2} + \left(\begin{Bmatrix} r \\ \mu v \end{Bmatrix} - \begin{Bmatrix} 0 \\ \mu v \end{Bmatrix}\frac{dr}{dt}\right)\frac{dx^\mu}{dt}\frac{dx^v}{dt} \qquad (1.26)$$

$$= \frac{d^2 r}{dt^2} - \frac{3}{2}\frac{1}{\alpha}\partial_r\alpha\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\alpha\partial_r\alpha \qquad (1.27)$$

Where, $\begin{Bmatrix} r \\ \mu v \end{Bmatrix}$ are the Christoffel symbols. We can safely neglect the term quadratic in the velocities. Then to first order $in\ M$

$$\frac{d^2 r}{dt^2} = -\frac{M}{r^2} \qquad (1.28)$$

This leads to $\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = E + \frac{M}{r}$ (1.29)

Where, E is an integration constant.

The Doppler measurement in terms of the measured velocity is

$$\frac{v^n - v_0}{v_0} = -2\frac{v^r}{1+v^r} \approx -2 = -2\frac{dr/dt}{\sqrt{\alpha(r)}+dr/dt}$$

We assume , $M/r \ll E$ and make an expansion with respect to M .Furthermore,

$$\frac{v^n-v_0}{v_0} = -2\frac{\sqrt{2E}}{1+\sqrt{2E}} - \frac{2M}{r}\frac{1+2E}{(1+\sqrt{2E})^2\sqrt{2E}} \qquad (1.30)$$

Since $2E$ corresponds to v the non-relativistic limit is

$$\frac{v^n-v_0}{v_0} = -2\sqrt{2E} - \frac{2M}{r}\frac{1}{\sqrt{2E}} \qquad (1.31)$$

For any non-Newtonian gravitational force the distance r of the planetary orbits as function of the angle φ is given by

$$r(\varphi) = \frac{r_0(1+\epsilon)}{1+\epsilon\ \cos((1+\epsilon)\varphi)} \qquad (1.32)$$

Where, ϵ is the parameter related to the non-Newtonian part of the gravitational interaction describing the perihelion shift and the change in the planetary orbital parameter r_0 (which is related to the semi-major axis). As a consequence, we can Newtonian or non-Einsteinian gravitational forces, there is an additional perihelion shift of the order 400μ as per century (which is the present day accuracy, then $\epsilon \approx 10^{-12}$ so that $\epsilon r_0 \approx 15\ cm$ .This is clearly beyond any observational access for any planet in the Solar system.

IX. Conclusion

We have calculated the Solar system effects in the frame of the Schwarzschild space-time. We considered the gravitational field of a spherically symmetric mass, described by the Schwarzschild space-time. We calculated how the absence of such a cosmological constant would affect Solar system effects like gravitational redshift,light deflection, gravitational time delay and geodetic precession. Comparing the theoretical results with the observations gives Solar system estimates on the magnitude of such an additional cosmological term in the Einstein equations. We conclude that the cosmic acceleration therefore does not lead to observable Solar system effects.

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