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The Consistent Estimators for Homogeneous Gaussian Fields Statistical Structures

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Abstract*: The present theory of consistent estimators of the parameters of statistical structures of homogeneous Gaussian fields can be used, for example, in the reliability predication of different engineering designs. In the paper there are discussed Gaussian homogeneous fields statistical structures $\{E, S, \mu_i, i \in I\}$ in Banach space of measures. We prove necessary and sufficient conditions for existence of such estimators.*
Keywords*: consistent estimators, orthogonal, weakly separable, strongly separable statistical structures.*
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I. Introduction

Let there is given (E, S) measurable space and on this space there given $\{\mu_i, i \in I\}$ family of probability measures depended on S, the I set of parameters.
Let bring some definition (see [1]-[10]).
Definition 1. A statistical structure is called object $\{E, S, \mu_i, i \in I\}$ where $i \in I$ some parameter associated with probability measure μ_i, I set of parameters.
Definition 2. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if μ_i and μ_j are orthogonal for each $i \neq j, \ i \in I, \ j \in I$.
Definition 3. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists family S-measurable sets $\{X_i, i \in I\}$ such that relations are fulfilled:

$$(\forall i)(\forall j)(i \in I \ \& \ j \in I) \Rightarrow \mu_i(X_j) = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}.$$

Definition4. A statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable if there exists disjoint family S-measurable sets $\{X_i, i \in I\}$ such that the relation are fulfilled. $\mu_i(X_i) = 1, \ \forall i \in I.$
Remark 1. A strong separable there follows weakly separable. From weakly separable there follows orthogonal but not vice versa (see [1]-[4]).
Remark 2. In the general theory of statistical decisions there often arises a problem of transition from a weakly separated family of probability measures to the corresponding strongly separated family. In 1981, A. Skozokhod (see [1]) proved (in $ZFC \& CH$ theory) proved that if the Continuum Hypothesis is true, then an arbitrary weakly separated family of probability measures,whose cardinality is not greater than the cardinality of the continuum is strongly separable.The validity of the inverse relation was established in (see [9]-[10]). In particular, It was shown there that if an arbitrary weakly separated family of probability measures whose cardinality is less than or equal to the cardinality of the continuum is strongly separated, then the continuum Hypothesis (CH) is true Applying Martin's axiom *(MA)* in 1984 Z. Zerekidze(see [2]-[4]) proved that an arbitrary weakly separated family of Borel probability measures defined in a separable completely metrizable space (i. e. Polish space) is strongly separated if its cordinality is not prated than the cardinality of the continuum. G. Pantsulaia proved (see [9]-[10]) this result is extended to all complete metric spaces whose topological weights are not measurable in a wider sence. Z. Zerekidze proved in ZF theory (see [8]) for the statistical structure $\{E, S, \mu_i, i \in I\}$ where N set at natural numbers, orthogonality, weak separability, separability and strong separability equivalent concepts.

Let I be set of parameters and B (I) σ -algebra of subsets of I which contains all finite subsets I.

Definite 5. A statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit a consistent estimators of parameter $i \in I$ if there exists at least one measurable map $\delta : (E, S) \to (I, B(I))$ such that $\overline{\mu}_i \{(x : \delta(x)) = i\} = 1, \quad \forall i \in I.$

We denote by $\overline{\mu}_i$ the completion of the measure μ_i and dom $\overline{\mu}_i$ the σ-algebra of all $\overline{\mu}_i$ measureable subsets of E

Remark 3. By A. Skorokhod was introduced definition a consistent estimators of parameters (see [1]).

Definite 6. A statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit a consistent estimators of any parametric function if for any real bounded measurable function $g : (I, B(I)) \to R$ there exists at least one measurable function $f : (E, S) \to R$ such that $\overline{\mu}_i \{x : f(x) = g(i)\} = 1, \quad \forall i \in I.$.

Definition 7. A statistical structure $\{E, S, \mu_i, i \in I\}$ will be said to admit an unbiased estimators of any parametric function if for any real bounded measurable function $g : (I, B(I)) \to R$ there exists at least one measurable function $\beta : (E, S) \to R,$ such that $\int_B \beta(x) \overline{\mu}_i(dx) = g(i), \quad \forall i \in I$.

Remark 4. If a statistical structure $\{E, S, \mu_i, i \in I\}$ admitting a consistent estimators of parameters $i \in I$ then this statistical structure $\{E, S, \mu_i, i \in I\}$ which admits a consistent estimators for any parametric function and a statistical structure which admits an unbiased estimators of any parametric function (see [6]-[8]).

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 8.A linear subset $M_\beta \subset M^\sigma$ is called a Banach space of measures if:

1) A norm can be defined on M_β so that M_β will be a Banach space with respect to this norm,and for any orthogonal measures $\mu, \nu \in M_\beta,$ and real number $\lambda \neq 0$ the enequality $\|\mu + \lambda \nu\| \geq \|\mu\|$ is fulfilled;
2) If $\mu \in M_B$, $|f(x)| \leq 1$ than $\nu_f(A) = \int_A f(x) \mu(dx) \in M_B$ and $\|\nu_f\| \leq \|\mu\|$;
3) If $\nu_n \in M_B, \ \nu_n > 0, \ \nu_n(E) < +\infty, \ n = 1,2,...$ and $\nu_n \downarrow 0$ then for any linear functional $\ell^* \in M_B^* \lim_{n\to\infty} \ell^*(\nu_n) = 0$, where M_B^* conjugate to M_B linear space.

The definition and construction of the Banach space of measures is studied Z.Zerekidze(see [7]).

Definition 9. Let I some set of indexes and M_{B_i} Banach space $\forall i \in I$. We set $M_B = \left\{ \{X_i\}_{i\in I}, X_i \in M_{B_i}, \sum_{i\in I} \|X_i\|_{M_{B_i}} < \infty \right\}.$

Then the M_B with norm $\|\{X_i\}_{i\in I}\| = \sum_{i\in I} \|X_i\|_{M_{B_i}} < \infty$ is the Banach space. It is called the direct sum of Banach spaces M_{B_i} and denoted so $M_B = \bigoplus_{i\in I} M_{B_i}$.

By Z. Zerakidze was introduced definition and is studied construction the Banach space of measures (see [7])
The following theorem has also been proved in this paper (see [7]).

Theorem 1. Let M_B be a Banach space of measures, then in M_B there exists a family of pairwise orthogonal probability measures $M = \{\mu_i, i \in I\}$ such that $M_B = \bigoplus_{i\in I} M_{B_i}$, where M_{B_i} is the Banach space of elements ν of the norm : $\nu(B) = \int_B f(x) \mu_i(dx) \ B \in S$, $\int_E |f(x)| \mu_i(dx) < \infty, \ \ \|\nu\|_{M_{B_i}} = \int_E |f(x)| \mu_i(dx)$.

Let $t = (t_1, t_2, ..., t_n) \in T,$ where T be closed bounded subset of R^n, $t \in T, i \in I$ Gaussian real homogenous field on T with zero means $E[\xi_i(t)] = 0, \forall i \in I$ and correlation function

$E[\xi_i(t)\xi_i(S)] = R_i(t-S), t, S \in T, i \in I.$ Let $\{\mu_i, i \in I\}$ be the corresponding probability measures given on S and $f_i(\lambda), \ \lambda \in R^n, \ \forall i \in I$ be spectral densities.

We be called the Fourier transformation of generation Fourier transformation.

Let $\int_{R^n}\int_{R^n} \frac{\left|\tilde{b}_{i,j}(\lambda,\mu)\right|^2}{f_i(\lambda) f_i(\mu)} d\lambda d\mu = +\infty \ \ \forall i, j \in I,$ where $\tilde{b}_{i,j}(\lambda,\mu), \ \lambda, \mu \in R^n, \ \forall i, j \in I$ the Generalization Fourier transformation of the following functions $b_{i,j}(S,t) = R_i(S,t) - R_j(S,t), \ S, t \in T, \ \forall i, j \in I$.

Then the corresponding probability measures μ_i and μ_j are pairwise orthogonal $\forall i, j \in I$ (see [5]) and $\{E, S, \mu_i, i \in I\}$ are Gaussian orthogonal homogeneous fields statistical structures.Next, we consider S-measurable $g_i(x), \ \forall i \in I$ function such that $\sum_{i \in I}\int_E |g_i(x)|\mu_i(dx) < +\infty$. Let M_B the set measures defined by formula $\nu(B) = \sum_{i \in I_1}\int_B g_i(x)\mu_i(dx)$, where $I_1 \subset I$ a countable subsets in I and $\sum_{i \in I_1}\int_E |g_i(x)|\mu_i(dx) < +\infty$, define a norm on M_B by formula $\|\nu\| = \sum_{i \in I}\int_E |g_i(x)|\mu_i(dx)$, then M_B is a Banach space of measures and $M_B = \oplus_{i \in I} M_{B_i}$, where M_{B_i} in Banach space of elements the form $\nu(B) = \sum_{i \in I_1} g_i(x)\mu_i(dx), \ \ B \in S, \ \int_E |g_i(x)|\mu_i(dx) < +\infty,$ with the norm on M_{B_i} $\|\nu\|_{M_{B_i}} = \int_E |g_i(x)|\mu_i(dx).$

It is also well known that in the (ZFC), (CH), (MA) theory there exists a weakly separable statistical structure which is not strongly separable.Here and in the equal we denote by (MA) the Martin's axiom (see [2]-[8]).

Theorem 2. Let $M_B = \oplus_{i \in I} M_{B_i}$ be a Banach space of measures, E be the complete separable metric space. S be the Borel σ-algebra in E and $cardI \le c$, where c denotes a power of continuum. Then in the theory (ZFC)&(MA) the Gaussian Homogeneous fields orthogonal statistical structures $\{E, S, \mu_i, i \in I\}$ admits a consistent estimators of parameters $i \in I$ if and only if the correspondence $f \to l_f$ defined by the equality $\int_E f(x)\mu_i(dx) = l_f(\mu_i), \ \ \forall \mu_i \in M_B$ is one-to-one. Here l_f is a linear continuous functional on M_B, $f \in F(M_B)$. Denote by $F = F(M_B)$ the set of real functions f for which $\int_E f(x)\mu_i(dx)$ is defined $\forall \mu_i \in M_B$.

Proof.Necessity. The existence of a consistent estimator $\delta : (E, S) \to (I, B(I))$ of the parameter $i \in I$ implies that $(\forall i)(i \in I \to \overline{\mu}_i(\{x : \delta(x) = i\}) = 1)$.

Setting $X_i = \{x : \delta(x) = i\}$ for $i \in I$, we get:

1) $\overline{\mu}_i(X_i) = \overline{\mu}_i(\{x : \delta(x) = 1\})$ for $i \in I$;
2) $X_{i_1} \cap X_{i_2} = \varnothing$ for all different parameters i_1 and i_2 from I because $(X_{i_1} = \{x : \delta(x) = i_1\}) \cap (\{x : \delta(x) = i_2\} = X_{i_2}) = \varnothing;$
3) $\bigcup_{i \in I} X_i = \{x : \delta(x) \in I\} = E.$

Therefore a statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable, so there exist S-measurable sets $\{X_i\}, i \in I$ such that $\mu_i(X_j) = \begin{cases} 1, \text{if } = j \\ 0, \ \text{if } \ne j \end{cases}$

We put the linear continuous functional l_{X_i} into the correspondence to a function $I_{X_i} = F(M_B)$ by the formula: $\int_E I_{X_i}(x)\mu_i(dx) = l_{I_{X_i}}(\mu_i) = \|\mu_i\|_{M_{B_i}}$.

We put the linear continuous functional $l_{\bar{f}_1}$ into the correspondence to the function $\bar{f}_1(x) = f_1(x)I_{X_i}(x)$ Then for any $\mu_{i_1} \in M_B(\mu_i) \int_E \bar{f}_1(x)\mu_{i_1}(dx) = \int_E f_1(x)I_{X_i}(x)\mu_{i_1}(dx) = \int_E f(x)f_1(x)I_{X_i}(x)\mu_i(dx) = l_{\bar{f}_1}(\mu_{i_1}) = \|\mu_{i_1}\|_{M_{B_i}}$.

Let Σ be the collection of extensions of functional satisfying the condition $l_f \le p(x)$ on those subspaces where they are defined.

Let us introduce on Σ a partial ordering having assumed $l_{f_1} < l_{f_2}$ if l_{f_2} is defined on large set then l_{f_1} and $l_{f_1} = l_{f_2}$ there where both of them are defined.

Let $\{l_{f_i}\}_{i\in I}$ be a linear ordered subset in Σ. Let M_{B_i} be the subspace on which l_{f_i} is defined. Define l_f on $\bigcup_{i\in I} M_{B_i}$ having assumed $l_f(\mu) = l_{f_i}(\mu)$ if $\mu \in M_{B_i}$.

It is obvious, that $l_{f_i} < l_f$. Since any linearly ordered subset in Σ has an upper bound by virtue of Chorn's lemma Σ contains a maximal element λ defined on some set X' satisfying the condition $\lambda(x) \le p(x)$ for $x \in X'$. But X' must coincide with the entire space M_B because otherwise. We could extended λ to a wider space by adding as above one more dimension. This contradicts the maximality of λ hence $X' = M_B$. Therefore the extension of the functional is defined everywhere.The extension of the functional is defined everywhere.

It we put the linear continuous functional l_f into correspondence to the function $f(x) = \sum_{i\in I} g_i(x)I_{X_i}(x) \in F(M_B)$ then obtain $\int_E f(x)\mu(dx) = \|\mu\| = \sum_{i\in I} \|\mu_i\|_{M_{B_i}}$, where $\mu(B) = \sum_{i\in I}\int_B g_i(x)\mu_i(dx), \quad B \in S$

The necessity is proved.

Sufficiency. For $f \in F(M_B)$ we define linear continuous functional l_f by the equality $\int f(x)\mu(dx) = l_f(\mu)$. Denote I_f a countable subset in I, for which $\int_E f(x)\mu_i(dx) = 0$ for $i \notin I_f$. Let us consider functional l_{f_i} on M_{B_i} to which there corresponds. Then for $\mu_{i_1}, \mu_{i_2} \in M_{B_{i_1}}$ have $\int_E f_{i_1}(x)\mu_{i_2}(dx) = l_{f_{i_1}}(\mu_{i_2}) = \int_E f_1(x)f_2(x)\mu_{i_1}(dx) = \int_E f_{i_1}(x)\mu_{i_1}(dx)$ therefore $f_{i_1} = f_1$ a.e. with respect measure μ_{i_1}. Let $f_i > 0$ a. e. with respect to the measure μ_i and $\int_E f_i(x)\mu_i(dx) < \infty,\ \mu_i(c) = \int_c f_i(x)\mu_i(dx)$, then $\int_E f_i(x)\mu_j(dx) = l_{f_i}(\mu_j) = 0 \quad \forall j \ne i.$

Denote $C_i = \{x | f_i(x) > 0\}$, then $\int_E f_i(x)\mu_i(dx) = 0 \quad \forall j \ne 0.$

Hence it follows that $\mu_j(C_i) = 0 \quad \forall j \ne i.$ On the other hand $\mu_i(E - C_i) = 0$ therefore the statistical structure $\{E, S, \mu_i, i \in I\}$ is weekly separable. A Boral orthogonal family of probability measures $\{\mu_i, i \in I\},\ cardI \le c$ is weakly separable. Represent $\{\mu_i, i \in I\}$ as an inductive sequence $\mu_i < w_1$ where w_1 denotes the first ordinal number of the power of the set I.

Sense the statistical structure $\{E, S, \mu_i, i \in I\}$ is weakly separable, there exists a family S-measurable sets $\{X_i\}, i \in I$ such that the following relations are fulfilled: $\mu_i(X_j) = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$

For all $i \in [0, w_1)$ and $j \in [0, w_1)$

We define w_1 sequence of parts Z_i of the space E so that the following relations are fulfilled:

1) Z_i is Borel subset in E for all $i < w_1$;
2) $Z_i \subset X_i$ for all $i < w_1$;
3) $Z_i \cap Z_j = \varnothing$ for all $i < w_1,\ j < w_1,\ i \neq j$;
4) $\mu_i(Z_i) = 1$ for all $i < w_1$.

Assume that $Z_0 = X_0$. Let further the partial sequence $\{Z_j\}_{j<i}$ be already defined for $i < w_1$.

It is clear, that $\mu^*\left(\bigcup_{j<i} Z_j\right) = 0$. Thus there exists a Borel subset Y_i of the space E such that the following relations are valid: $\bigcup_{j<i} Z_j \subset Y_i$ and $\mu(Y_i) = 0$. Assume $Z_i = X_i - Y_i$, thereby the w_1 sequence of $\{Z_i\}_{i<w_1}$ disjunctive measurable subsets of space E is constructed. Therefore $\mu_i(Z_i) = 1$. For all $i < w_1$. A statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separable.

The statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separated there exists a family $(Z_i)_{i \in I}$ of elements of σ-algebra $S_1 = \bigcap_{i \in I} dom(\overline{\mu}_i)$ such that:

1. $\overline{\mu}_i(Z_i) = 1, \quad \forall i \in I;$
2. $Z_{i_1} \cap Z_{i_2} = \varnothing$ for all different parameters i_1 and i_2 from I;
3. $\bigcup_{i \in I} Z_i = E.$

For $x \in E,$ we put $\delta(x) = i,$ where i is unique parameter from the set I for which $x \in Z_i$. The existence of such a unique parameter I can be proved by using conditions (2), (3).

Now let $Y \in B(I)$. Then $\{x : \delta(x) \in Y\} = \bigcup_{i \in Y} Z_i.$ We have to show that $\{x : \delta(x) \in Y\} = dom(\overline{\mu}_{i_0})$ for each $i_0 \in I$.

If $i_0 \in Y$ then $\{x : \delta(x) \in Y\} = \bigcup_{i \in Y} Z_i = Z_{i_0} \cup \bigcup_{i \in Y - i_0} Z_i.$

On the other hand, from the validity of the condition that the statistical structure $\{E, S, \mu_i, i \in I\}$ is strongly separated it follows that $Z_{i_0} \in S_1 = \bigcap_{i \in I} dom(\overline{\mu}_i) \subseteq dom(\overline{\mu}_{i_0}).$

On the other hand, the validity of the condition $\bigcup_{i \in Y - i_0} Z_i \subseteq (E - Z_{i_0})$ implies that $\overline{\mu}_{i_0}\left(\bigcup_{i \in Y - i_0} Z_i\right) = 0.$ The latter equality yields that $\bigcup_{i \in Y - i_0} Z_i \in dom(\overline{\mu}_{i_0}).$

Since $dom(\overline{\mu}_{i_0})$ is σ-algebra, we deduce that $\{x : \delta(x) \in Y\} = Z_{i_0} \cup \bigcup_{i \in Y - i_0} Z_i \in dom(\overline{\mu}_{i_0}).$

If $i_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \bigcup_{i \in Y} Z_i \subseteq (E - Z_{i_0})$ and we claim that $\overline{\mu}_{i_0}\{x : \delta(x) \in Y\} = 0.$ The letter relation implies that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{i_0})$.

Thus we have shown the validity of the condition $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{i_0})$ for an arbitrary $i_0 \in I$. Hence $\{x : \delta(x) \in Y\} \in \bigcap_{i \in I} dom(\overline{\mu}_{i_0}) = S_1$. We have shown that the map $\delta : (E, S) \to (I, B(I))$ is measurable map.

Since $B(I)$ contain all singletons of I, we claim that $(\forall i)(i \in I \to \overline{\mu}_i(\{x : \delta(x) = i\}) = \overline{\mu}_i(Z_i) = 1)$.

Theorem 2 is proved.

Analogously is proved

Theorem3. Let $M_B = \bigoplus_{i \in I} M_{B_i}$ Be a Banach space of measures.E be the complete metric space, whose topological weights are not measurable in a wider sense. S be the Borel σ-algebra in E and $cardI \leq c$. Then in the theory (ZFC)&(MA) the Gaussian homogeneous fields orthogonal statistical structures $\{E, S, \mu_i, i \in I\}$ admits a consistent estimators of parameters $i \in I$ and only if the correspondence $f \to l_f$ Defined by the equality $\int_E f(x)\mu_j(dx) = l_f(\mu_i)$ $\forall \mu_i \in M_B$ is one-to-one. Here l_f is linear continuous functional on M_B, $f \in F(M_B)$ Denote by $F = F(M_B)$ the set of real functions f for which $\int_E f(x)\mu_j(dx)$ is defined $\forall \mu_i \in M_B$.

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