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Decomposition formulas for H_A - hypergeometric functions of three variables

*Mosaed M. Makky
Mathematics Department, Faculty of Science (Qena) South Valley University (Qena, Egypt)
Corresponding Author: Mosaed M. Makky

Abstract: *In this paper we investigate several decomposition formulas associated with hypergeometric functions H_A in three variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By means of these operator identities, as many as 5 decomposition formulas are then found, which express the aforementioned triple hypergeometric functions in terms of such simpler functions as the products of the Gauss and Appell hypergeometric functions.*

Keywords: *Decomposition formulas; hypergeometric functions; Multiple hypergeometric functions; Gauss hypergeometric function; Appell's hypergeometric functions.*

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I. Introduction

In the present work we aim to find differential equations to describe experimental data. The obtained equation would be practically helpful in the evaluation of the equality of experimental results.

The hypergeometric functions help solution many practical problems, such as partial differential equations, which can be obtained with the help of hypergeometric functions (see [9, 10, 16]).

Initially we acting by the neutral operator $D = \sum_{j=1}^{3} d_j \; , \; d_j = z_j \frac{\partial}{\partial z_j}$ a differential equations is found. One can then recall that the hypergeometric function is a solutions of such an equation.

Suppose that a hypergeometric function in the form (c.f. [4, 11])

(1.1) $$ {}_2F_1(\alpha, \beta; \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!(\gamma)_n} z^n $$

for γ neither zero nor a negative integer.

Now we consider H_A - hypergeometric function defined in [16] as follows

(1.2) $$ H_A = H_A(\alpha, \beta, \beta'; \gamma, \gamma'; -z_1, -z_2, -z_3) = \sum_{n_1, n_2, n_3} \frac{(\alpha)_{n_1+n_3} (\beta)_{n_1+n_2} (\beta')_{n_2+n_3}}{n_1! n_2! n_3! (\gamma)_{n_1} (\gamma')_{n_2+n_3}} (-z_1)^{n_1} (-z_2)^{n_2} (-z_3)^{n_3} $$

The study of H_A - hypergeometric function, where it is regular in the unit hypersphere (c.f. [7,11]), for the H_A - function, we can define as contiguous to it each of the following functions, which are samples by uppering or lowering one of the parameters by unity.

This study begins with an applied example for the idea of the research. we consider The H_A - hypergeometric function as in (1.2)

$$ D = \sum_{j=1}^{3} d_j \; , \; d_j = z_j \frac{\partial}{\partial z_j} $$

and the way we effect it with the recursions relations as it is found in the second part of the research, we obtain, as a result of acting by D on this function a differential equation, some special cases for a group of differential equations are the functions that are effected by the differential operator. There is a numerical example for one of these cases.

II. The Symbolic Operators

Burchnall and Chaundy [1,2] and Chaundy [3] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

(2.1) $$\nabla_{z_1 z_2}(h) = \frac{\Gamma(h)\Gamma(d_1+d_2+h)}{\Gamma(d_1+h)\Gamma(d_2+h)} = \sum_{k=0}^{\infty} \frac{(-d_1)_k (-d_2)_k}{(h)_k\, k!}$$

(2.2) $$\Delta_{z_1 z_2}(h) = \frac{\Gamma(d_1+h)\Gamma(d_2+h)}{\Gamma(h)\Gamma(d_1+d_2+h)} = \sum_{k=0}^{\infty} \frac{(-d_1)_k (-d_2)_k}{(1-h-d_1-d_2)_k\, k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-d_1)_k (-d_2)_k}{(h+k-1)_k (d_1+h)_k (d_2+h)_k\, k!}$$

and

(2.3) $$\nabla_{z_1 z_2}(h)\Delta_{xy}(g) = \frac{\Gamma(h)\Gamma(d_1+d_2+h)\Gamma(d_1+g)\Gamma(d_2+g)}{\Gamma(d_1+h)\Gamma(d_2+h)\Gamma(g)\Gamma(d_1+d_2+g)}$$

$$= \sum_{k=0}^{\infty} \frac{(g-h)_k (g)_{2k} (-d_1)_k (-d_2)_k}{(g+k-1)_k (d_1+g)_k (d_2+g)_k\, k!} = \sum_{k=0}^{\infty} \frac{(g-h)_k (-d_1)_k (-d_2)_k}{(h)_k (1-g-d_1-d_2)_k\, k!}$$

since $d_j = z_j \dfrac{\partial}{\partial z_j}$, j=1,2.

We now recall here the following multivariable analogues of the Burchnall–Chaundy symbolic operators $\nabla_{z_1 z_2}(h)$ and $\Delta_{z_1 z_2}(h)$ defined by (2.1) and (2.2), respectively (cf. [6]; see also [15] for the case when $r=3$):

(2.4) $$\nabla_{z_1:z_2 z_3}(h) = \frac{\Gamma(h)\Gamma(d_1+d_2+d_3+h)}{\Gamma(d_1+h)\Gamma(d_2+d_3+h)} = \sum_{n_2,n_3=0}^{\infty} \frac{(-d_1)_{n_2+n_3} (-d_2)_{n_2} (-d_3)_{n_3}}{(h)_{n_2+n_3}\, n_2!\, n_3!}$$

since $d_j = z_j \dfrac{\partial}{\partial z_j}$, j=1,2,3 and

(2.5) $$\Delta_{z_1:z_2 z_3}(h) = \frac{\Gamma(d_1+h)\Gamma(d_2+d_3+h)}{\Gamma(h)\Gamma(d_1+d_2+d_3+h)} = \sum_{n_2,n_3=0}^{\infty} \frac{(-d_1)_{n_2+n_3} (-d_2)_{n_2} (-d_3)_{n_3}}{(1-h-d_1-d_2-d_3)_{n_2+n_3}\, n_2!\, n_3!}$$

$$= \sum_{n_2,n_3=0}^{\infty} \frac{(-1)^{n_2+n_3}}{n_2!\, n_3!} \frac{(h)_{2(n_2+n_3)} (-d_2)_{n_2} (-d_3)_{n_3}}{(h+n_2+n_3-1)_{n_2+n_3}\, n_2!\, n_3!} \frac{(-d_1)_{n_2+n_3} (-d_2)_{n_2} (-d_3)_{n_3}}{(d_1+h)_{n_2+n_3} (d_2+d_3+h)_{n_2+n_3}}$$

since $d_j = z_j \dfrac{\partial}{\partial z_j}$, j=1,2,3.

where we have applied such known multiple hypergeometric summation formulas as (cf. [8,1])

$$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$

$$= \sum_{n_1,n_2,n_3} \frac{(\alpha)_{n_1+n_3} (\beta)_{n_1+n_2} (\beta')_{n_2+n_3}}{(\gamma)_{n_1} (\gamma')_{n_2+n_3}} \frac{(-z_1)^{n_1}}{n_1!} \frac{(-z_2)^{n_2}}{n_2!} \frac{(-z_3)^{n_3}}{n_3!}$$

since

$$R(\gamma+\gamma'-\alpha-\beta-\beta') > 0 \quad ; \quad \left(\max\{|z_1|,|z_2|,|z_3|\} < 1\right)$$

III. Some Operators For H_A - Hypergeometric Functions.

By applying the pairs of symbolic operators in (2.1) to (2.5), we find the following set of operator identities involving the Gauss function ${}_2F_1$ and $H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$ defined by (1.2):

(3.1) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\,{}_2F_1(\alpha,\beta;\gamma;-z_1)F_1(\beta',\beta,\alpha;\gamma';-z_2,-z_3)$$

(3.2) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\nabla_{z_2z_3}(\gamma')\,{}_2F_1(\alpha,\beta;\gamma;-z_1)F_2(\beta',\beta,\alpha;\gamma',\gamma';-z_2,-z_3)$$

(3.3) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\nabla_{z_2z_3}(\beta')\,{}_2F_1(\alpha,\beta;\gamma;-z_1)F_3(\beta,\beta',\beta',\alpha;\gamma';-z_2,-z_3)$$

(3.4) $$H_A(\alpha,\alpha,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$
$$=\nabla_{z_1z_2}(\alpha)\nabla_{z_1z_3}(\alpha)\nabla_{z_2z_3}(\alpha)\nabla_{z_2z_3}(\gamma')\,{}_2F_1(\alpha,\alpha;\gamma';-z_1)F_4(\alpha,\beta';\gamma',\gamma';-z_2,-z_3)$$

(3.5) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$
$$\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\nabla_{z_2z_3}(\beta')\nabla_{z_2z_3}(\gamma')\,{}_2F_1(\alpha,\beta;\gamma;-z_1)\,{}_2F_1(\beta,\beta';\gamma';-z_2)\,{}_2F_1(\alpha,\beta';\gamma';-z_3)$$

IV. Decompositions For H_A - Hypergeometric Functions.

Using of the principle of superposition of operators, from the operator identities (3.1) to (3.5) we can derive the following decomposition formulas for hypergeometric functions H_A :

(4.1) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$
$$\cdot\sum_{n_1,n_2}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1+n_2}(\beta')_{n_1+n_2}}{(\gamma)_{n_1+n_2}(\gamma')_{n_1+n_2}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2}(-z_3)^{n_1}}{n_1!\,n_2!}$$
$$\cdot\,{}_2F_1(\alpha+n_1+n_2,\beta+n_1+n_2;\gamma+n_1+n_2;-z_1)$$

$$.\,F_1(\beta'+n_1+n_2,\beta+n_1+n_2,\alpha+n_1;\gamma'+n_1+n_2;-z_2,-z_3)$$

(4.2) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3}(-1)^{-n_3}\frac{(\gamma')_{2n_3}(\alpha)_{n_1+n_2+n_3}(\beta)_{n_1+n_2+n_3}(\beta')_{n_1+n_2+2n_3}}{(\gamma'+n_3-1)_{n_3}(\gamma)_{n_1+n_2}(\gamma')_{n_1+2n_3}(\gamma')_{n_2+2n_3}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2+n_3}(-z_3)^{n_1+n_3}}{n_1!\,n_2!\,n_3!}$$

$$\cdot\,{}_2F_1(\alpha+n_1+n_2+n_3,\beta+n_1+n_2+n_3;\gamma+n_1+n_2;-z_1)$$

$$.\,F_2(\beta'+n_1+n_2+2n_3,\beta+n_2+n_3,\alpha+n_1+n_2+n_3;\gamma'+n_2+2n_3,\gamma'+n_1+2n_3;-z_2,-z_3)$$

$$(4.3)\qquad H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3}\frac{(\alpha)_{n_1+n_2+n_3}(\beta)_{n_1+n_2+n_3}(\beta')_{n_1+n_2}(\beta')_{n_1+n_3}}{(\beta')_{n_1}(\gamma)_{n_2+n_3}(\gamma')_{2n_1+n_2+n_3}}\frac{(-z_1)^{n_2+n_3}(-z_2)^{n_1+n_2}(-z_3)^{n_1+n_3}}{n_1!n_2!n_3!}$$

$$\cdot\,{}_2F_1(\alpha+n_1+n_2+n_3,\beta+n_1+n_2+n_3;\gamma+n_2+n_3;-z_1)$$

$$.\,F_3(\beta+n_1+n_2,\beta'+n_1+n_3,\beta'+n_1+n_2,\alpha+n_1+n_2+n_3;\gamma'+2n_1+n_2+n_3;-z_2,-z_3)$$

$$(4.4)\qquad H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2,n_3}\frac{(\alpha)_{n_1+n_2}(\alpha)_{n_1+n_3}(\beta)_{n_1+n_2+n_3}(\beta')_{n_1+n_2+n_3}(\gamma'-\beta')_{n_3}}{(\gamma'+n_1+n_2+n_3-1)_{n_3}(\alpha)_{n_1}(\gamma)_{n_1+n_2}(\gamma')_{n_1+n_2+2n_3}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2+n_3}(-z_3)^{n_1+n_3}}{n_1!n_2!n_3!}.$$

$$\cdot\,{}_2F_1(\alpha+n_1+n_2,\beta+n_1+n_2;\gamma+n_1+n_2;-z_1)$$

$$\cdot\,{}_2F_1(\beta'+n_1+n_2+n_3,\beta+n_1+n_2+n_3;\gamma'+n_1+n_2+2n_3;-z_2)$$

$$\cdot\,{}_2F_1(\beta'+n_1+n_2+n_3,\alpha+n_1+n_3;\gamma'+n_1+n_2+2n_3;-z_3)$$

$$(4.5)\qquad H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1+n_2}(\beta')_{n_1+n_2}}{(\gamma)_{n_1+n_2}(\gamma')_{n_1+n_2}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_1+n_2}}{n_1!n_2!}$$

$$\cdot\,{}_2F_1(\alpha+n_1+n_2,\beta+n_1+n_2;\gamma+n_1+n_2;-z_1)$$

$$\cdot\,{}_2F_1(\beta'+n_1+n_2,\alpha+\beta+2n_1+n_2;\gamma'+n_1+n_2;-z_2)$$

Now we shall use apply superposition's of operators for Hypergeometric function, for instance, we consider decomposition (3.3).
It's easy to see, that equality takes place Decomposition (3.3) can be proved by means of equality

$$(4.6)\qquad \nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\nabla_{z_2z_3}(\beta')=$$

$$.\frac{1}{(\beta')_j\,(\beta')_k\,(\alpha)_k}\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\beta')_{n+n_2}(\beta')_{p+n_1}(\alpha)_{p+n_1}(-\delta_1)_{n_1+n_2}(-\delta_2)_{n_1+n_3}(-\delta_3)_{n_2+n_3}}{(\alpha)_{n_1+n_2}(\beta)_{n_1}(\beta')_{n_1+n_2+n_3}n_1!n_2!n_3!}$$

Taking into account the identities (4.6), from parity (3.3), we have

(4.7) $H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$

$$\sum_{n_1,n_2,n_3=0}^{\infty}\frac{(\beta')_{n_2}(\beta')_{n_1}(\alpha)_{n_1}(-d_1)_{n_1+n_2}(-d_2)_{n_1+n_3}(-d_3)_{n_2+n_3}}{(\alpha)_{n_1+n_2}(\beta)_{n_1}(\beta')_{n_1+n_2+n_3}n_1!n_2!n_3!}$$

$$.F(\alpha,\beta;\gamma;-z_1)F_3(\beta,\beta'+n_1,\beta'+n_2,\alpha+n_1;\gamma';-z_2,-z_3)$$

By virtue of the formula:

(5.8) $(\delta+a)(\delta+a+1).......(\delta+a+r-1)f(\xi)=\xi^{1-a}\frac{d^r}{d\xi^r}\left[\xi^{a+r-1}f(\xi)\right],$

where f (x) - analytic function, we find that

$$(-\delta)_r f(\xi)=(-1)^r\xi^r\frac{d^r}{d\xi^r}f(\xi).$$

we have

(4.9) $(-d_1)_{n_1+n_2}F(\alpha,\beta;\gamma;-z_1)=$

$$(-1)^{n_1+n_2}(-z_1)^{n_3+n_4}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1+n_2}(\alpha_3)^2_{n_1}}{(\gamma)_{n_1+n_2}}(-z_1)^{n_1+n_2}$$

$$.F(\alpha+n_1+n_2,\beta+n_1+n_2;\gamma+n_1+n_2;(-z_1))$$

and

(4.10) $(-d_2)_{n_1+n_3}(-d_3)_{n_2+n_4}F_3(\beta,\beta'+n_1,\beta'+n_2,\alpha_1+n_1;\gamma';(-z_2),(-z_3))$

$$=(-1)^{n_1+n_2}(-z_2)^{n_1+n_3}(-z_3)^{n_2+n_3}\frac{(\alpha_1)_{n_1+n_2+n_3}(\beta)_{n_1+n_3}(\beta')^2_{n_1+n_2+n_3}}{(\alpha)_{n_1}(\beta)_{n_2}(\beta')_{n_1}(\gamma')_{n_1+n_2+2n_3}}$$

$$.F_3\begin{pmatrix}\beta+n_1+n_3,\beta'+n_1+n_2+n_3,\beta'+n_1+n_2+n_3,\alpha+n_1+n_2+n_3;\\ \gamma'+n_1+n_2+2n_3;(-z_2),(-z_3)\end{pmatrix}$$

Substituting identities (4.9) and (4.10) into equality (4.7), we get

$$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$

$$=\nabla_{z_1z_3}(\alpha)\nabla_{z_1z_2}(\beta)\nabla_{z_2z_3}(\beta')\,{}_2F_1(\alpha,\beta;\gamma;-z_1)F_3(\beta,\beta',\beta',\alpha;\gamma';-z_2,-z_3)$$

Our operational derivations of the decomposition formulas (4.1) to (4.5) would indeed run parallel to those presented in the earlier works which we have already cited in the preceding sections.

V. Alternative Derivations Of The Above Decomposition Formulas

First of all, we prove the decomposition formula (4.1) with the help of the following known integral representation for H_A [13] :

(5.1) $H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}$

$$.\int_0^1\int_0^1\xi^{\beta-1}\eta^{\beta'-1}(1-\xi)^{\gamma-\beta-1}(1-\eta)^{\gamma'-\beta'-1}(1+z_2\eta)^{\alpha-\beta}\left[(1+z_2\eta)(1+z_3\eta)+z_1\xi\right]^{-\alpha}d\xi\, d\eta$$

since

$$R(\gamma)>R(\beta)>0;\ R(\gamma')>R(\beta')>0.$$

Now,

(5.2) $\left[(1+z_2\eta)(1+z_3\eta)+z_1\xi\right]^{-\alpha}=$

$$\left[(1+z_1\xi)(1+z_2\eta)(1+z_3\eta)\right]^{-\alpha}\sum_{n_1,n_2=0}^{\infty}\frac{(\alpha)_{n_1+n_2}}{n_1!\,n_2!}\sigma_1^{n_1}\sigma_2^{n_2}$$

since

$$\sigma_1=\frac{(-z_1)(-z_3)\,\xi\,\eta}{(1+z_1\xi)(1+z_2\eta)(1+z_3\eta)}\qquad;\qquad\sigma_2=\frac{(-z_1)(-z_3)\,\xi\,\eta}{(1+z_1\xi)(1+z_2\eta)}$$

By substituting from (5.2) into the integral representation (5.1), we find that

$$(5.3)\qquad H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=\sum_{n_1,n_2,n_3}\frac{(\alpha)_{n_1+n_2}}{n_1!\,n_2!}(-z_1)^{n_1+n_2}(-z_2)^{n_2}(-z_3)^{n_1}$$

$$.\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1\xi^{\beta+n_1+n_2-1}(1-\xi)^{\gamma-\beta-1}(1+z_1\xi)^{-\alpha-n_1-n_2}\,d\xi$$

$$.\frac{\Gamma(\gamma')}{\Gamma(\beta')\Gamma(\gamma'-\beta')}\int_0^1\eta^{\beta'+n_1+n_2-1}(1-\eta)^{\gamma'-\beta'-1}(1+z_2\eta)^{-\beta-n_1-n_2}(1+z_3\eta)^{-\alpha-n_1}\,d\eta.$$

From the above we get

$$(5.4)\qquad\int_0^1\xi^{\beta}(1-\xi)^{\gamma-\beta-1}(1+z_1\xi)^{-\alpha}\,d\xi=\frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\,{}_2F_1(\alpha,\beta;\gamma;-z_1)$$

since $\quad R(\gamma)>R(\beta)>0\quad$ and

$$(5.5)\qquad\int_0^1\eta^{\alpha-1}(1-\eta)^{\gamma-\alpha-1}(1+z_2\eta)^{-\beta}(1+z_3\eta)^{-\beta'}\,d\eta$$

$$=\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F_1(\alpha,\beta,\beta';\gamma;-z_2,-z_3)$$

since $\quad R(\gamma)>R(\alpha)>0$

Therefore the decomposition formula (4.1) is

$$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=$$

$$\sum_{n_1,n_2}\frac{(\alpha)_{n_1+n_2}(\beta)_{n_1+n_2}(\beta')_{n_1+n_2}}{(\gamma)_{n_1+n_2}(\gamma')_{n_1+n_2}}\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2}(-z_3)^{n_1}}{n_1!\,n_2!}$$

$$.\,{}_2F_1(\alpha+n_1+n_2,\beta+n_1+n_2;\gamma+n_1+n_2;-z_1)$$

$$.\,F_1(\beta'+n_1+n_2,\beta+n_1+n_2,\alpha+n_1;\gamma'+n_1+n_2;-z_2,-z_3)$$

VI. Integral Representations Decomposition Formulas

For hypergeometric function H_A, Srivastava [14,15] gave several ordinary as well as contour integral representations of the Eulerian, Laplace, Mellin– Barnes, and Pochhammer's double-loop types. Here, in this section, we first observe that several known integral representations of the Eulerian type can be deduced also from the corresponding decomposition formulas of Section 4, (see [14]) .

(6.1) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}$$

$$\cdot\int_0^1\int_0^1 \xi^{\beta-1}\eta^{\beta'-1}(1-\xi)^{\gamma-\beta-1}(1-\eta)^{\gamma'-\beta'-1}(1+z_2\eta)^{-\beta}(1+z_1\xi+z_3\eta)^{-\alpha}$$

$$\cdot\left(1-\frac{(-z_1)(-z_2)\,\xi\,\eta}{(1+z_2\eta)(1+z_1\xi+z_3\eta)}\right)^{-\alpha} d\xi\, d\eta$$

since

$$R(\gamma)>R(\beta)>0;\ R(\gamma')>R(\beta')>0,$$

Srivastava [14] deduced from his single-integral representation:

(6.2) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$

$$=\frac{\Gamma(\gamma')}{\Gamma(\beta')\Gamma(\gamma'-\beta')}\int_0^1 \eta^{\beta'-1}(1-\eta)^{\gamma'-\beta'-1}(1+z_2\eta)^{-\beta}(1+z_3\eta)^{-\alpha}$$

$$\cdot\ {}_2F_1\left(\alpha,\beta;\gamma;\frac{(-z_1)}{(1+z_2\eta)(1+z_3\eta)}\right)d\eta$$

since

$$R(\gamma')>R(\beta')>0,$$

Next we turn to a set of known double-integral representations of the Laplace type for H_A, each of which was derived by Srivastava [15] from the following rather elementary formula :

(6.3) $$(\lambda)_n=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}t^{\lambda+n-1}dt$$

since

$$R(\lambda)>0;\quad n\in N_0,$$

(6.4) $$H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty e^{-s-t}t^{\alpha-1}s^{\beta-1}\ {}_0F_1(-;\gamma;(-z_1)s\,t)\ {}_1F_1(\beta';\gamma';(-z_2)s+(-z_2)t)ds\,dt$$

since

$$\min\{R(\alpha),R(\beta)\}>0;\ \max\{R(-z_2),R(-z_3)\}<1,$$

which, in view of the elementary integral formula:

(6.5) $${}_1F_1(\lambda;\mu;(-z_3))=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-t}t^{\lambda-1}\ {}_0F_1(-;\mu;(-z_3)t)dt$$

since $\Re(\lambda)>0$,

immediately yields the following triple-integral representation of the Laplace type for H_A :

$$(6.6)\qquad H_A(\alpha,\beta,\beta';\gamma,\gamma';-z_1,-z_2,-z_3)$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta')}\int_0^\infty\int_0^\infty\int_0^\infty e^{-s-t-u}t^{\alpha-1}s^{\beta-1}u^{\beta'-1}$$

$$\cdot\ {}_0F_1(-;\gamma;\ (-z_1)s\ t)\ {}_0F_1(-;\gamma';\ (-z_2)u\ s+(-z_3)u\ t)ds\ dt\ du$$

since

$$\min\{R(\alpha),R(\beta),R(\beta')\}>0\ ,$$

In each of the integral representations presented in this as well as the preceding sections, it is tacitly assumed that both sides of the result exist.

VII. Concluding Remarks And Observations

By suitably specializing the decomposition formulas (4.1) to (4.5), we can deduce a number of decomposition formulas including those given by Burchnall and Chaundy [1,2]. For instance, we find the following results:

$$(7.1)\qquad F_1(\alpha,\beta;\gamma;-z_1,-z_2)=\sum_{i,j=0}^{\infty}\frac{(\alpha)_{2i+2j}(\beta)_{i+j}(\gamma)_{2i}}{(\gamma+i-1)_i\left[(\gamma)_{2i+j}\right]^2}\frac{(-z_1)^{i+j}(-z_2)^{i+j}}{i!\,j!}$$

$$.F_4(\alpha+2i+2j,\beta+i+j;\gamma+2i+j,\gamma+2i+j;-z_1,-z_2)$$

and

$$(7.2)\qquad F_1(\alpha,\beta,\beta';\gamma,\gamma;-z_1,-z_2)=\sum_{i,j=0}^{\infty}\frac{(\alpha)_{2i+j}(\beta)_{i+j}(\beta')_{i+j}}{(\gamma)_i(\gamma)_{2i+2j}}\frac{(-z_1)^{i+j}(-z_2)^{i+j}}{i!\,j!}$$

$$.F_3(\alpha+2i+j,\alpha+2i+j,\beta+i+j,\beta'+i+j;\gamma+2i+j;-z_1,-z_2)$$

Furthermore, by making use of the decompositions (4.2), we can derive the following known reduction formulas for Srivastava's triple hypergeometric function H_A [14]:

$$(7.3)\qquad H_A(\alpha,\beta,\beta';\gamma,\beta';-z_1,-z_2,-z_3)$$

$$=(1+z_2)^{-\beta}(1+z_3)^{-\alpha}F_4\left(\alpha,\beta;\gamma,\beta';\frac{-z_1}{(1+z_2)(1+z_3)},\frac{(-z_2)(-z_3)}{(1+z_2)(1+z_3)}\right)$$

Some of the most recent contributions in the theory of Srivastava's H_A - hypergeometric series include a paper by Harold Exton [5] and a paper by Rathie and Kim [12].

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