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Solution of Fractional Order Boundary Value Problems Using Least-Square Method

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Abstract: *The main objective of this paper is to explain how to use the least square method to solve two-point boundary value problems of fractional order, in which three type boundary value problems are considered. The results of the proposed method show high accuracy and reliability in comparison with the exact solution if given.*

Keywords: *Fractional order boundary value problems, Boundary value problem, least-square method.*

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I. Introduction

The history of fractional calculus [1–4] returns to more than 300 years old, in which in the recent decades the applied scientists and the engineers realize that such fractional differential equations provides a better approach to describe the nature of complex phenomena's, such as, signal processing, systems identification, control, viscoelastic materials and polymers Although the origin of the subject dates back to almost a hundred years ago, recently, this subject has been broadly employed in the various fields of engineering and science. Because of the non-local property of the fractional derivative, it can be used to describe those complex systems which involve long-memory in time in a better way.

Based on these requirements, a numerical and approximate approach has become very desired to analyses the experimental data which are described in a fractional way. Some papers have discussed the numerical approaches for fractional calculus and fractional differential equations.

Fractional calculus basically deals with a generalization of the concept of the ordinary and partial derivative (or differentiation) and integration to an arbitrary order including a fractional order, therefor fractional calculus is that branch of mathematics with a long history. In earlier work, the main application of fractional calculus has been as a technique for solving integral equations. This seems first to have many vague notions and poor defined concepts to the readers who are interested in this branch of mathematics. Recently, fractional differential and integral equations have many possible applications in areas like mechanics, laws, diffusion, processes and materials [1].

Many studies concerned with the existence and a uniqueness of solution of ordinary, partial integral and integro-differential equations fractional order, such as Bahuguna and Dabas [2], Hu et al. [3], Ibrahimin [4], Karthikeyan and Trujillo[5], Li et al.[6], Lizama and Pozo [7], Chalishajar and Karthikeyan [8], Chuong et al.[9],Nishimoto K.[10], Oldham K.B[11].

The main theme of this paper is to solve boundary value problems fractional order using the least-square method.

The general form of the consider problems is given by:

$$\mathrm{D}^{\mathrm{q}}\mathrm{y}(\mathrm{x}) = \mathrm{f}(\mathrm{x}, \mathrm{y}, \mathrm{D}^{\mathrm{p}}\mathrm{y}) \quad , \quad 0 < p \leq 1 \, , \; 1 < q \leq 2 \, , \qquad \ldots (1)$$

With boundary condition

$$y(a) = y_a \quad , \quad y(b) = y_b$$

Where D stands for Caputo fractional order derivatives.

This paper is organized as follows; in section 2 some basic and fundamental definitions of fractional calculus are presented for completers purpose.

In section 3, we describe the formulation of the least square method for solving two-points boundary value problems and then some illustrative examples are given in section 4.

II. Preliminaries

In this section, basic definitions and fundamental concepts of fractional differentiation and integration are given in order to following the vague notions, if exist.

The Riemann-Liouville fractional order integral, [6]:

Riemann-Liouville definition of fractional integral operator of order $\alpha \geq 0$ for a continuous function f is defined as:

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (t-s)^{\alpha-1} f(s) ds \quad , \alpha > 0, x > 0 \qquad \dots (2)$$

Where Γ is the well-known Euler's gamma function, defined by:

$\Gamma(x) = \int_0^x e^{-t} t^{x-1} dt \qquad , \quad x > 0$

The basic properties that are satisfied by Riemann-Liouville fractional order integration may be summarized for all $\alpha, \beta \geq 0$ and $\gamma \geq -1$ as follows:

1. $I^0 f(x) = f(x)$.
2. $I^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$
3. $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x) = I^{\alpha+\beta} f(x)$.

Caputo's definition of the fractional order derivatives, [6], [12]:

Caputo fractional derivative of order $\alpha \geq 0$ for a continuous function f is defined as follows:

$$D^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(x-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+1-m}} d\tau & , \quad m-1 < \alpha \leq m \\ \frac{d^m}{dx^m} f(x) & , \quad \alpha = m \end{cases} \qquad \dots (3)$$

The Caputo fractional order derivative satisfies the following properties:

1. $D^{\alpha} c = 0$, where c is a constant.
2. $D^{\alpha} x^{\beta} = 0$, for $\beta \leq \alpha - 1$, $\alpha, \beta \in \mathbb{R}^+$
3. $D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$, for $\beta > \alpha - 1$, $\alpha, \beta \in \mathbb{R}^+$.
4. $D^{\alpha} I^{\alpha} f(x) = f(x)$.
5. $I^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}$, where $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$

III. The Least Square Method (LSM). for solving BVPs of fractional order.

The least square method **[11]** is one of the classical powerful methods that may be used to solve several types of problems in linear algebra, numerical analysis, integral equation, ordinary and partial differential equations. This method depends on the normed space of the solutions of the problem under consideration.

Two types of least-square methods may be considered depending in the problem that have to be solved, namely discrete and continues least square methods.

3.1 Continuous Least Square Approximation.

Now we turn our attention to what is probably the most widely used type of approximation called least square, there are a great many variations on this approach and we will explore just some of them.

Initially, we concentrate on continuous of least square approximation in which we seek to approximation f, in the simplest case we seek polynomial approximation from Π_n, that is to find $y_n(x) \in \Pi_n$.

Now consider the problem of solving the two-points BVP (1) of fractional order and to find $p \in \Pi_n$, which minimizes:

$$\|f - p\|_2^2 = \int_a^b (f(x, y, D^p y) - D^q y_n(x))^2 dx \qquad \dots (4)$$

The procedure is started by writing y(x) in its standard from as:

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad \dots (5)$$

and it will follows that we seek coefficients $a_0, a_1, a_2, \dots, a_n$ such that

$E(a_0, a_1, a_2, \dots, a_n) = \int_a^b (f(x, y, D^p y) - D^q(y))^2 dx$

$= \int_a^b (f(x, (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n), D^p(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)) - D^q(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n))^2 dx$

$= \int_a^b (f(x, (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n), D^p a_0 + D^p a_1 x + D^p a_2 x^2 + \cdots + D^p a_n x^n)) - D^q a_0 - D^q a_1 x - D^q a_2 x^2 - \dots - D^q a_n x^n)^2 dx \qquad \dots (6)$

Since $D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}$, for $\beta > \alpha - 1$
is minimized.
Now;

$$\frac{\partial E}{\partial a_i} = -2\int_a^b x^i(f(x, y, D^p y) - D^q(y))dx$$

$$= -2\int_a^b x^i(f(x, a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, D^p(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)) - D^q(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n))dx$$

$$= -2\int_a^b x^i(f(x, a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, D^pa_0 + D^pa_1x + D^pa_2x^2 + \cdots + D^pa_nx^n)) - D^q a_0 - D^q a_1x - D^q a_2x^2 - \ldots - D^q a_nx^n)dx \quad \ldots (7)$$

for each i so E will be minimized when $\frac{\partial E}{\partial a_i} = 0$ for every i. Rearranging (7), the following normal equations will be obtained.

$$a_0\int_a^b x^i dx + a_1\int_a^b x^{i+1}dx + \cdots + a_n\int_a^b x^{i+n}dx = \int_a^b x^i f(x, a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, D^p(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n))dx \quad \ldots (8)$$

Or

$$Xa = f \quad \ldots (9)$$

Where the matrix X and right – hand – side vector f have the following entries

$$x_{ij} = \int_a^b x^{i+j-2}dx \qquad f_i = \int_a^b x^{i-1}f(x, y, D^py)dx \quad \text{, for all i, i=1, 2,…,n} \quad \ldots (10)$$

Again the problem is reduced to the solution of a system of linear equations.

3.2 Discrete least Square Approximation.

In this subsection we continue with the theme of using the least square approximations but in the special context where only data at a discrete set of points is available. The simplest form of this problem was outlined in the introduction to this section where we again found a system of normal equation, we concentrate here on polynomial approximations. The problem then is to find a polynomial $y_n(x) \in \Pi_n$ of degree at most n, which minimizes the weighted discrete sum of square errors.
Now consider the problem of solving the following two BVP (1) of fractional order:

$$\sum_{i=0}^{N} E(D^q(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) - f(x, a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, D^p(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)))^2 = \sum_{i=0}^{N} E_i(f(x, a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, D^p(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n)) - a_0 - a_1x_i - a_2x_i - \ldots - a_nx_i^n)^2 \quad \ldots (11)$$

Where $x_0, x_1, \ldots, x_N$ are the data point and $E_0, E_1, \ldots, E_N$ the associated positive weights it is assumed that $N > n$ since otherwise this error measure can force to zero by using interpolation polynomial.

In practice it is often the case that the number of data points is significantly greater than the degree of the approximation sought.

As for the continuous case, we can differentiate the sum (11) with respect to each of the coefficients to obtain the normal equations

$$a_0\sum_{i=0}^{N} E_ix_i^j + a_1\sum_{i=0}^{N} E_ix_i^{j+1} + \cdots + a_n\sum_{i=0}^{N} E_ix_i^{j+n} = \sum_{i=0}^{N} E_ix_i^j f(x, y, D^py) \quad \ldots (12)$$

Where (j=0,1, …, n).
Which may be written in matrix form as:

$$Xa = f \quad \ldots (13)$$

IV. Numerical simulation

To demonstrate the effectiveness of the proposed method, we consider several examples for linear and non-linear BVPs of fractional order.

Example (1):

Consider the linear fractional order BVP:

$$D^{1.5}y + D^{1/3}y + y = \frac{8(\sqrt{x-1})(2x-1)}{3\sqrt{\pi}} + \frac{6(x-1)^{\frac{2}{3}}(3x+2)}{5\Gamma(\frac{2}{3})} + 1 + 2x^2 \quad , \ x \in [1,2] \quad \ldots (14)$$

With boundary condition

y (1) =3 , y (2) =9

For comparison purpose the exact solution is given by:

$$y(x) = 1 + 2x^2$$

Hence the approximate solution using the least-square method is taken by:

$$y_n(x) = c_0 + c_1x + c_2x^2$$

where c_0, c_1, c_2 are constants to be determined, hence, the problem now is reduced to find the coefficients c0, c1, c2. A necessary condition for the coefficient c0, c1, c2 which minimizes E, the results are found to be residual error:

$$c_0 = 1 \ , \ c_1 = 6.935(10)^{-6} \quad , \quad c_2 = 2$$

Table (1)

Comparison between the exact and approximate solution of the BVP (14)

Where q=1.5 , p=1/3

x	Exact solution	approximate solution
0	3	2.99999731
0.1	3.42	3.41999758
0.2	3.88	3.87999781
0.3	4.38	4.37999799
0.4	4.92	4.91999814
0.5	5.5	5.49999825
0.6	6.12	6.11999831
0.7	6.78	6.77999834
0.8	7.48	7.47999833
0.9	8.22	8.21999827
1	9	8.99999818

Example (2):

Consider the linear fractional order BVP:

$$y'' + D^{1/2} \ y + y = 4x + \frac{6(x-1)^{\frac{2}{3}}(3x+2)}{5\Gamma(\frac{2}{3})} + 1 + 2x^2 \quad , \quad x \in [1,2] \qquad \dots (15)$$

With boundary condition

$$y(1) = 3 \quad , \quad y(2) = 9$$

For comparison purpose the exact solution is given by:

$$y(x) = 1 + 2x^2$$

Hence the approximate solution using the least-square method is taken by:

$$y_n(x) = c_0 + c_1 x + c_2 x^2$$

where c_0, c_1, c_2 are constants to be determined, hence, the problem now is reduced to find the coefficients c0, c1, c2. A necessary condition for the coefficient c0, c1, c2 which minimizes E, the results are found to be residual error:

$$c_0 = 1 \ , \ c_1 = 6.361(10)^{-7} \quad , \quad c_2 = 2$$

Table (2)

Comparison between the exact and approximate solution of the BVP (15)

Where q=1/2

X	Exact solution	Approximate solution
0	3	3.009456
0.1	3.42	3.430402
0.2	3.88	3.891347
0.3	4.38	4.392293
0.4	4.92	4.933238
0.5	5.5	5.514184
0.6	6.12	6.13513
0.7	6.78	6.796075
0.8	7.48	7.497021
0.9	8.22	8.237966
1	9	9.018912

Example (3):

Consider the non-linear fractional order BVP:

$$D^{1.5}y = yy^{1/3} + 1.50455561273500985\sqrt{x-1}(2x+1) - (1+2x^2)\frac{6(x-1)^{\frac{2}{3}}(3x+2)}{5\Gamma\left(\frac{2}{3}\right)} \ , x \in [1,2] \quad \dots (16)$$

With boundary condition

$$y(1)=3 \quad , \quad y(2)=9$$

For comparison purpose the exact solution is given by:

$$y(x) = 1 + 2x^2$$

Hence the approximate solution using the least-square method is taken by:

$$y_n(x) = c_0 + c_1 x + c_2 x^2$$

where c_0, c_1, c_2 are constants to be determined, hence, the problem now is reduced to find the coefficients c0, c1, c2. A necessary condition for the coefficient c0, c1, c2 which minimizes E, the results are found to be residual error:

$$c_0 = 0.817 \quad , \quad c_1 = 0.142 \quad , \quad c_2 = 1.965$$

Table (3)

Comparison between the exact and approximate solution of the BVP (16)

Where q=1.5 , p=1/3

Table (3)

X	Exact solution	Approximate solution
0	3	2.923077
0.1	3.42	3.349847
0.2	3.88	3.815913
0.3	4.38	4.321273
0.4	4.92	4.865929
0.5	5.5	5.44988
0.6	6.12	6.073126
0.7	6.78	6.735667
0.8	7.48	7.437504
0.9	8.22	8.178635
1	9	8.959062

Example (4):

Consider the non-linear fractional order BVP given in [14].

$$D^2 y + D^{3/2} y + y = x^2 + 2 + 4\sqrt{\frac{x}{\pi}} \qquad x \in [0,5] \qquad \dots (17)$$

with boundary condition

$$y(0) = 0 \quad , \quad y(5) = 25$$

For comparison purpose the exact solution is given by:

$$y(x) = x^2$$

Figures (1) and (2) present the results of the exact and approximate solutions of equation(17) and the comparison with the results obtained upon using the collocation shooting method for solving fractional boundary value problems of equation (17).

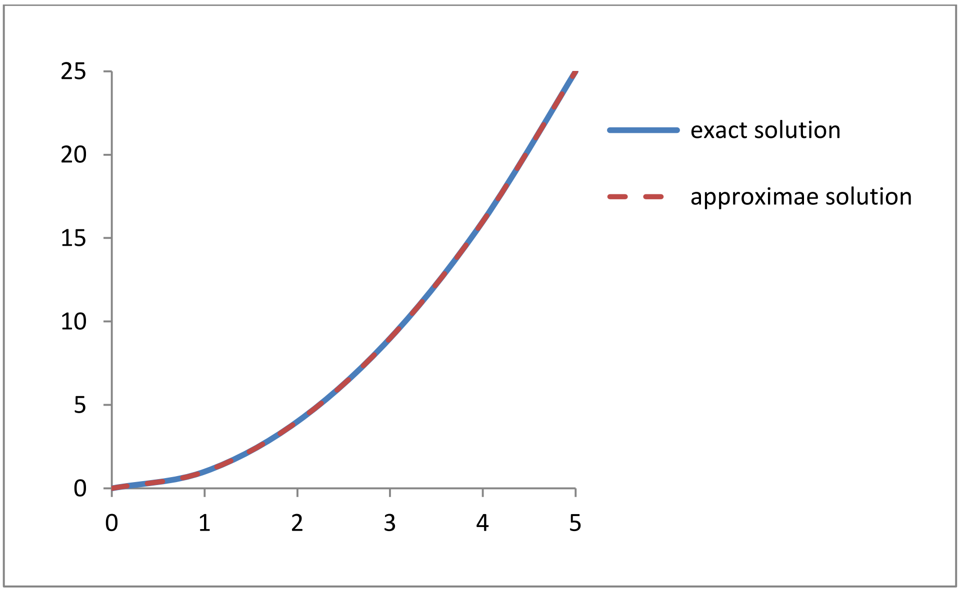

Figure (1)

The approximate solution for BVP (17) by used the least square method given in [14].

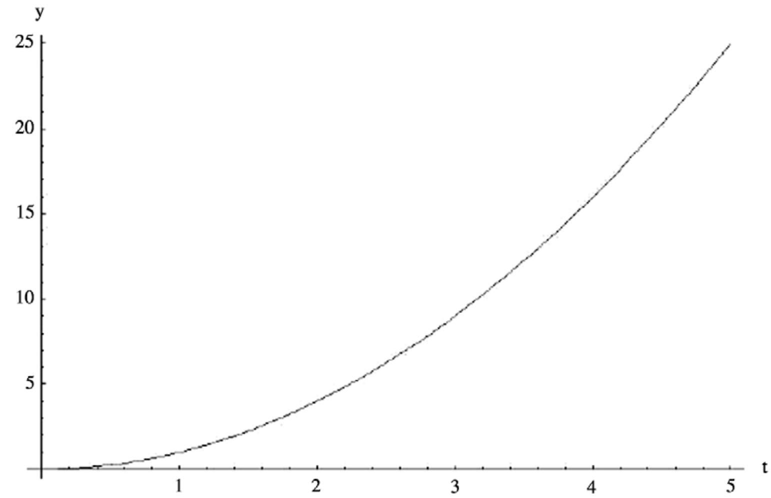

Figure (2)

Also approximate solution for BVP (17), by collocation shooting method given in [14].

The comparisons error between the exact and approximate solution for BVP (17) using the absolute error

Table (4)

X	$\lvert y(x)_{LSME} - y(x)_{exact} \rvert$	$\lvert y(x)_{Shooting} - y(x)_{exact} \rvert$ [14]
0	1.24469910^{-6}	3.7810^{-12}
1	316.47597010^{-9}	3.6110^{-12}
2	369.69119010^{-9}	2.9410^{-12}
3	813.80238910^{-9}	7.2210^{-12}
4	1.01585810^{-6}	1.1010^{-12}
5	975.85689910^{-9}	2.5510^{-12}

V. Conclusions

The obtained results from the consider test examples show the simplicity very high accuracy and the powerful approach for solving two-points fractional order BVPs. Therefore this method and its improvement may be recommended to solve other type of problems in fractional calculus and fractional order ODEs.

References

[1] M. Kurulay, A. Secer, Int. J. of Computer Science and Emerging Technologies 2(1), 18-20 (2011)..
[2] D.Bahuguna, J. Dabas, EJQTDE, 4, 1-12 (2008).
[3] L. Hu,Y. Ren and R. Sakthivel, In Semigroup Forum (Vol. 79, No. 3, pp. 507-514). Springer-Verlag,(2009).
[4] R. W. Ibrahim, ANZIAM Journal 52, 1-21 (2010).
[5] K. Karthikeyan, J.J. Trujillo, Communications in Nonlinear Science and Numerical Simulation 17(11), 4037-4043 (2012).
[6] F. Li, J. Liang and H. K. Xu, Journal of Mathematical Analysis and Applications 391(2), 510-525 (2012).
[7] C. Lizama and J. C. Pozo, Abstract and Applied Analysis 2012, 507-514 (2012).
[8] D. N. Chalishajar and K. Karthikeyan, Acta Mathematica Scientia 33(3), 758-772 (2013).
[9] N. M.Chuong, T. D. Ke and N. N. Quan, Journal of Integral Equations and Applications 26(2), 145-170 (2014).
[10] Nishimoto K., "Fractional Calculus: Integrations and Differentiations of Arbitrary Order", Descartes Press Co. Koriyama Japan, 1983.
[11] K. Diethelm, An improvement of a nonclassical numerical method for the computation of fractional derivatives, Numer. Algorithms 47 (2008) 190– 361.
[12] D.A. Murio, On the stable numerical evaluation of Caputo fractional derivatives, Comput. Math. Appl. 51 (2006) 1539–1550
[13] James L., and Peter R. T., "Numerical Methods And Analysis", , Delaware State University, 1992.
[14] Qasem M. Al-Mdallal, Muhammed I. Syam, M.N. Anwar , , Communications in Nonlinear Science and Numerical Simulation . December 2010.

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