IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 15, Issue 2 Ser. I (Mar – Apr 2019), PP 21-27
www.iosrjournals.org

A Study of Distortion Theorem and Inclusion Relations for a new class of Meromorphic Functions

Mr. Vidyadhar Sharma[1], Dr. Nisha Mathur[2]
[1](Research Scholar, *Department of mathematics,M .L.V. Government P G College, India)*
[2]*(Assistant Professor, Department of mathematics,M .L.V. Government P G College, India)*

Abstract:*By having used of differential subordination, it has been investigated in the present paper, subordination relations, inclusion relations, distortion theorem and inequality properties are discussed of the class*$\mathbb{MB}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\ell},\boldsymbol{\mu},\boldsymbol{A},\boldsymbol{B})$. *In this paper it has been introduced some new classes* $\mathbb{MB}(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{\ell},\boldsymbol{\mu},\boldsymbol{A},\boldsymbol{B})$ *of meromorphic functions which are defined by means a meromorphic function using a new operator.*

Date of Submission: 05-03-2019 Date of acceptance: 22-03-2019

I. Introduction

Let Ω_{m+1} be the class of function of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (1)$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z: z \in \mathbb{C} \text{ and } 0 < z < 1\} = \mathbb{U}\backslash\{0\}$

If $f(z)$and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows $f(z) \prec g(z)$or $f \prec g \quad (z \in \mathbb{U})$. If there exist a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in \mathbb{U})$ such that$f(z) = g(\omega(z)) \quad (z \in \mathbb{U})$. If the function $g(z)$ is univalent in $\mathbb{U}$, it has the following equivalence (cf., e.g., [11]; [12])

$$f(z) \prec g(z)(z \in \mathbb{U})$$
$$\Leftrightarrow f(0) = g(0) \text{and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

We define the convolution or Hadamard product of the functions $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} a_k b_k z^k = (g * f)(z) \ (p \in \mathbb{N}, z \in \mathbb{U} \quad (2)$$

Following the current work of Liu and Srivastava [7] see also [8], [9] for a function $f(z)$ in the class Ω_{m+1} given by (1), now it is defined the integral operator

$$Q_p^m(\ell,\lambda)f(z) = \frac{1}{z^p} + \sum_{k=p}^{\infty} \left[\frac{\ell}{\ell+\lambda(p+k)}\right]^m a_k z^k \quad \{\lambda\ell > 0\} \quad (3)$$

The above integral operator converts into the following operator with $p = 1$

$$Q_1^m(\ell,\lambda)f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left[\frac{\ell}{\ell+\lambda(1+k)}\right]^m a_k z^k \ , (\ell > 0, \lambda \geq 0, m \in N_0; z \in \mathbb{U}^*) \quad (4)$$

It is easily verified from (4)

$$\lambda(z)\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)' = \ell Q_1^m(\ell,\lambda)f(z) - (\lambda + \ell)Q_1^{m+1}(\ell,\lambda)f(z), (\lambda > 0) \quad (5)$$

Making use of the principle of differential subordination as well as the operator$Q_1^{m+1}(\ell,\lambda)$, now it is introduced a subclass of the function classΩ_{m+1} as follows.

II. Definition

Suppose that$\alpha \geq 0, \mu > 0, -1 \leq B \leq 1, A \neq B, A \in R$, we say that a function $f(z) \in \Omega_{m+1}$ is in the class $\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$ if is satisfies

$$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U}$$

In particular, we claim $\mathbb{MB}\ (\alpha,\lambda,\mu,-2\rho-1) \equiv \mathbb{MB}\ (\alpha,\lambda,\ell,\mu,\rho)$denote the subclass of $\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$ for $A = 1-2\rho, B = -1$ and $0 \leq \rho < 1$

It is clear that $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,\rho) \Leftrightarrow f(z) \in \Omega_{m+1}$ and satisfies$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} > \rho, (z \in \mathbb{U})$

In present study, it will discussed distortion theorem, inclusion relations, inequalities properties and subordination relations of the class $\mathbb{MB}(\alpha, \lambda, \ell, \mu, \mathrm{A}, \mathrm{B})$.

III. Preliminary Lemmas

In proving the chief results, the following lemma is applied:

Suppose the function $h(z)$ be analytic and convex in $\mathbb{U}$ with the condition $h(0) = 1$ and also let the function $\varphi(z)$ given by

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{6}$$

is analytic in $\mathbb{U}$. If

$$\varphi(z) + \frac{z\varphi'(z)}{\gamma} \prec h(z) \text{where } (z \in \mathbb{U}, \Re e\{\gamma\} \geq 0, \gamma \neq 0) \tag{7}$$

then, $\varphi(z) \prec \phi(z) = \frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1} h(t)dt \prec h(z)(z \in \mathbb{U}),$

and$\phi(z)$ is the best dominant of (7).

IV. Main Results

Theorem1. Let $\mu > 0, \alpha \geq 0, -1 \leq B \leq 1, A \in R, f(z) \in \mathbb{MB}(\alpha, \lambda, \ell, \mu, \mathrm{A}, \mathrm{B})$, then

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu \prec \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz}$$

Proof: Consider the function

$$\varphi(z) = \left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu, \ (z \in \mathbb{U}) \tag{8}$$

Now $\varphi(z)$ is analytic in $\mathbb{U}$ with $\varphi(0) = 1$

Differentiating (8) with respect to z, we have

$$\frac{\mu}{z} + \mu\frac{\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)'}{Q_1^{m+1}(\ell,\lambda)f(z)} = \frac{\varphi'(z)}{\varphi(z)}$$

Multiplying by λz and using the recurrence, we have

$$\lambda(z)\frac{\varphi'(z)}{\varphi(z)} = -\mu\ell + \frac{\mu\ell Q_1^m(\ell,\lambda)f(z)}{Q_1^{m+1}(\ell,\lambda)f(z)} \tag{9}$$

From equation (8) and (9) we obtain

$$\varphi(z) + z\frac{\lambda\alpha}{\mu\ell}\varphi'(z) = (1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1}$$

Since $f(z) \in \mathbb{MB}(\alpha, \lambda, \ell, \mu, \mathrm{A}, \mathrm{B})$ so $\varphi(z) + z\frac{\lambda\alpha}{\mu\ell}\varphi'(z) \prec \frac{1+Az}{1+Bz}$

Now using the lemma (6) with $\gamma = \frac{\mu\ell}{\alpha\lambda}$ we have

$$\varphi(z) \prec \phi(z) = \frac{\mu\ell}{\alpha\lambda} z^{\frac{\mu\ell}{\alpha\lambda}} \int_0^z . t^{\frac{\mu\ell}{\alpha\lambda}-1} dt$$

$$\text{or } \varphi(z) \prec \phi(z) = \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz} \tag{10}$$

Corollary1. : $\mu > 0, \alpha \geq 0, \rho \neq 1$. If

$$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1}$$

$\prec z\frac{[1+(1-2\rho)]}{1-z}(z \in \mathbb{U})$ (11)

then, $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu \prec \rho + (1-\rho)\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+zu}{1-zu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$

Proof: By the definition (6) and equation (10), we obtain

$$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} \prec \frac{1+Az}{1+Bz}$$

$$\Rightarrow zQ_1^m(\ell,\lambda)f(z) \prec \frac{1+Az}{1+Bz}$$

Taking $A = 1-2\rho, B = -1$ then we obtain

$$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^\mu + \alpha\left(zQ_1^m(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} \prec z\frac{[1+(1-2\rho)]}{1-z}$$

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} = \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+(1-2\rho)zu}{1-zu} u^{\frac{\mu\ell}{\alpha\lambda}-1}du = \rho+(1-\rho)f(z) \in \mathbb{MB}(\alpha_1,\lambda,\ell,\mu,\rho), (z\in\mathbb{U})$$

Corollary2. : Let $\mu > 0, \alpha \geq 0$ then $\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B) \subset \mathbb{MB}(0,\lambda,\ell,\mu,A,B)$

Proof**:** If $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$,

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz} \tag{12}$$

If $\alpha = 0$ then $f(z) \in \mathbb{MB}(0,\lambda,\ell,\mu,A,B)$, so

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz} \tag{13}$$

From equation (12) and (13), we have $\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B) \subset \mathbb{MB}(0,\lambda,\ell,\mu,A,B)$

Theorem2. Let $f(z) \in \mathbb{MB}(0,\lambda,\ell,\mu,\rho)$ for $(z\in\mathbb{U})$ then

$$f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,\rho) \text{for } |z| < R(\alpha,\mu,\ell), \text{ Where } \Re(\alpha,\mu,\ell) = \frac{\ell\mu}{\alpha+\sqrt{\alpha^2+\ell^2\mu^2}} \tag{14}$$

Proof: Setting $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} = (1-\rho)h(z)+\rho \quad (z\in\mathbb{U})$

Differentiating with respect to z, we get $\frac{\mu}{z} + \mu\frac{\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)'}{Q_1^{m+1}(\ell,\lambda)f(z)} = \frac{(1-\rho)h'(z)}{(1-\rho)h(z)+\rho}$

Multiplying by λz and using recurrence relation (5), we have

$$-\mu\ell + \mu\ell\,\frac{Q_1^{m}(\ell,\lambda)f(z)}{Q_1^{m+1}(\ell,\lambda)f(z)} = \frac{\lambda z(1-\rho)h'(z)}{(1-\rho)h(z)+\rho}$$

Again multiplying by α and using the definition of $h(z)$, we obtain

$$(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} - \rho = (1-\rho)\left\{h(z) + z\frac{\lambda\alpha}{\mu\ell}h'(z)\right\}$$

or

$$\frac{\left[(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} - \rho\right]}{(1-\rho)}$$

$$= \left\{h(z) + z\frac{\lambda\alpha}{\mu\ell}h'(z)\right\}$$

Using the known estimate, $|zh'(z)| \leq \frac{2r}{1-r^2}\Re\{h(z)\}, |z| = r < 1$

We have

$$\frac{(1-\alpha)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1} - \rho}{(1-\rho)}$$

$$= \Re e\left\{h(z) + z\frac{\lambda\alpha}{\mu\ell}h'(z)\right\} \geq \Re e\left\{h(z) - \frac{\lambda\alpha}{\mu\ell}|zh'(z)|\right\} \geq \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\,\{h(z)\}\left\{1 - \frac{2\alpha\lambda r}{\mu\ell(1-r^2)}\right\}$$

Since $kr^2 + 2rk - k < 0, r \in (\alpha,\beta)$

The right hand side of the inequality is positive if $r < \Re(\alpha,\mu,\ell)$ where $\Re(\alpha,\mu,\ell)$ is given (11) consequently it follows from (10) that $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,\rho)$ for $|z| < R(\alpha,\mu,\ell)$.

Theorem3. Let $0 \leq \alpha_2 \leq \alpha_1$. Then $\mathbb{MB}(\alpha_1,\lambda,\ell,\mu,A,B) \subset \mathbb{MB}(\alpha_2,\lambda,\ell,\mu,A,B)$

Proof: First suppose that $f(z) \in \mathbb{MB}(\alpha_1,\lambda,\ell,\mu,A,B)$ then by theorem1, we point out $f(z) \in \mathbb{MB}(0,\lambda,\ell,\mu,A,B)$ and also by the definition (14), we have

$$\left\{(1-\alpha_2)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha_2\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1}\right\} \prec \left\{\frac{1+Az}{1+Bz}\right\}(z\in\mathbb{U})$$

Multiplying by $\frac{\alpha_2}{\alpha_1}$, we have

$$\frac{\alpha_2}{\alpha_1}\left\{(1-\alpha_2)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} + \alpha_2\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1}\right\} \prec \frac{\alpha_2}{\alpha_1}\left\{\frac{1+Az}{1+Bz}\right\} \tag{15}$$

Since we know that

$$\left(1-\frac{\alpha_2}{\alpha_1}\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \left(1-\frac{\alpha_2}{\alpha_1}\right)\left(\frac{1+Az}{1+Bz}\right) \tag{16}$$

Now by the equation (6) and (7), we get

$$\frac{\alpha_2}{\alpha_1}\left\{(1-\alpha_2)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}+\alpha_2\left(zQ_1^{m}(\ell,\lambda)f(z)\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu-1}\right\}$$
$$+\left(1-\frac{\alpha_2}{\alpha_1}\right)\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}$$
$$\prec\left(\frac{1+Az}{1+Bz}\right),\qquad f(z)\in\mathbb{MB}(\alpha_2,\lambda,\ell,\mu,\rho)$$

So, $f(z)\in\mathbb{MB}(\alpha_1,\lambda,\ell,\mu,\rho)\subset f(z)\in\mathbb{MB}(\alpha_2,\lambda,\ell,\mu,\rho)$.

Theorem4. Let $\mu>0,\ \alpha\geq 0,-1\leq B<A\leq 1$ and $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$. Then

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<Re\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

and the above inequality is sharp with extremal function defined by

$$\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)=z^{-1}\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}^{\frac{1}{\mu}}$$

Proof: Since$f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$, now by theorem1, we get

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\prec\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\prec\frac{1+Az}{1+Bz}$$

Therefore it follows from the definition of subordination and $A>B$

$$\boldsymbol{\Re e}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\leq\sup(z\in\mathbb{U})\Re e\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}$$
$$\leq\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Also, $\boldsymbol{\Re e}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\geq Inf(z\in\mathbb{U})\Re e\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}$

$$\geq\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Note that the function $\left(\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)\right)$ defined in theorem4 and belongs to the class$\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$, then we get the inequality $\mathbb{MB}(\alpha_1,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})\subset\mathbb{MB}(\alpha_2,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$ of theorem3 is sharp.

Theorem5. Let $\mu>0,\ \alpha\geq 0,-1\leq B<A\leq 1$ and $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$then

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<Re\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

and the above inequality is sharp with extremal function defined by

$$\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)=z^{-1}\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}^{\frac{1}{\mu}}$$

Proof: Since$f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,\mathrm{A},\mathrm{B})$, now by theorem1, we get

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\prec\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\prec\frac{1+Az}{1+Bz}$$

Therefore it follows from the definition of subordination and $A>B$

$$\boldsymbol{\Re e}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\leq\sup(z\in\mathbb{U})\Re e\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}$$
$$\leq\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Also,

$$\boldsymbol{\Re e}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\geq Inf(z\in\mathbb{U})\Re e\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}$$
$$\geq\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Note that the function $\left(\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)\right)$ defined in theorem (4.4) and belongs to the class $\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, then we get the inequality $\mathbb{MB}(\alpha_1,\lambda,\ell,\mu,A,B)\subset\mathbb{MB}(\alpha_2,\lambda,\ell,\mu,A,B)$ of theorem3 is sharp.

Theorem6. Let $\mu>0,\ \alpha\geq 0,-1\leq B<A\leq 1$ and $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, then

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<Re\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

and the above inequality is sharp with extremal function defined by

$$\left(Q_1^{m+1}(\ell,\lambda)f(z)\right)=z^{-1}\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}^{\frac{1}{\mu}}$$

Proof: Since $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, now by theorem1, we get

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\prec\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\prec\frac{1+Az}{1+Bz}$$

Therefore it follows from the definition of subordination and $B>A$

$$\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}>\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

and also we have,

$$\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Therefore $$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu l}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu l}{\alpha\lambda}-1}du \qquad (17)$$

Corollary3.Let $\mu>0,\ \alpha\geq 0,\ 0\leq\rho<1$ and $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, then

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-(1-2\rho)}{1+u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+(1-2\rho)}{1-u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

and equivalent to

$$\rho+(1-\rho)\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-u}{1+u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}$$
$$<\rho+(1-\rho)\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+u}{1-u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du \qquad (18)$$

Proof: By setting $A=(1-2\rho)$ *and* $B=1$ in inequality of theorem5, we obtain

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-(1-2\rho)}{1+u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+(1-2\rho)}{1-u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Also we have,

$$\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{(1-\rho)(1-u)+\rho(1+u)}{1+u}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}$$
$$<\frac{\mu\ell}{\alpha\lambda}\int_0^1\left\{\rho+\frac{(1-\rho)(1+u)}{1-u}\right\}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$
$$\rho+\frac{\mu\ell}{\alpha\lambda}(1-\rho)\int_0^1\frac{(1-u)}{(1+u)}u^{\frac{\mu\ell}{\alpha\lambda}-1}du<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}$$
$$<\rho+\frac{\mu\ell}{\alpha\lambda}(1-\rho)\int_0^1\frac{(1+u)}{(1-u)}u^{\frac{\mu\ell}{\alpha\lambda}-1}du$$

Theorem7. Let $\mu>0,\ \alpha\geq 0,-1\leq B<A\leq 1$ and $f(z)\in\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, Then

$$\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}^{\frac{1}{2}}<\mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\frac{\mu}{2}}<\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\mu\ell}{\alpha\lambda}-1}du\right\}^{\frac{1}{2}}$$

Proof: Since we know that by the theorem1 we have $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}\prec\frac{1+Az}{1+Bz}$. Since $-1\leq B<A\leq 1$, so from inequality of theorem4 for $u=1$

$$0\leq\frac{1-A}{1-B}<\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu}<\frac{1+A}{1+B}\text{ and}$$

$$0 \le \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du < Re\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} < \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$$

$$\text{or } \left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right\}^{\frac{1}{2}} < \mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\frac{\mu}{2}} < \left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right\}^{\frac{1}{2}}$$

Corollary4. Let $\mu > 0$, $\alpha \ge 0, -1 \le B < A \le 1$ and $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, then

$$\left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right\}^{\frac{1}{2}} < \mathfrak{Re}\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\frac{\mu}{2}} < \left\{\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right\}^{\frac{1}{2}}$$

Theorem8. Let > 0, $\alpha \ge 0, -1 \le B < A \le 1$ and $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, Then

Case1. If $\alpha = 0$, when $|z| < 1$, we have

$$r^{-1}\left(\frac{1-Ar}{1-Br}\right)^{\frac{1}{\mu}} \le |Q_1^{m+1}(\ell,\lambda)f(z)| \le r^{-1}\left(\frac{1+Ar}{1+Br}\right)^{\frac{1}{\mu}} \quad (19)$$

and inequality (19) is sharp, with the extremal function defined by

$$Q_1^{m+1}(\ell,\lambda)f(z) \le z^{-1}\left(\frac{1+Az}{1+Bz}\right)^{\frac{1}{\mu}} \quad (20)$$

Case2. If $\alpha \ne 0$, when $|z| = r < 1$, we have

$$r^{-1}\left(\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Aru}{1-Bru} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right)^{\frac{1}{\mu}} \le |Q_1^{m+1}(\ell,\lambda)f(z)| \le r^{-1}\left(\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Aru}{1+Bru} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right)^{\frac{1}{\mu}} \quad (21)$$

and inequality (21) is sharp, with the external function defined by

$$Q_1^{m+1}(\ell,\lambda)f(z) = z^{-1}\left(\frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Aru}{1+Bru} u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right)^{\frac{1}{\mu}} \quad (22)$$

Proof**:** Case1.If $\alpha = 0$, since $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$ and $\mu > 0$, $\alpha \ge 0, -1 \le B < A \le 1$ and now by the definition of $\mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$ that $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$

Therefore it follows from the definition of the subordination that

$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+A\omega(z)}{1+B\omega(z)}$where$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ is analytic in $\mathbb{U}$

and$|\omega(z)| < |z|$,so when $|z| = r < 1$ we have $|zQ_1^{m+1}(\ell,\lambda)f(z)|^{\mu} = \left|\frac{1+A\omega(z)}{1+B\omega(z)}\right| \le \frac{1+Ar}{1+Br}$ and

$|zQ_1^{m+1}(\ell,\lambda)f(z)|^{\mu} \ge \Re e\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \ge \frac{1-Ar}{1-Br}$

It is obvious that (19) is sharp with the extremal function defined by (20).

Case2. If $\alpha \ne 0$, by the theorem1 we have

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz}$$

Therefore it follows from the definition of the subordination where

$$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$$

is analytic in $\mathbb{U}$ and $|\omega(z)| < |z|$,so when $|z| = r < 1$ we have

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} = \frac{\mu\ell}{\alpha\lambda}\int_0^1 \left|\frac{1+A\omega(z)u}{1+B\omega(z)u}\right| u^{\frac{\mu\ell}{\alpha\lambda}-1} du \le \frac{\mu\ell}{\alpha\lambda}\int_0^1 \left|\frac{1+Aur}{1+Bur}\right| u^{\frac{\mu\ell}{\alpha\lambda}-1} du$$

and $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \ge \Re e|zQ_1^{m+1}(\ell,\lambda)f(z)|^{\mu} > \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$.

Theorem9. Let > 0, $\alpha \ge 0, -1 \le B < A \le 1$ and $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, Then

Case1. If $\alpha = 0$, when $|z| = r < 1$, we have

$$r^{-1}\left(\frac{1+Ar}{1+Br}\right)^{\frac{1}{\mu}} \le |Q_1^{m+1}(\ell,\lambda)f(z)| r^{-1}\left(\frac{1-Ar}{1-Br}\right)^{\frac{1}{\mu}} \quad (23)$$

and inequality (23) is sharp, with the extremal function defined by (20)

Case2. If $\alpha \ne 0$, when $|z| = r < 1$, we have

$$r^{-1}\left(\frac{\mu\ell}{\alpha\lambda}\int_0^1 \left(\frac{1+Aru}{1+Bru}\right) u^{\frac{\mu\ell}{\alpha\lambda}-1} du\right)^{\frac{1}{\mu}} \le |Q_1^{m+1}(\ell,\lambda)f(z)| \le r^{-1}\left(\frac{\mu l}{\alpha\lambda}\int_0^1 \left(\frac{1-Aru}{1-Bru}\right) u^{\frac{\mu l}{\alpha\lambda}-1} du\right)^{\frac{1}{\mu}} \quad (24)$$

and inequality (24) is sharp, with extremal function defined by (22).

Proof: Case (i) if $\alpha = 0$, since $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$, and $-1 \le B < A \le 1$, we get from the definition of $f(z) \in \mathbb{MB}(\alpha,\lambda,\ell,\mu,A,B)$ that $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+Az}{1+Bz}$ it follows from the definition of the subordination that $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{1+A\omega(z)}{1+B\omega(z)}$ where $\omega(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$is analytic in $\mathbb{U}$ and $|\omega(z)| < |z|$,so when $|z| = r < 1$ we have $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \geq \frac{1+A|\omega(z)|}{1+B|\omega(z)|} \geq \frac{1+Ar}{1+Br}$ and $|zQ_1^{m+1}(\ell,\lambda)f(z)|^{\mu} \leq \Re e\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \leq \frac{1-Ar}{1-Br}$

It is clear that (17) is sharp with the external function defined by (18), so

$$r^{-1}\left(\frac{1+Ar}{1+Br}\right)^{\frac{1}{\mu}} \leq |Q_1^{m+1}(\ell,\lambda)f(z)| \leq r^{-1}\left(\frac{1-Ar}{1-Br}\right)^{\frac{1}{\mu}} \qquad (25)$$

Case2. If $\alpha \neq 0$, and by the theorem1, we have

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\ell}{\alpha\lambda}-1} du \prec \frac{1+Az}{1+Bz}$$

Therefore it follows from the definition of subordination that

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \prec \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+A\omega(z)}{1+B\omega(z)} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$$

where$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ is analytic in $\mathbb{U}$ and
$|\omega(z)| < |z|$,so when $|z| = r < 1$ we have

$$\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \geq \frac{\mu\ell}{\alpha\lambda}\int_0^1 \left|\frac{1+A\omega(z)u}{1+B\omega(z)u}\right| u^{\frac{\mu\ell}{\alpha\lambda}-1} du \geq \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1+Aur}{1+Bur} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$$

and $\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \leq \Re e\left(zQ_1^{m+1}(\ell,\lambda)f(z)\right)^{\mu} \leq \frac{\mu\ell}{\alpha\lambda}\int_0^1 \frac{1-Aur}{1-Bur} u^{\frac{\mu\ell}{\alpha\lambda}-1} du$

References

[1]. Aouf, M. K. and Mostafa, A. O. (2008) Certain subclass of p-valent meromorphic functions involving certain operator, *J. Inequal. Pure and Appl. Math.*,, (9), Article 45, 8pp.

[2]. Aqlan, E., Jahangiri, J. M. and Kulkarni, S. R. (2003) Certain integral operator applied to meromorphic p-valent functions, *J. of Nat. Geom.*, 24, 111-120.

[3]. Dziok, J. and Srivastava, H. M. (1993) Classes of analytic functions associated with the generalized hypergeometric functions, *Appl. Math. Comput.*, 103, 1-13.

[4]. Dziok, J. and Srivastava, H. M. (2002) Some subclasses of analytic functions with fixed arguments of coefficients associated with generalized hypergeometric function. *Adv. Stud, Contemp. Math.*, 5, 115-125.

[5]. Dziok, J. and Srivastava, H. M. (2003) Certain subclasses of analytic functions associated with the generalized hypergeometric functions. *IntegralTrans. Spec. Funct.*, 14, 7-18.

[6]. El-Ashwah, R. M. (2008) Some properties of certain subclasses of meromorphically multivalent functions. *Appl. Math. Comput.* 20, 824-832.

[7]. Liu, J. L. and Srivastava, H. M. (2004) Subclasses of meromorphically multivalent functions associated with a certain linear operator, *Math. Comput. Modelling*, 39, 35-44.

[8]. Liu, J. L. and Srivastava, H. M. (2001) A linear operator and associated families of meromorphically multivalent functions. *J. Math. Anal. Appl.*, 259, 566-581.

[9]. Liu, J. L. and Srivastava, H. M. (2004) Class of meromorphically multivalent functions associated with the generalized hypergeometric function. *Math. Comut. Modell.*, 39, 21-34.

[10]. Liu, J. L. *(2001)* The noor integral and strongly starlike functions, *J. Math. Anal. Appl. 261, 441-447.*

[11]. Miller, S. S. and Mocanu, P. T. (1981) Differential subordinations and univalent functions, *Michigan Math. J.* 28, 157-171.

[12]. Miller, S. S. and Mocanu, P. T. (2000) Differential monographs and textbooks in pure and applied Mathematics, Vol. 225, Marcel Dekker, New York.

[13]. Muhammad, A. (2010) On certain classes of meromorphic functions defined by means a linear operator, *Acta, Universitatis Apulensis*, 23, 251-262.

[14]. PAP, M. (1998) On certain subclasses of meromorphic m-valent close-to- convex functions, pure Math. Appl., 9, 155-163.

[15]. Srivastava, H. M. and Patel, J. (2005) Application of differential subordination to certain subclasses of meromorphically multivalent functions, *Jipam-Vu-edu. Au.*, Volume 6, Article 88, 30C45, 30D30 and 30C20.

[16]. Srivastava, H. M. and Owa, S. (1987) Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions. Hadamard products, linear operator, and certain subclasses of analytic functions, *Nagoya Math. J.* 106, 1-28.

Mr. Vidyadhar Sharma. " A Study of Distortion Theorem and Inclusion Relations for a new class of Meromorphic Functions." IOSR Journal of Mathematics (IOSR-JM) 15.2 (2019): 21-27.