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Circulant Graphs without Cayley Isomorphism Property with $m = 3$

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Abstract: *A circulant graph $C_n(R)$ is said to have the Cayley Isomorphism (CI) property if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which S = aR. In this paper, we prove that $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are isomorphic circulant graphs without CI-property where R = {1, 9n-1, 9n+1, $3p_1$, $3p_2$, . . . , $3p_{k-2}$}, S = {3n+1, 6n-1, 12n+1, $3p_1$, $3p_2$, . . . , $3p_{k-2}$}, T = {3n-1, 6n+1, 12n-1, $3p_1$, $3p_2$, . . . , $3p_{k-2}$}, $k \geq 3$, gcd(p_1, p_2,..., p_{k-2}) = 1 and $n, p_1, p_2, ..., p_{k-2} \in \mathbb{N}$ and also obtain new abelian groups from these isomorphic circulant graphs.*

AMS Subject Classification: ***05C60, 05C25.***
Keywords: *Adam's isomorphism or Type-1 isomorphism, Type-2 isomorphism, Cayley Isomorphism (CI) property, symmetric equidistance condition, Type-1 group on $C_n(R)$, Type-2 group on $C_n(R)$ w.r.t. m.*

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I. Introduction

Circulant graphs have been investigated by many authors [1]-[16]. An excellent account can be found in the book by Davis [3] and in [6]. A circulant graph $C_n(R)$ is said to have the *Cayley Isomorphism (CI) property* if whenever $C_n(S)$ is isomorphic to $C_n(R)$ there is some $a \in \mathbb{Z}_n$ for which $S = aR$. Finding circulant graphs without CI-property is difficult. Type-2 isomorphism, a new type of isomorphism of circulant graphs, other than already known Adam's isomorphism, was defined and studied in [10,13]. Type-2 isomorphic circulant graphs have the property that they are isomorphic circulant graphs without CI-property.

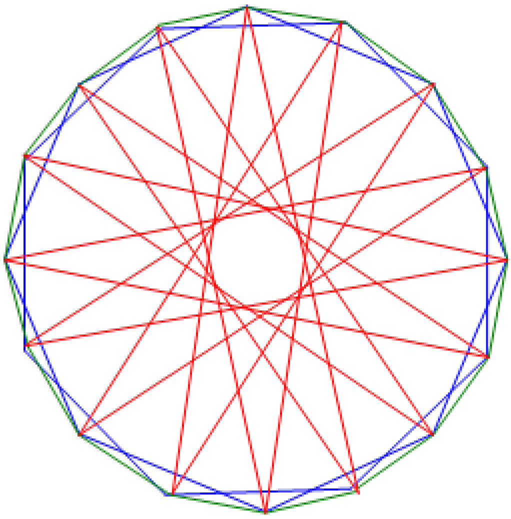
Families of isomorphic circulant graphs of Type-2, each circulant graph of a family with $m_j = gcd(n,r_j)$ number of copies of a circulant subgraph for $m_j = 2$, 5 or 7 are obtained in [14]-[16]. In this paper, we prove that for $n \in \mathbb{N}, k \geq 3$, $R = \{1, 9n\text{-}1, 9n+1, 3p_1, 3p_2, . . . , 3p_{k-2}\}$, $S = \{3n+1, 6n\text{-}1, 12n+1, 3p_1, 3p_2, . . . , 3p_{k-2}\}$ and $T = \{3n\text{-}1, 6n+1, 12n\text{-}1, 3p_1, 3p_2, ..., 3p_{k-2}\}$, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphic with $m_i = 3$ where $gcd(p_1,p_2,...,p_{k-2}) = 1$ and $p_1, p_2,..., p_{k-2} \in \mathbb{N}$ and obtain abelian groups $(Ad_{27n}(C_{27n}(R)), \circ) = (T1_{27n}(C_{27n}(R)), \circ)$, $(V_{27n,3}(C_{27n}(R)), \circ)$ and $(T2_{27n,3}(C_{27n}(R)), \circ)$.

Through-out this paper, for a set $R = \{r_1, r_2, ..., r_k\}$, $C_n(R)$ denotes circulant graph $C_n(r_1, r_2, ..., r_k)$ where $1 \leq r_1 < r_2 < \cdots < r_k \leq [n/2]$. We consider only connected circulant graphs of finite order, $V(C_n(R)) = \{v_0, v_1, v_2, ..., v_{n-1}\}$ with v_i adjacent to v_{i+r} for each $r \in R$, subscript addition taken modulo n and all cycles have length at least 3, unless otherwise specified, $0 \leq i \leq n\text{-}1$. However when $\frac{n}{2} \in R$, edge $v_i v_{i+\frac{n}{2}}$ is taken as a single edge for considering the degree of the vertex v_i or $v_{i+\frac{n}{2}}$ and as a double edge while counting the number of edges or cycles in $C_n(R)$, $0 \leq i \leq n\text{-}1$.

Circulant graph is also defined as a Cayley graph or digraph of a cyclic group. If a graph G is circulant, then its adjacency matrix $A(G)$ is circulant. It follows that if the first row of the adjacency matrix of a circulant graph is $[a_1, a_2, ..., a_n]$, then $a_1 = 0$ and $a_i = a_{n-i+2}$, $2 \leq i \leq n$ [3]. We will often assume, with-out further comment, that the vertices are the corners of a regular *n-gon*, labeled clockwise. Circulant graphs $C_{16}(1,2,7)$ and $C_{16}(2,3,5)$ are shown in Figures 1 and 2, respectively.

Now, we present a few definitions and results that are required in this paper.

Theorem 1.1 [10] *If $C_n(R) \cong C_n(S)$, then there is a bijection f from R to S so that for all $r \in R$, gcd(n, r) = gcd(n, f(r)).* □



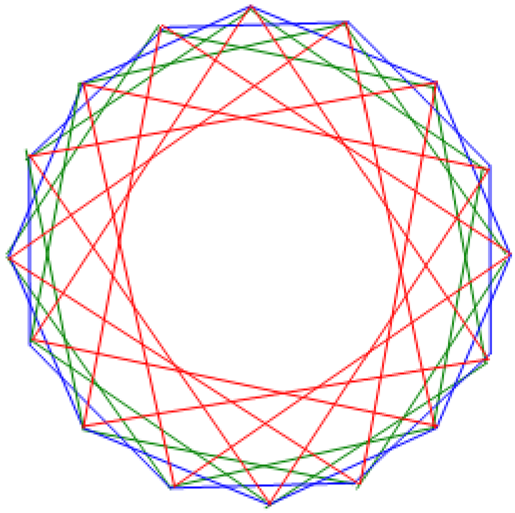


Fig.1. $C_{16}(1,2,7)$ **Fig.2.** $C_{16}(2,3,5)$

Definition 1.2 [9] A circulant graph $C_n(R)$ is said to have the *CI-property* if whenever $C_n(S)$ is isomorphic to $C_n(R)$, there is some $a \in Z_n^*$ for which $S = aR$.

Lemma 1.3 [13] *Let S be a non-empty subset of* $\mathbb{Z}_n$ *and* $x \in \mathbb{Z}_n$. *Define a mapping* $\Phi_{n,x}$: $S \to \mathbb{Z}_n$ *such that* $\Phi_{n,x}(s) = xs$ *for every* $s \in S$ *under multiplication modulo n. Then* $\Phi_{n,x}$ *is bijective if and only if* $S = \mathbb{Z}_n$ *and* $gcd(n, x) = 1$. □

Definition 1.4 [1] Circulant graphs, $C_n(R)$ and $C_n(S)$ for $R = \{r_1, r_2, \ldots, r_k\}$ and $S = \{s_1, s_2, \ldots, s_k\}$ are *Adam's isomorphic* or *Type-1 isomorphic* if there exists a positive integer x relatively prime to n with $S = \{xr_1, xr_2, \ldots, xr_k\}_n^*$ where $< r_i >_n^*$, the *reflexive modular reduction* of a sequence $<r_i>$ is the sequence obtained by reducing each r_i modulo n to yield $r_i^{'}$ and then replacing all resulting terms $r_i^{'}$ which are larger than $\frac{n}{2}$ by n-$r_i^{'}$.

Lemma 1.5 [13] *Let* $m,r,t \in \mathbb{Z}_n$ *such that* $gcd(n, r) = m > 1$ and $0 \leq t \leq \frac{n}{m} - 1$. *Then the mapping* $\theta_{n,r,t}$: $\mathbb{Z}_n \to \mathbb{Z}_n$ *defined by* $\theta_{n,r,t}(x) = x + jtm$ *for every* $x \in \mathbb{Z}_n$ *under arithmetic modulo n is bijective where* $x = qm + j$, $0 \leq j \leq m$-1, $0 \leq q \leq \frac{n}{m} - 1$ *and* $j, q \in \mathbb{Z}_n$. □

Theorem 1.6 [13] *Let* $V(C_n(R)) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$, $V(K_n) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$, $R = \{r_1, r_2, \ldots, r_k, n - r_k, n - r_{k-1}, \ldots, n - r_1\}$ *and* $r \in R$ *such that* $gcd(n, r) = m > 1$. *Then the mapping* $\theta_{n,r,t}$: $V(C_n(R)) \to V(C_n(1,2,\ldots,n\text{-}1)) = V(K_n)$ *defined by* $\theta_{n,r,t}(v_x) = u_{x+jtm}$ *and* $\theta_{n,r,t}((v_x, v_{x+s})) = (\theta_{n,r,t}(v_x), \theta_{n,r,t}(v_{x+s}))$ *for every* $x \in \mathbb{Z}_n$, $x = qm + j$, $0 \leq j \leq m$-1, $0 \leq q, t \leq \frac{n}{m} - 1$ *and* $s \in R$, *under subscript arithmetic modulo n, is one-to-one, preserves adjacency and* $\theta_{n,r,t}(C_n(R)) \cong C_n(R)$ *for* $t = 0,1,2,\ldots,\frac{n}{m} - 1$. □

Definition 1.7 [13] *For a given circulant graph* $C_n(R)$ *and for a particular value of t,* $0 \leq t \leq \frac{n}{m} - 1$ *if* $\theta_{n,r,t}(C_n(R)) = C_n(S)$ *for some* $S \subseteq [1, \frac{n}{2}]$ *and* $S \neq xR$ *for all* $x \in \phi_n$ *under reflexive modulo n, then* $C_n(R)$ *and* $C_n(S)$ *are called Type-2 isomorphic circulant graphs w.r.t. r,* $r \in R$. *In this case, subsets R and S of* $\mathbb{Z}_n$ *are called Type-2 isomorphic subsets of* $\mathbb{Z}_n$ *w.r.t. r.*

Thus, clearly Type-2 isomorphic circulant graphs are circulant graphs without CI-property.

Theorem 1.8 [13] *For* $n \geq 2$, $k \geq 3$, $1 \leq 2s\text{-}1 \leq 2n\text{-}1$, $n \neq 2s\text{-}1$, $R = \{2s\text{-}1, 4n\text{-}2s\text{+}1, 2p_1, 2p_2, \ldots, 2p_{k-2}\}$ and $S = \{2n\text{-}2s\text{+}1, 2n\text{+}2s\text{-}1, 2p_1, 2p_2, \ldots, 2p_{k-2}\}$, *circulant graphs* $C_{8n}(R)$ *and* $C_{8n}(S)$ *are Type-2 isomorphic (and without CI-property) where* $gcd(p_1, p_2, \ldots, p_{k-2}) = 1$ and $n, s, p_1, p_2, \ldots, p_{k-2} \in \mathbb{N}$. □

Theorem 1.9 [13] *For* $R = \{2, 2s\text{-}1, 2s'\text{-}1\}$, $1 \leq t \leq [\frac{n}{2}]$, $1 \leq 2s\text{-}1 < 2s'\text{-}1 \leq [\frac{n}{2}]$ *and* $n, s, s', t \in \mathbb{N}$ *if* $C_n(R)$ *and* $\theta_{n,2,t}(C_n(R))$ *are Type-2 isomorphic circulant graphs for some t, then* $n \equiv 0 \pmod 8$, $2s\text{-}1 + 2s'\text{-}1 = \frac{n}{2}$, $t = \frac{n}{8}$ *or* $\frac{3n}{8}$, $2s'\text{-}1 \neq \frac{n}{8}$, $1 \leq 2s\text{-}1 \leq \frac{n}{4}$ *and* $n \geq 16$. □

Definition 1.10 [13] Let $Ad_n(C_n(R)) = T1_n(C_n(R)) = \{\Phi_{n,x}(C_n(R)): x \in \Phi_n\} = \{C_n(xR): x \in \Phi_n\}$ for a set $R = \{r_1, r_2, \ldots, r_k, n - r_k, n - r_{k-1}, \ldots, n - r_1\}$. Define 'o' in $Ad_n(C_n(R))$ such that $\Phi_{n,x}(C_n(R))$ o $\Phi_{n,y}(C_n(R)) = \Phi_{n,xy}(C_n(R))$ and $C_n(xR)$ o $C_n(yR) = C_n((xy)R)$ for every $x, y \in \Phi_n$, under arithmetic modulo n. Clearly,

$Ad_n(C_n(R))$ is the set of all circulant graphs which are Adam's isomorphic to $C_n(R)$and($Ad_n(C_n(R))$, o)= ($T1_n(C_n(R))$, o)is an abelian group called *the Adam's group* or *theType*-1 *group on* $C_n(R)$ under 'o'.

Definition 1.11 [13] *Let* $V(C_n(R))$ = $\{v_0, v_1, v_2, \ldots, v_{n-1}\}$, $V(K_n)$ = $\{u_0, u_1, u_2, \ldots, u_{n-1}\}$, *r*∈*R*, *m,q,t,t',x*∈$\mathbb{Z}_n$ *such thatgcd*(*n, r*) = *m*> 1, *x* = *qm*+*j*, 0 ≤*j*≤*m*-1 and 0 ≤*q,t,t'* ≤$\frac{n}{m}$ -1. Define $\theta_{n,r,t}$:$\mathbb{Z}_n$→$\mathbb{Z}_n$ and $\theta_{n,r,t}$: $V(C_n(R))$ → $V(C_n(1,2,\ldots,n\text{-}1))$ = $V(K_n)$such that $\theta_{n,r,t}(x)$ = *x*+*jtm*, $\theta_{n,r,t}(v_x)$ = u_{x+jtm} and $\theta_{n,r,t}((v_x, v_{x+y}))$ = ($\theta_{n,r,t}(v_x)$, $\theta_{n,r,t}(v_{x+y})$) for every *x*∈$\mathbb{Z}_n$ and *y*∈*R*,under subscript arithmetic modulo *n*. Let *s*∈$\mathbb{Z}_n$, $V_{n,r}$ = {$\theta_{n,r,t}$: *t* = 0,1,...,$\frac{n}{m}$ -1}, $V_{n,r}(s)$ = {$\theta_{n,r,t}(s)$: *t* = 0,1,...,$\frac{n}{m}$ -1} and $V_{n,r}(C_n(R))$ = {$\theta_{n,r,t}(C_n(R))$: *t* = 0,1,...,$\frac{n}{m}$ -1}. Define 'o' in $V_{n,r}$ such that $\theta_{n,r,t}\, o\, \theta_{n,r,t'}$ = $\theta_{n,r,t+t'}$, $(\theta_{n,r,t}\, o\theta_{n,r,t'})(x)$ (= $\theta_{n,r,t}(\theta_{n,r,t'}(x))$ = $\theta_{n,r,t}(x+jt'm)$ = (*x*+*jt'm*)+*jtm* = *x*+*j*(*t*+*t'*)*m*) = $\theta_{n,r,t+t'}(x)$ and $\theta_{n,r,t}(C_n(R))\, o\, \theta_{n,r,t'}(C_n(R))$ = $\theta_{n,r,t+t'}(C_n(R))$ for every $\theta_{n,r,t}$,$\theta_{n,r,t'}$∈$V_{n,r}$ where *t*+*t'* is calculated under addition modulo $\frac{n}{m}$. Clearly, ($V_{n,r}(s)$, o) and ($V_{n,r}(C_n(R))$, o)are *abelian groups for all s*∈$\mathbb{Z}_n$.

Properties of $\boldsymbol{\theta_{n,r,t}(C_n(R))}$

1.1 Let $\theta_{n,r,t}(C_n(R))$= $C_n(S)$ and r_i∈$\mathbb{Z}_n$ such that *gcd*(*n*,r_i) = *gcd*(*n*,*r*). Then, r_i∈*R* if and only if r_i∈*S*, follows from the definition of $\theta_{n,r,t}$.

1.2 For a given circulant graph $C_n(R)$ and for a particular value of *t*, if $\theta_{n,r,t}(C_n(R))$= $C_n(S)$ for some *S*⊆ [1, [$\frac{n}{2}$]], then $\theta_{n,r,t+t'}(C_n(R))$ = $\theta_{n,r,t'}(C_n(S))$ for every *t'*, 0 ≤t,t'≤$\frac{n}{m}$ -1 where *gcd*(*n*, *r*) = *m*> 1. This follows from the fact, $\theta_{n,r,t+t'}(C_n(R)) = \theta_{n,r,t'+t}(C_n(R)) = \theta_{n,r,t'}(\theta_{n,r,t}(C_n(R))) = \theta_{n,r,t'}(C_n(S))$.

1.3 Let $C_n(R)$ and $C_n(S)$ be isomorphic circulant graphs. Then $C_n(S)$ = $\theta_{n,r,t}(C_n(R))$ for some *t*, 0 ≤ t ≤$\frac{n}{m} - 1$ ifand only if $C_n(R)$ = $\theta_{n,r,\frac{n}{m}-t}(C_n(S))$.This follows from the fact that $\theta_{n,r,\frac{n}{m}-t}(C_n(S))$ =$\theta_{n,r,\frac{n}{m}-t}(\theta_{n,r,t}(C_n(R)))$ = $\theta_{n,r,\frac{n}{m}-t+t}(C_n(R))$= $\theta_{n,r,0}(C_n(R))$= $C_n(R)$ if and only if $C_n(S)$ = $\theta_{n,r,t}(C_n(R))$.

1.4 For isomorphic circulant graphs $C_n(R)$and $C_n(S)$, $C_n(S)$∈$T2_{n,r}(C_n(R))$ if and only if$C_n(S)$ = $\theta_{n,r,t}(C_n(R))$ for some *t*, 0 ≤ t ≤$\frac{n}{m} - 1$ and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. *r*if and only if $C_n(R)$ = $\theta_{n,r,\frac{n}{m}-1}(C_n(S))$ for some *t*, 0 ≤ t ≤$\frac{n}{m} - 1$ and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. *r*if and only if$C_n(R)$∈$T2_{n,r}(C_n(S))$.

1.5 Let $C_n(R)$, $C_n(S)$ be two isomorphic circulant graphs of Type-2 w.r.t. *r*, *r*∈*R*,*S* and *R*≠*S*. Then, $T2_{n,r}(C_n(R)) = T2_{n,r}(C_n(S))$ follows from Property 1.4.

1.6 Let $C_n(R)$ and $C_n(S)$ be two isomorphic circulant graphs and *R*≠*S*. Then, at least one of the following statements is true.

(i) $C_n(S)$= $C_n(xR)$, *x*∈ϕ_n.That is $C_n(R)$ and $C_n(S)$ are Adam's isomorphic.

(ii)$T2_{n,r}(C_n(R))$= $T2_{n,r}(C_n(S))$.This implies that$C_n(R)$ and $C_n(S)$ are Type-2 isomorphic circulant graphs w.r.t. *r*.

(iii)$C_n(S)$≠$C_n(xR)$for all *x*∈ϕ_n and $T2_{n,r}(C_n(R))$≠$T2_{n,r}(C_n(S))$ for any particular*r*∈$\mathbb{Z}_n$. That is circulant graphs $C_n(R)$ and $C_n(S)$ are neither Adam's isomorphic nor Type-2 isomorphic w.r.t. any particular *r*∈$\mathbb{Z}_n$. But their isomorphism is connected by a sequence of isomorphic transformations involving Type-2 isomorphisms w.r.t. different *r*'s or Type-2 isomorphisms w.r.t. different *r*'s as well as Adam's isomorphism.

As an example the two circulant graphs $C_{27}(1,3,8,10)$and$C_{27}(2,7,11,12)$ are isomorphic but they are neither Adam's nor Type-2 isomorphic w.r.t. 3 or 12 (or w.r.t. any particular *r* whose *gcd* with 27 is > 1) because of the following.

a) $\phi_{27,x}(C_{27}(1,3,8,10))$≠$C_{27}(2,7,11,12)$for every *x*∈$\phi_{27}$ (See Table-1). This implies, $C_{27}(1,3,8,10)$and$C_{27}(2,7,11,12)$ are not Adam's isomorphic.

b) Even though *gcd*(27, 3) = 3 = *gcd*(27, 12), the two circulant graphs $C_{27}(1,3,8,10)$and$C_{27}(2,7,11,12)$don't have common jump size, say m, such that *gcd*(27, *m*) = 3 or *gcd*(27, *m*) = 12 and so they can't be Type-2 isomorphic w.r.t. any *m*.

c) $\phi_{27,2}(C_{27}(2,7,11,12))$= $\phi_{27,2}(C_{27}(2,7,11,12,15,16,20,25))$=$C_{27}(4,14,22,24,30,32,40,50)$ =$C_{27}(4,14,22,24,3,5,13,23)$ =$C_{27}(3,4,5,13)$ which implies that $C_{27}(3,4,5,13)$ and $C_{27}(2,7,11,12)$are Adam's isomorphic.

d) $\theta_{27,3,1}(C_{27}(1,3,8,10))$ = $\theta_{27,3,1}(C_{27}(1,3,8,10,17,19,24,26))$=$C_{27}(4,3,14,13,23,22,24,32)$ =$C_{27}(4,3,14,13,23,22,24,5)$= $C_{27}(3,4,5,13)$ which implies, $C_{27}(3,4,5,13)$≅$C_{27}(1,3,8,10)$. Also, $\theta_{27,3,2}(C_{27}(1,3,8,10))$ =$\theta_{27,3,2}(C_{27}(1,3,8,10,17,19,24,26))$ =$C_{27}(7,3,20,16,2,25,24,11)$= $C_{27}(2,3,7,11)$. $\theta_{27,3,3}(C_{27}(1,3,8,10))$ =

$\theta_{27,3,3}(C_{27}(1,3,8,10,17,19,24,26))$=$C_{27}(10,3,26,19,8,1,24,17)$=$C_{27}(1,3,8,10)$.Thus, $C_{27}(3,4,5,13)\cong C_{27}(2,7,11,12)$ and $C_{27}(3,4,5,13)\cong C_{27}(1,3,8,10)$which implies, $C_{27}(1,3,8,10)\cong C_{27}(2,7,11,12)$but theyare not Type-2 isomorphic w.r.t. any particular r.

Thus, we could see that for a given a circulant graph $C_n(R)$ one can make sequence of isomorphic transformations involving Adam's isomorphism as well as Type-2 isomorphisms w.r.t. different r's and obtain an isomorphic circulant graph $C_n(S)$which may not be Adam's isomorphic or Type-2 isomorphic w.r.t. a particular r to $C_n(R)$. And thus a new study is needed to find the sequence of isomorphisms involved among isomorphic circulant graphs.

Table 1. Calculation of xrunder arithmetic modulo 27,$x \in \phi_{27}$and $r \in R$.

	Jump Size r							
Multiplier x	**1**	**3**	**8**	**10**	**17**	**19**	**24**	**26**
2	2	6	16	20	7	11	21	25
4	4	12	5	13	14	22	15	23
5	5	15	13	23	4	14	12	22
7	7	21	2	16	11	25	6	20
8	8	24	10	26	1	17	3	19
10	10	3	26	19	8	1	24	17
11	11	6	7	2	25	20	21	16
13	13	12	23	22	5	4	15	14

Moreover,$V_{n,r}(C_n(R))$ contains all isomorphic circulant graphs of Type 2 of $C_n(R)$ w.r.t. r, if exist. Let $T2_{n,r}(C_n(R))= \{C_n(R)\} \cup \{C_n(S)$: $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. $r\}$. Thus, $T2_{n,r}(C_n(R)) = \{C_n(R)\} \cup\{\theta_{n,r,t}(C_n(R))$: $\theta_{n,r,t}(C_n(R)) = C_n(S)$ and $C_n(S)$ is Type-2 isomorphic to $C_n(R)$ w.r.t. r, $0 \leq t \leq \frac{n}{m}$ - 1$\}\subseteq V_{n,r}(C_n(R))$ and $(T2_{n,r}(C_n(R))$, o) is a subgroup of $(V_{n,r}(C_n(R))$, o) (See Theorem 1.12.). Clearly, $T1_n(C_n(R)) \cap T2_{n,r}(C_n(R)) = \{C_n(R)\}$. $C_n(R)$has Type-2 isomorphic circulant graph w.r.t. r if and only if $T2_{n,r}(C_n(R)) \neq \{C_n(R)\}$ if and only if $T2_{n,r}(C_n(R)) \cap \{C_n(R)\} \neq\Phi$ if and only if $|T2_{n,r}(C_n(R))| > 1$.

Theorem 1.12 [11]Let$C_n(R)$ be any circulant graph, $r\in R$ and $gcd(n, r) > 1$. Then, $(T2_{n,r}(C_n(R))$, o)is a subgroup of $(V_{n,r}(C_n(R))$, o).

ProofClearly, $T2_{n,r}(C_n(R))\subseteq V_{n,r}(C_n(R))$.In $T2_{n,r}(C_n(R))$,$C_n(R) = \theta_{n,r,0}(C_n(R))$. If $T2_{n,r}(C_n(R)) = \{\theta_{n,r,0}(C_n(R))= C_n(R)\}$, then $(T2(\theta_{n,r,0}(C_n(R)))$, o) is a group that contains identity element only.

If $T2_{n,r}(C_n(R))\neq \{\theta_{n,r,0}(C_n(R))= C_n(R)\}$, then let $C_n(S)\in T2_{n,r}(C_n(R))$ with $R\neq S$. This implies, $C_n(S) = \theta_{n,r,t}(C_n(R))$ for some $t$and $C_n(R)$ and $C_n(S)$ are Type-2 isomorphic w.r.t. r, $1 \leq t\leq \frac{n}{m}$ -1. And $T2_{n,r}(C_n(R)) = T2_{n,r}(C_n(S))$, $R\neq S$using the Property 1.5.

This implies, for $1 \leq t,t' \leq \frac{n}{m}$ -1 and $R\neq S$, $\theta_{n,r,t}(C_n(R)) = C_n(S)$ and $C_n(R) = \theta_{n,r,t'}(C_n(S)) = \theta_{n,r,t'}(\theta_{n,r,t}(C_n(R))) = \theta_{n,r,t'+t}(C_n(R)) = \theta_{n,r,t'}(C_n(R))$ o $\theta_{n,r,t}(C_n(R))$, using the definition of $\theta_{n,r,t}$. This implies, $\theta_{n,r,t'}(C_n(R))$o$\theta_{n,r,t}(C_n(R)) = C_n(R) = \theta_{n,r,o}(C_n(R))$, using the definition of $\theta_{n,r,t}$,$\theta_{n,r,t}(C_n(R)) = C_n(S)$,$\theta_{n,r,t'}(C_n(S)) = C_n(R)\in T2_{n,r}(C_n(R))$, $0 \leq t,t' \leq \frac{n}{m}$ -1. This implies that $t+t' \equiv 0$ $(mod \frac{n}{m})$ and also $\theta_{n,r,t'}(C_n(R))$ and $\theta_{n,r,t}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) which implies that $C_n(S)$ and $\theta_{n,r,t'}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) for some t', $1\leq t,t'\leq \frac{n}{m}$ -1 and $t+t' \equiv 0$ (mod $\frac{n}{m}$). This implies, $t'=\frac{n}{m}-t$and $\theta_{n,r,t'}(C_n(R))\in T2_{n,r}(C_n(R))$, $1\leq t,t'\leq \frac{n}{m}$ -1.

Also, we have if $C_n(R)$ and $\theta_{n,r,t}(C_n(R))$ are Type-2 isomorphic for a particular t, then $C_n(R)$ and $\theta_{n,r,\frac{n}{m}-t}(C_n(R))$are also Type-2 isomorphic circulant graphs. This implies, $\theta_{n,r,t'}(C_n(R))\in T2_{n,r}(C_n(R))$ and hence $C_n(S)$ and $\theta_{n,r,t'}(C_n(R))$ are inverse elements in $(T2_{n,r}(C_n(R))$, o) for some t' where$1\leq t,t' \leq \frac{n}{m}$ -1 and $t+t' \equiv 0$ (mod $\frac{n}{m}$).

Other laws of Abelian group are easy to prove. Hence the result follows. □

Definition 1.13 [15] For any circulant graph$C_n(R)$, if group $(T2_{n,r}(C_n(R))$, o) exists, then itis called *the Type-2 group of*$C_n(R)$ *w.r.t. r*under 'o'.

Theorem 1.14 [14] For $n\geq 2$, $k\geq 3$, $1 \leq 2s\text{-}1 \leq 2n\text{-}1$, $n\neq 2s\text{-}1$, $R = \{2s\text{-}1, 4n\text{-}2s\text{+}1, 2p_1, 2p_2,...,2p_{k-2}\}$ and $S = \{2n\text{-}(2s\text{-}1), 2n\text{+}2s\text{-}1, 2p_1,2p_2,...,2p_{k-2}\}$, $T2_{8n,2}(C_{8n}(R)) = T2_{8n,2}(C_{8n}(S))$,$(T2_{8n,2}(C_{8n}(R))$, o)= $(T2_{8n,2}(C_{8n}(S))$, o) is a Type-2 group of order 2 and $(T2_{8n,2}(C_{8n}(R \cup 8n - R))$, o)= $(T2_{8n,2}(C_{8n}(S \cup 8n - S))$, o)where $gcd(p_1,p_2,...,p_{k-2}) = 1$ and $n, s, p_1,p_2,...,p_{k-2} \in N$. □

Obtaining new families of circulant graphs without CI-property is the motivation for this work. For all basic ideas in graph theory, we follow**[5]**.

2 Family of Type-2 Isomorphic Circulant Graphs and Abelian Groups

Theorem2.1*For* $n\in N$, R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1} *and* T = {3, 3n-1, 6n+1, 12n-1}, $C_{27n}(R)$, $C_{27n}(S)$ *and* $C_{27n}(T)$ *are isomorphic circulant graphs.*

Proof: Here, we prove, $\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$ and $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(T)$ when R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1}and T = {3, 3n-1, 6n+1, 12n-1}. To simplify our calculation let us consider R = {1, 3, 9n-1, 9n+1, 18n-1, 18n+1, 27n-3, 27n-1}, S = {3, 3n+1, 6n-1, 12n+1, 15n-1, 21n+1, 24n-1, 27n-3}and T = {3, 3n-1, 6n+1, 12n-1, 15n+1, 21n-1, 24n+1, 27n-3}.

Clearly, $\theta_{n,r,t}$: $V(C_n(R))\rightarrow V(K_n)$ is a bijective function andbythe definition of$\theta_{n,r,t}$, we get $\theta_{27n,3,n}(3)$= 3, $\theta_{27n,3,n}(27n-3)$ = 27n-3, $\theta_{27n,3,n}(1)$ = 3n+1, $\theta_{27n,3,n}(9n+1)$ = 12n+1, $\theta_{27n,3,n}(18n+1)$ = 21n+1, $\theta_{27n,3,n}(9n-1)$ = 15n-1, $\theta_{27n,3,n}(18n-1)$ = 24n-1 and $\theta_{27n,3,n}(27n-1)$ = 6n-1. This implies, $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(S)andC_{27n}(R)\cong C_{27n}(S)$.

Similarly, $\theta_{27n,3,2n}(3)$ = 3, $\theta_{27n,3,2n}(27n-3)$= 27n-3, $\theta_{27n,3,2n}(1)$ = 6n+1, $\theta_{27n,3,2n}(9n+1)$ = 15n+1, $\theta_{27n,3,2n}(18n+1)$ = 24n+1, $\theta_{27n,3,2n}(9n-1)$ = 21n-1, $\theta_{27n,3,2n}(18n-1)$ = 3n-1 and $\theta_{27n,3,2n}(27n-1)$ = 12n-1. This implies, $\theta_{27n,3,2n}(C_{27n}(R)) = C_{27n}(T)andC_{27n}(R)\cong C_{27n}(T)$.This implies that$C_{27n}(R)\cong C_{27n}(S)\cong C_{27n}(T)$. Hence the result.□

Theorem 2.2*For* $n\in N$, R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1} *and* T = {3, 3n-1, 6n+1, 12n-1},$\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S)) = C_{27n}(T)$*and* $\theta_{27n,3,n}(C_{27n}(T)) = C_{27n}(R)$*and*$C_{27n}(R)$, $C_{27n}(S)$ *and* $C_{27n}(T)$ *are Type-*2 *isomorphic circulant graphs.*

Proof: *For* $n\in N$,R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1} *and* T = {3, 3n-1, 6n+1, 12n-1},$\theta_{27n,3,n}(C_{27n}(R)) = C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S)) = C_{27n}(T)$, $\theta_{27n,3,n}(C_{27n}(T)) = C_{27n}(R)$and $C_{27n}(R)\cong C_{27n}(S)\cong C_{27n}(T)$using Theorem 2.1.Also, for a given $n\in N$, the set of jump sizes of the three circulant graphs are different. Here, $R\cap S$ = {3} and so if $C_{27n}(R)$ and$C_{27n}(S)$are Type-2 isomorphic, then they are Type-2 isomorphic w.r.t. m = 3 only.

Claim: For R={1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1}and$n\in N$,$C_{27n}(R)$*and* $C_{27n}(S)$ *are Type-*2 *isomorphic w.r.t.* m = 3.

If not, they are of Adam's isomorphic. This implies, there exists $s\in N$ such that $gcd(27n, s)$ = 1 and $C_{27n}(sR) = C_{27n}(S)$where s = 3x-2 or s = 3x-1, $x\in N$. Now, let s = 3x-2 such that $gcd(27n, 3x-2)$ = 1, $C_{27n}((3x-2)R) = C_{27n}(S)$and $s\in N$.This implies, (3x-2){1, 3, 9n-1, 9n+1, 18n-1, 18n+1, 27n-3, 27n-1} = {3, 3n+1, 6n-1, 12n+1, 15n-1, 21n+1, 24n-1, 27n-3}, under arithmetic modulo 27n. This implies, 3(3x-2), (3x-2)(27n-3), 3+27np_1 and 27n-3+27np_2 are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2\in\mathbb{N}_0$. Thus the following two cases arise.

Case i. 3(3x-2) = 3+27np_1,$p_1\in\mathbb{N}_0$, 1 ≤ 3x-2 ≤ 27n-1.

In this case, p_1 = 0 or 1 or 2 since 1 ≤ 3x-2 ≤ 27n-1 and $n,x\in\mathbb{N}$. When p_1= 0, 3x-2 = 1; p_1 = 1, 3x-2 = 9n+1; p_1 = 2, 3x-2 = 18n+1 and in each case, the two graphs are the same. The jump sizes of the circulant graph corresponding to Adam's isomorphism when s = 3x-2 = 9n+1 and s = 3x-2 = 18n+1 are given in Table 2.

Case ii. 3(3x-2) = 27n-3+27np_2,$p_2\in\mathbb{N}_0$, $x\in\mathbb{N}$, 1 ≤ 3x-2 ≤ 27n-1.

In this case, p_2 = 0 or 1 or 2 since 1 ≤ 3x-2 ≤ 27n-1 and $n,x\in\mathbb{N}$. When p_2= 0, 3x-2 = 9n-1; p_2= 1, 3x-2 = 18n-1; p_2 = 2, 3x-2 = 27n-1 and in each case, the two graphs are the same. The jump sizes of the circulant graph corresponding to Adam's isomorphism when s = 3x-2 = 9n-1, s = 3x-2 = 18n-1 and s = 3x-2 = 27n-1 are given in Table 2.

Table 2. Calculation of rs under arithmetic modulo 27n where s = 3x-2 or 3x-1

	Jump Size *r*					
Multiplier *s*	**1**	**9*n*-1**	**9*n*+1**	**18*n*-1**	**18*n*+1**	**27*n*-1**
9*n*-1	9*n*-1	9*n*+1	27*n*-1	1	18*n*-1	18*n*+1
9*n*+1	9*n*+1	27*n*-1	18*n*+1	9*n*-1	1	18*n*-1
18*n*-1	18*n*-1	1	9*n*-1	18*n*+1	27*n*-1	9*n*+1
18*n*+1	18*n*+1	18*n*-1	1	27*n*-1	9*n*+1	9*n*-1
27*n*-1	27*n*-1	18*n*+1	18*n*-1	9*n*+1	9*n*-1	1

Now, consider the case when s = 3x-1 with $gcd(27n, 3x-1)$ = 1, $C_{27n}(sR)$= $C_{27n}(S)$and $x\in\mathbb{N}$. This implies, (3x-1){1, 3, 9n-1, 9n+1, 18n-1, 18n+1, 27n-3, 27n-1} = {3, 3n+1, 6n-1, 12n+1, 15n-1, 21n+1, 24n-1,

$27n$-3}, under arithmetic modulo $27n$. This implies, $3(3x$-1), $(3x$-1)($27n$-3), $3+27np_1$ and $27n$-3+$27np_2$ are the only numbers, each multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. The following two cases arise.
Case i. $3(3x$-1) = $3+27np_1$,$p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x$-1 $\le 27n$-1.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x$-1 $\le 27n$-1 and $n, x \in \mathbb{N}$. When $p_1 = 0$, $3x$-1 = 1; $p_1 = 1$, $3x$-1= $9n$+1; $p_1 = 2$, $3x$-1 = $18n$+1 and in each case, $C_{27n}(sR) = C_{27n}((3x-1)R) = C_{27n}(S)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x$-1 = $9n$+1 and $s = 3x$-1 = $18n$+1 are given in Table 2.
In this case, $p_2 = 0$ or 1 or 2 since $1 \le 3x$-1 $\le 27n$-1 and $n, x \in \mathbb{N}$. When $p_2 = 0$, $3x$-1 = $9n$-1; $p_2 = 1$, $3x$-1 = $18n$-1; $p_2 = 2$, $3x$-1 = $27n$-1 and in each case, $C_{27n}(sR) = C_{27n}((3x-1)R) = C_{27n}(S)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x$-1 = $9n$-1, $s = 3x$-1 = $18n$-1 and $s = 3x$-1 = $27n$-1 are given in Table 2.
Case ii. $3(3x$-1) = $27n$-3+$27np_2$,$p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x$-1 $\le 27n$-1.
This shows that the isomorphic circulant graphs $C_{27n}(R)$and $C_{27n}(S)$ for R = {1, 3, $9n$-1, $9n$+1} and S = {3, $3n$+1, $6n$-1, $12n$+1} are not of Type-1, $n \in \mathbb{N}$.
Now consider isomorphic circulant graphs$C_{27n}(S)$and $C_{27n}(T)$ for S = {3, $3n$+1, $6n$-1, $12n$+1} and T = {3, $3n$-1, $6n$+1, $12n$-1}, $n \in \mathbb{N}$.Here, $S \cap T$ = {3} and so if $C_{27n}(S)$ and$C_{27n}(T)$are Type-2 isomorphic, then they are Type-2 isomorphic circulant graphs w.r.t. $m = 3$ only.
Claim: For$n \in \mathbb{N}$,S = {3, $3n$+1, $6n$-1, $12n$+1} and T = {3, $3n$-1, $6n$+1, $12n$-1},$C_{27n}(S)$ *and* $C_{27n}(T)$ *are Type-2 isomorphic.*
If not, they are of Adam's isomorphic. This implies, there exists $s \in \mathbb{N}$ such that $gcd(27n, s) = 1$ and $C_{27n}(sS) = C_{27n}(T)$where $s = 3x$-2 or $s = 3x$-1, $x \in \mathbb{N}$. Now, let $s = 3x$-2 such that $gcd(27n, 3x$-2) = 1,$C_{27n}(sS)$ =$C_{27n}((3x-2)S) = C_{27n}(T)$,$x \in \mathbb{N}$.This implies, $(3x$-2){3, $3n$+1, $6n$-1, $12n$+1, $15n$-1, $21n$+1, $24n$-1, $27n$-3} = {3, $3n$-1, $6n$+1, $12n$-1, $15n$+1, $21n$-1, $24n$+1, $27n$-3}, under arithmetic modulo $27n$. Now, $3(3x$-2), $(3x$-2)($27n$-3), $3+27np_1$ and $27n$-3+$27np_2$ are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. Thus the following two cases arise.
Case i. $3(3x$-2) = $3+27np_1$,$p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x$-2$\le 27n$-1.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x$-2$\le 27n$-1 and $n, x \in \mathbb{N}$. This implies,when $p_1 = 0$, $3x$-2 = 1; $p_1 = 1$, $3x$-2= $9n$+1; $p_1 = 2$, $3x$-2 = $18n$+1 and in each case, $C_{27n}(sS) = C_{27n}((3x-2)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x$-2 = $9n$+1 and $s = 3x$-2 = $18n$+1 are given in Table 3.
Case ii. $3(3x$-2) = $27n$-3+$27np_2$,$p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x$-2 $\le 27n$-1.
In this case, $p_2 = 0$ or 1 or 2 since $1 \le 3x$-2 $\le 27n$-1 and $n, x \in \mathbb{N}$. When $p_2 = 0$, $3x$-2 = $9n$-1; $p_2 = 1$, $3x$-2 = $18n$-1; $p_2 = 2$, $3x$-2 = $27n$-1 and in each case, $C_{27n}(sS) = C_{27n}((3x-2)S) = C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when $s = 3x$-2 = $9n$-1, $s = 3x$-2 = $18n$-1 and $s = 3x$-2 = $27n$-1 are given in Table 3.

Table 3. Calculation of rs under arithmetic modulo $27n$ where $s = 3x - 2$ or $3x - 1$.

	Jump Size r					
Multiplier s	**$3n$+1**	**$6n$-1**	**$12n$+1**	**$15n$-1**	**$21n$+1**	**$24n$-1**
$9n$-1	$6n$-1	$12n$+1	$24n$-1	$3n$+1	$15n$-1	$21n$+1
$9n$+1	$12n$+1	$24n$-1	$21n$+1	$6n$-1	$3n$+1	$15n$-1
$18n$-1	$15n$-1	$3n$+1	$6n$-1	$21n$+1	$24n$-1	$12n$+1
$18n$+1	$21n$+1	$15n$-1	$3n$+1	$24n$-1	$12n$+1	$6n$-1
$27n$-1	$24n$-1	$21n$+1	$15n$-1	$12n$+1	$6n$-1	$3n$+1

This shows that the isomorphic circulant graphs $C_{27n}(R)$and $C_{27n}(S)$ for R = {1, 3, $9n$-1, $9n$+1} and S = {3, $3n$+1, $6n$-1, $12n$+1} are not of Type-1, $n \in \mathbb{N}$.
Now consider the case when $s = 3x$-1 with $gcd(27n, 3x$-1) = 1, $C_{27n}((3x-1)S) = C_{27n}(T)$and $x \in \mathbb{N}$. This implies, $(3x$-1){3, $3n$+1, $6n$-1, $12n$+1, $15n$-1, $21n$+1, $24n$-1, $27n$-3} = {3, $3n$-1, $6n$+1, $12n$-1, $15n$+1, $21n$-1, $24n$+1, $27n$-3}, under arithmetic modulo $27n$. This implies, $3(3x$-1), $(3x$-1)($27n$-3), $3+27np_1$ and $27n$-3+$27np_2$ are the only numbers, each is a multiple of 3, in the two sets for some $p_1, p_2 \in \mathbb{N}_0$. The following two cases arise.
Case i. $3(3x$-1) = $3+27np_1$, $p_1 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \le 3x$-1 $\le 27n$-1.
In this case, $p_1 = 0$ or 1 or 2 since $1 \le 3x$-1 $\le 27n$-1 and $n, x \in \mathbb{N}$. When $p_1 = 0$, $3x$-1 = 1; $p_1 = 1$, $3x$-1 = $9n$+1; $p_1 = 2$, $3x$-1 = $18n$+1 and in each case, $C_{27n}(sS) = C_{27n}((3x-1)S) = C_{27n}(T)$. The jump sizes of

the circulant graph corresponding to Adam's isomorphism when $s = 3x$-1 = 9n+1 and $s = 3x$-1 = 18n+1 are given in Table 3.

Case ii. $3(3x\text{-}1) = 27n\text{-}3+27np_2, p_2 \in \mathbb{N}_0$, $x \in \mathbb{N}$, $1 \leq 3x\text{-}1 \leq 27n\text{-}1$.

In this case, p_2 = 0 or 1 or 2 since $1 \leq 3x\text{-}1 \leq 27n\text{-}1$ and $n, x \in \mathbb{N}$. When p_2= 0, 3x-1 = 9n-1; p_2= 1, 3x-1 = 18n-1; p_2 = 2, 3x-1 = 27n-1 and in each case, $C_{27n}(sS)$= $C_{27n}((3x-1)S)$= $C_{27n}(T)$. The jump sizes of the circulant graph corresponding to Adam's isomorphism when s = 3x-1 = 9n-1, s = 3x-1 = 18n-1 and s = 3x-1 = 27n-1 are given in Table 3.

This shows that the isomorphic circulant graphs $C_{27n}(S)$and $C_{27n}(T)$ for S= {3, 3n+1, 6n-1, 12n+1} and T= {3, 3n-1, 6n+1, 12n-1} are not of Type-1, $n \in \mathbb{N}$.

Similarly, we can prove that isomorphic circulant graphs $C_{27n}(R)$and $C_{27n}(T)$ for R = {1, 3, 9n-1, 9n+1} and T= {3, 3n-1, 6n+1, 12n-1} are not of Type-1, $n \in \mathbb{N}$.

Thus, all the 3 different isomorphic circulant graphs$C_{27n}(R)$, $C_{27n}(S)$and $C_{27n}(T)$ for R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1} and T = {3, 3n-1, 6n+1, 12n-1} are not of Type-1. Moreover, $\theta_{27n,3,n}(C_{27n}(R))$= $C_{27n}(S)$, $\theta_{27n,3,n}(C_{27n}(S))$ = $C_{27n}(T)$ *and* $\theta_{27n,3,n}(C_{27n}(T))$ = $C_{27n}(R)$, $n \in \mathbb{N}$.Hence the result follows.□

Theorem 2.3 For $k \geq 3$, R = {1, 9n-1, 9n+1, 3p_1, 3p_2, . . . , 3p_{k-2}}, S = {3n+1, 6n-1, 12n+1, 3p_1, 3p_2, . . . , 3p_{k-2}} and T = {3n-1, 6n+1,12n-1, 3p_1, 3p_2, . . . , 3p_{k-2}}, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphicwith m_i = 3and without CI-property where $gcd(p_1, p_2,...,p_{k-2})$ = 1 and $n, p_1, p_2,..., p_{k-2} \in \mathbb{N}$.

Proof:When R = {1, 3, 9n-1, 9n+1}, S = {3, 3n+1, 6n-1, 12n+1} and T = {3, 3n-1, 6n+1,12n-1},$C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphiccirculant graphs, using Theorem 2.2,$n \in \mathbb{N}$. Lemma 1.5 helps us while searching for possible value(s) of t such that the transformed graph $\theta_{n,r,t}(C_n(R))$is circulant of the form $C_{27n}(S)$ for some $S \subseteq [1, \frac{n}{2}]$, the calculation on r_js which are integer multiples of m = $gcd(n, r)$ need not be done as there is no change in these r_js under the transformation $\theta_{n,r,t}$. This implies when$R$ = {1, 9n-1, 9n+1, 3p_1, 3p_2, . . . , 3p_{k-2}}, S = {3n+1, 6n-1, 12n+1, 3p_1, 3p_2, . . . , 3p_{k-2}} and T = {3n-1, 6n+1,12n-1, 3p_1, 3p_2, . . . , 3p_{k-2}}, circulant graphs $C_{27n}(R)$, $C_{27n}(S)$ and $C_{27n}(T)$ are Type-2 isomorphicwhere $k \geq 3$,$gcd(p_1, p_2,..., p_{k-2})$ = 1 and $n, p_1, p_2,..., p_{k-2} \in \mathbb{N}$. Type-2 isomorphic circulant graphs are graphs without CI-property. Hence the result follows.□

Type 2 isomorphic circulant graphs C_{27}(1,3,8,10), C_{27}(3,4,5,13) and C_{27}(2,3,7,11) are given in Figures 3,4,5, respectively.

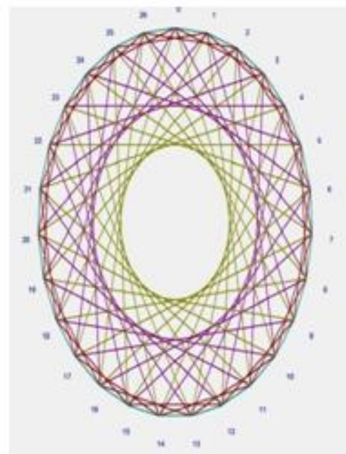
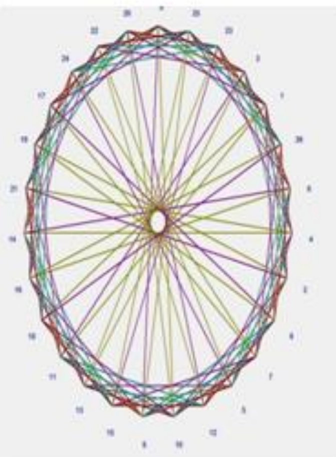
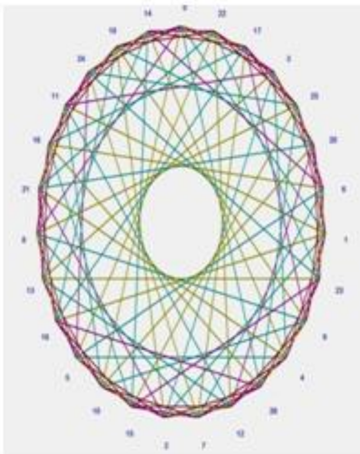




Fig.3.C_{27}(1,3,8,10).**Fig.4**. C_{27}(3,4,5,13).**Fig.5.**C_{27}(2,3,7,11)

II. Conclusion

The results derived in this paper and in [13] on circulant graphs of Type-2 isomorphism and without CI-property are based on circulant graphs with three and two copies of isomorphic circulant subgraphs, respectively. One can try similar results on circulant graphs with m = $gcd(n, r)$ is odd and > 3.

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